

Domain Theory of Ferromagnetics Under Stress: Part I

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The statistical domain theory of ferromagnetism, introduced by Heisenberg and extended by others, is developed in a general form capable of application to any ferromagnetic, crystalline or polycrystalline. Formulas are derived by which the magnetization and strain components can be computed to the first order in the stresses, provided the magnetization curve at zero stress is known. The analysis is valid at any magnetization below that at which the rotation process begins, and the six stress components may have arbitrary values. The formulas are then specialized to nickel crystals; the results reduce to those of Gans and v. Harlem, and of Akulov and Kondorsky, for the special cases treated by them, except for one formula of the former authors and one of the latter. In these cases the original formula is shown to be in error, and the corrected formula leads to better agreement with experiment.

INTRODUCTION

THE statistical domain theory by which Heisenberg¹ calculated the magnetostriction curves of iron crystals has been successfully extended to other fields by Gans and v. Harlem² and by Bozorth.³ A similar calculation carried out by Akulov and Kondorsky,⁴ and leading to formulas for the variation of Young's modulus with magnetization and of magnetostriction with tension, has been less successful; for although the authors found fair agreement with certain data, and although Girenchin⁵ has recently presented data apparently in good agreement with the theoretical formulas, the results of Siegel and Quimby⁶ indicate a need for emendation of the theory. Moreover the formulas are valid only to the third order in the magnetization.

It was recently pointed out by the writer⁷ that because of the magnetic skin effect, measurements of Young's modulus by a dynamical method give approximately the value at constant flux density (B), while static methods give the value at constant magnetizing field (H). It is therefore important, before applying theoretical formulas to such data, to ascertain to which of these two values the formulas correspond.

Another source of uncertainty is the inaccuracy of any method of averaging over the constituent crystals of polycrystalline material; the true average may be expected to lie somewhere between the values obtained by assuming uniform stress and by assuming uniform strain, but Akulov and Kondorsky give only the first of these. It seems desirable, therefore, to carry out the theoretical calculation under various assumptions in regard to the constancy of magnetic quantities during the strain, and by both methods of averaging over the crystals. To do this by the original method of Akulov and Kondorsky would involve tedious calculations, as would any attempt to apply the method to the calculation of the variation of rigidity with magnetization, or to obtain a higher degree of approximation in the formulas. The need for a more flexible procedure is also evident from the fact that so far, every writer who has attempted a further application of Heisenberg's method has gone through a complete derivation of a distribution function for the particular type of crystal and the particular magnetic and mechanical situation under investigation. The method has now proved useful in such a variety of problems that it seems desirable to obtain, once for all, general formulas applicable to any ferromagnetic crystal under the action of any field and stress system.

The application of the theory to polycrystalline material may be made in several different ways, depending on the nature of the material. If each of the constituent crystals contains many

¹ W. Heisenberg, *Zeits. f. Physik* **69**, 287 (1931).

² R. Gans and J. v. Harlem, *Ann. d. Physik* **15**, 516 (1932); **16**, 162 (1933).

³ R. M. Bozorth, *Phys. Rev.* **42**, 882 (1932).

⁴ N. Akulov and E. Kondorsky, *Zeits. f. Physik* **78**, 801 (1932); **85**, 661 (1933).

⁵ B. K. Girenchin, *Physik. Zeits. Sowjetunion* **10**, 689 (1936).

⁶ S. Siegel and S. L. Quimby, *Phys. Rev.* **49**, 664 (1936).

⁷ W. F. Brown, *Phys. Rev.* **50**, 1165 (1936).

domains, the results for crystals may be averaged; this is the procedure followed by Akulov and Kondorsky. If the size of the crystals is of the same order of magnitude as the size of the domains, then, following Gans,⁸ we may regard the substance as an aggregate of crystalline domains with their axes oriented at random. If the crystals are much smaller than the domains, the domains can no longer be regarded as crystalline; they may be isotropic, or may have an anisotropy due to nonrandom orientation of the crystal axes or to other causes. The theory should be flexible enough to apply to any of these cases.

The first object of the present paper is to derive general formulas by which the strains (or stresses) and certain other quantities can be calculated rigorously as functions of the magnetization, and to the first order in the stresses (or strains), for any ferromagnetic under various assumptions regarding the constancy of magnetic quantities during the strain. (The assumption of uniform stress or of uniform strain will enter implicitly, according as one or the other is taken as independent variable.) The formulas will then be applied to nickel crystals; the results will be compared with those of previous writers, especially of Akulov and Kondorsky, and with experimental data.

GENERAL THEORY

Using Heisenberg's simple model, we regard a ferromagnetic specimen as consisting of domains of equal size, N of them in unit volume; these may be divided into classes in accordance with the nature of their anisotropy, so that of the N domains, N^τ are of class τ : then $\sum_\tau N^\tau = N$. For a single crystal, all the domains will belong to a single class, so that the superscript τ and the summation sign \sum_τ may be omitted in all equations. For a polycrystalline specimen as viewed by Gans, the orientation of fixed axes with respect to the crystal axes of a domain may be specified by Eulerian angles ϕ, θ, ψ ; a particular class τ will then consist of those domains for which the direction ϕ, θ is within a solid angle $d\omega$ and for which ψ is within $d\psi$, and N^τ will be $Nd\omega d\psi/8\pi^2$.

⁸ R. Gans, *Physik. Zeits.* 33, 15 (1932).

Each domain of class τ will have certain directions of easy magnetization; we are concerned with magnetizations below the knee of the J - H curve, so that rotation of the magnetization vector out of these directions of easy magnetization may be neglected, and the magnetization process may be assumed to proceed entirely by changes of the spontaneous magnetization in individual domains from one of these directions to another. In the limiting case of isotropic domains, every direction is a direction of easy magnetization, and the transitions degenerate to continuous rotations. A particular microscopic state of the specimen could be described by giving the direction of the magnetization of each domain; those states for which the number of domains of class τ with magnetization in direction σ is a specified number N_σ^τ for each τ and σ are macroscopically indistinguishable and together constitute a macroscopic state of statistical weight

$$W = \prod_\tau (N^\tau!) / \prod_{\tau, \sigma} (N_\sigma^\tau!),$$

or by use of Stirling's approximation,

$$\log W = \text{const.} - \sum_\tau \sum_\sigma N_\sigma^\tau \log N_\sigma^\tau.$$

All variations of the partition numbers N_σ^τ are subject to the conditions

$$\sum_\sigma N_\sigma^\tau = N^\tau. \quad (1)$$

We take, as the independent variables of the system, quantities y_r , each of which is either a thermodynamic coordinate or the negative of a thermodynamic force. Let V be the thermodynamic potential per unit volume corresponding to this choice of independent variables, for isothermal or for adiabatic changes. Then $Y_r = \partial V / \partial y_r$ will be the thermodynamic force corresponding to y_r as coordinate, or the coordinate corresponding to $-y_r$ as force, and these quantities will consequently satisfy the thermodynamic relations

$$\partial Y_r / \partial y_s = \partial Y_s / \partial y_r. \quad (2)$$

V for the specimen as a whole will consist of a part made up of contributions from individual domains, and a part resulting from interaction of the domains. In the contribution of a par-

ticular domain to the former, terms independent of the direction of the spontaneous magnetization may be disregarded for present purposes, and the remainder, in all cases in which we are interested, may be regarded as linear in the y_r 's. Thus the value of V for a domain of class τ and magnetization-direction σ is

$$V_{\sigma}^{\tau} = \sum_r Y_{r\sigma}^{\tau} y_r, \quad (3)$$

where the quantities $Y_{r\sigma}^{\tau}$ are constants, in general different for different σ 's as well as for different τ 's. The value of Y_r for such a domain then consists of a part independent of the magnetization-direction plus a part $\partial V_{\sigma}^{\tau} / \partial y_r = Y_{r\sigma}^{\tau}$ dependent on this magnetization-direction.

If q is any quantity whose observed value for the specimen as a whole is an average of values for the individual domains and whose value for a domain of class τ and magnetization-direction σ is q_{σ}^{τ} , then

$$q = \sum_{\tau} n^{\tau} \sum_{\sigma} n_{\sigma}^{\tau} q_{\sigma}^{\tau}, \quad (4)$$

$$\text{where } n^{\tau} = N^{\tau} / N, \quad n_{\sigma}^{\tau} = N_{\sigma}^{\tau} / N^{\tau}. \quad (5)$$

In particular, the macroscopic quantities Y_r are given by

$$Y_r = Y_{r0} + \sum_{\tau} n^{\tau} \sum_{\sigma} n_{\sigma}^{\tau} Y_{r\sigma}^{\tau} = Y_{r0} + Y_r', \quad (6)$$

where Y_{r0} is the part independent of the distribution of magnetization-directions; and the value of the thermodynamic potential V for the specimen is

$$V = V_0 + \sum_{\tau} n^{\tau} \sum_{\sigma} n_{\sigma}^{\tau} V_{\sigma}^{\tau} + V'' = V_0 + V' + V'', \quad (7)$$

where V_0 is independent of the distribution of magnetization-directions, V'' is due entirely to interaction between domains, and V' is the part contributed by the separate domains and dependent on their magnetization-directions. V' is given by

$$\begin{aligned} V' &= \sum_{\tau} n^{\tau} \sum_{\sigma} n_{\sigma}^{\tau} V_{\sigma}^{\tau} = \sum_{\tau} y_r \sum_{\sigma} n^{\tau} \sum_{\sigma} n_{\sigma}^{\tau} Y_{r\sigma}^{\tau} \\ &= \sum_{\tau} Y_r' y_r. \end{aligned} \quad (8)$$

It is to be noted that the Y_{r0} 's and the Y_r 's satisfy (2) separately, since $Y_{r0} = \partial V_0 / \partial y_r$ and $Y_r' = \partial(V' + V'') / \partial y_r$.

The values of the N_{σ}^{τ} 's for which $\log W$ is a maximum, for a specified value of V' , are determined by

$$\frac{\partial}{\partial N_{\sigma}^{\tau}} \{ \log W + \sum_{\tau} L^{\tau} \sum_{\sigma} N_{\sigma}^{\tau} + L' V' \} = 0,$$

where the L^{τ} 's and L' are Lagrangian multipliers and the N_{σ}^{τ} 's are independent variables in the differentiation. We thus find for the most probable distribution consistent with a prescribed V' and with (1),

$$n_{\sigma}^{\tau} = \exp(LV_{\sigma}^{\tau}) / \sum_{\rho} \exp(LV_{\rho}^{\tau}), \quad (9)$$

where L is determined as a function of V' by

$$\sum_{\tau} n^{\tau} \{ \sum_{\sigma} V_{\sigma}^{\tau} \exp(LV_{\sigma}^{\tau}) / \sum_{\sigma} \exp(LV_{\sigma}^{\tau}) \} = V'. \quad (10)$$

It, of course, does not follow that this is the actual distribution; it can be said only that this is the one we are safest in predicting on the basis of a knowledge of V' , in the absence of definite knowledge of the form of the interaction term V'' . If V'' were known, we should merely find the distribution making $V' + V''$ a minimum.

Equation (6), together with (9) and (10), now determines the Y_r 's as functions of the y_r 's and of the prescribed value of V' . The completion of the analysis requires a knowledge of V' or of L as a function of the y_r 's. This we do not have. We do know, however, that whatever the nature of this function, it must lead to values of the Y_r 's satisfying (2). If one of the Y_r 's, say Y_0' , is known as a function of the corresponding variable y_0 when all the other y_r 's are zero, then V' and hence all the Y_r 's are determined when $y_r = 0$ ($r > 0$). The derivatives $\partial Y_r' / \partial y_s$ are then, as we shall see, completely determined by (2), and thus the Y_r 's are determined to the first order in the y_r 's. Thus from a knowledge of the magnetization curve at zero stress we can calculate the magnetization and the strains as functions of the field, not merely at zero stress, but to the first order in the stresses.

For convenience, we introduce the definitions

$$\begin{aligned} S^{\tau} &= \sum_{\sigma} \exp(LV_{\sigma}^{\tau}), \\ S_r^{\tau} &= \sum_{\sigma} Y_{r\sigma}^{\tau} \exp(LV_{\sigma}^{\tau}), \\ S_{rs}^{\tau} &= \sum_{\sigma} Y_{r\sigma}^{\tau} Y_{s\sigma}^{\tau} \exp(LV_{\sigma}^{\tau}). \end{aligned} \quad (11)$$

We then have, from (6),

$$Y_r - Y_{r0} = Y_r' = \sum_{\tau} n^{\tau} S_r^{\tau} / S^{\tau}, \quad (12)$$

and from (10),

$$\sum_{\tau} n^{\tau} \sum_{\tau} y_r S_r^{\tau} / S^{\tau} = V'. \quad (13)$$

When $y_r = 0$ ($r > 0$), LV_{σ}^{τ} becomes $L^0 Y_{0\sigma}^{\tau} y_0$, and (13) becomes, by virtue of (8),

$$\sum_{\tau} n^{\tau} S_0^{\tau 0} / S^{\tau 0} = Y_0'^0, \quad (13')$$

an equation identical with that one of Eqs. (12) for which $r = 0$. This determines L^0 and the S^{τ} 's, and the rest of Eqs. (12) then determines the quantities

$$Y_r'^0 = \sum_{\tau} n^{\tau} S_r^{\tau 0} / S^{\tau 0}. \quad (12')$$

To evaluate the Y_r' 's to the first order in the y_s 's we differentiate (12) with respect to y_s , using (11), and then set all the y_r 's except y_0 equal to zero. This gives (we omit the primes)

$$\partial Y_r / \partial y_s = LP_{rs} + (\partial L / \partial y_s) P_{r0} y_0, \quad (14)$$

where $P_{rs} = \sum_{\tau} n^{\tau} P_{rs}^{\tau}$,

$$P_{rs}^{\tau} = S_{rs}^{\tau} / S^{\tau} - S_r^{\tau} S_s^{\tau} / S^{\tau 2} = P_{sr}^{\tau}. \quad (15)$$

In these formulas, superscript zeros have been dropped, but it is to be understood that all quantities occurring in the equations are to be evaluated for $y_r = 0$ ($r > 0$), after the differentiations have been performed. We note also the useful relation

$$P_{r0} = \partial Y_r^0 / \partial (L y_0). \quad (15')$$

For a given Y_i and y_j , with i and $j \geq 0$, we can obtain from (14) four equations, by setting, in turn, $r = s = 0$; $r = j$, $s = 0$; $r = 0$, $s = j$; $r = i$, $s = j$. In addition we have, by (2), $\partial Y_0 / \partial y_j = \partial Y_j / \partial y_0$. By elimination of these two quantities and of $\partial L / \partial y_0$ and $\partial L / \partial y_j$, we get

$$\left(\frac{\partial Y_i}{\partial y_j} \right)_{y_0} = L \left(P_{ij} - \frac{P_{i0} P_{j0}}{P_{00}} \right) + \frac{P_{i0} P_{j0}}{P_{00}^2} \frac{\partial Y_0}{\partial y_0}, \quad (16)$$

where y_0 has been written as a subscript to indicate that it is the independent variable held constant in the differentiation.

For the derivative at constant Y_0 , we have

$$\begin{aligned} \left(\frac{\partial Y_i}{\partial y_j} \right)_{Y_0} &= \left(\frac{\partial Y_i}{\partial y_j} \right)_{y_0} - \frac{\partial Y_i}{\partial y_0} \frac{\partial Y_j}{\partial y_0} / \frac{\partial Y_0}{\partial y_0} \\ &= L \left(P_{ij} - \frac{P_{i0} P_{j0}}{P_{00}} \right). \end{aligned} \quad (17)$$

By a similar analysis we may obtain the formulas to be used if a certain number of the Y_r 's, say those for which $r \leq m$, are known as functions of the corresponding y_r 's when the other y_r 's ($r > m$) are zero. In this case the functions must of course be such as to satisfy the thermodynamic relations (2), and they must also be such that the m Eqs. (12), with $r \leq m$, give consistent values of L . For the derivatives at constant y_r ($r \leq m$) we get

$$\left(\frac{\partial Y_i}{\partial y_j} \right)_{y_r} = LP_{ij} + \frac{Q_i Q_j}{Q_r Q_s} \left(\frac{\partial Y_r}{\partial y_s} - LP_{rs} \right), \quad (18)$$

where $Q_i = \sum_{t=1}^m P_{it} y_t$. (19)

Here r and s may have any values (not necessarily different) $\leq m$; if the consistency requirements just mentioned are satisfied, the uniqueness of the result is insured. For the derivatives at constant Y_r ($r \leq m$), we get

$$\left(\frac{\partial Y_i}{\partial y_j} \right)_{Y_r} = L(P_{ij} - T_{ijtu}) - L^2 \sum_{\tau} \sum_{s} \frac{\partial y_r}{\partial Y_s} R_{ist} R_{jru}, \quad (20)$$

where

$$\begin{aligned} T_{ijtu} &= (Q_i Q_u P_{jt} + Q_j Q_t P_{iu} - Q_i Q_j P_{tu}) / Q_t Q_u, \\ R_{ist} &= P_{is} - Q_i P_{ts} / Q_t. \end{aligned} \quad (21)$$

In the differentiations, the independent variables are the Y_r 's ($r \leq m$) and the y_j 's ($j > m$). Formula (20) is required for the evaluation of the strains to the first order in the stresses when the *vector* magnetization remains constant during the strain.

We have, finally (restoring the primes),

$$Y_i = Y_{i0} + Y_i'^0 + \sum_j (\partial Y_i' / \partial y_j) y_j, \quad (22)$$

where the derivatives will be those at constant y_r or at constant Y_r , according as one or the other set of quantities is assumed to remain constant as the y_j 's vary.

NICKEL AND IRON CRYSTALS

To apply the general theory to single crystals of nickel and iron, we begin with the thermodynamic potential F_{He} per unit volume of a single domain, defined by

$$\delta F_{He} = -\sum J_i \delta H_i + \sum X_{ii} \delta e_{ii} + \sum X_{jk} \delta e_{jk}. \quad (23)$$

Here the H_i 's ($i=1, 2, 3$) are the components of magnetic field, the J_i 's the components of magnetization, along the crystal axes; $X_{ij}=X_{ji}$ is the x_i -component of stress exerted across a surface normal to Ox_j ; $e_{ii}=\partial u_i/\partial x_i$, $e_{jk}=\partial u_j/\partial x_k + \partial u_k/\partial x_j$ ($k \neq j$), where the u_i 's are components of displacement. The summation, in this and in subsequent expressions, is over the three sets of values $(i, j, k)=(1, 2, 3)$; $(2, 3, 1)$; $(3, 1, 2)$. For ferromagnetic crystals $J_i=J_s \alpha_i$, where J_s is the spontaneous magnetization and $\alpha_i=\cos(J_s, Ox_i)$. Keeping all the terms that it has been found necessary to introduce in order to account satisfactorily for experimental results,⁹ we have by virtue of the cubic symmetry

$$\begin{aligned} F_{He} = & -J_s \sum H_i \alpha_i + 2K_1 \sum \alpha_i^2 \alpha_k^2 + K_2 \alpha_1^2 \alpha_2^2 \alpha_3^2 \\ & + k_0 \sum e_{ii} + k_1 \sum e_{ii} \alpha_i^2 + 2k_2 \sum e_{jk} \alpha_j \alpha_k \\ & + k_3 (\sum e_{ii}) (\sum \alpha_i^4) + k_4 \sum e_{ii} \alpha_i^4 \quad (24) \\ & + k_5 \sum e_{ii} \alpha_j^2 \alpha_k^2 + k_6 \sum e_{jk} \alpha_i^2 \alpha_j \alpha_k \\ & + \frac{1}{2} c_{11} \sum e_{ii}^2 + c_{12} \sum e_{ij} e_{kk} + \frac{1}{2} c_{44} \sum e_{jk}^2. \end{aligned}$$

The thermodynamic potential F_{HX} defined by

$$\begin{aligned} \delta F_{HX} = & \delta (F_{He} - \sum X_{ii} e_{ii} - \sum X_{jk} e_{jk}) \\ = & -\sum J_i \delta H_i - \sum e_{ii} \delta X_{ii} - \sum e_{jk} \delta X_{jk} \quad (23') \end{aligned}$$

is given by a formula obtainable from (24) by substituting $-X_{ij}$ for e_{ij} , and new constants K_i' , κ_i , $-s_{ij}$ for K_i , k_i , c_{ij} . The relations between the two sets of constants are easily obtained, and need not be given here. The K 's are the constants of magnetic anisotropy, the c 's and s 's are the elastic constants and moduli, and the k 's and κ 's determine the magnetostrictive stresses or strains. F_{He} is to be used when the strain is assumed uniform, F_{HX} when the stress is assumed uniform.

⁹ F. Bitter, Phys. Rev. 42, 697 (1932); 43, 655 (1933); R. Gans, Physik. Zeits. 33, 924 (1932); reference 2. The h_2 of reference 2 is our $2\kappa_2$. The k_5 -term is really superfluous, since $\alpha_j^2 \alpha_k^2 = \frac{1}{2} - \alpha_i^2 + \alpha_i^4 - \frac{1}{2} \sum \alpha_i^4$.

The actual values of the α_i 's are those that make F_{He} or F_{HX} a minimum; but since we are neglecting the small rotation produced by the field and the strains, we have for nickel $\alpha_i=p_i/\sqrt{3}$, where p_i is either $+1$ or -1 for each value of i ; and for iron $\alpha_i=+\delta_{ir}$ or $-\delta_{ir}$, where $\delta_{ii}=1$, $\delta_{ij}=0$ ($j \neq i$), and r may be 1, 2, or 3. Omitting the terms that are the same for all the directions of easy magnetization, and using the condition that the average strains must vanish in the demagnetized state, we get for the part of F_{HX} dependent on the direction of magnetization of the domain

$$\begin{aligned} \text{(Ni)} \quad F_{HX}' = & \frac{1}{r} (J_s H / \sqrt{3}) \sum p_i l_i \\ & - (2\kappa_2/3 + \kappa_6/9) \sum p_i p_k X_{jk}, \quad (25) \end{aligned}$$

$$\begin{aligned} \text{(Fe)} \quad F_{HX}' = & \mp J_s H l_r \\ & - (\kappa_1 + \kappa_4) \sum (\delta_{ir} - \frac{1}{3}) X_{ii}, \quad (25') \end{aligned}$$

where the l_i 's are the direction-cosines of the field. For the parallel component of magnetization, J_p , and for the part of the strains depending on the magnetization-direction, we have

$$\begin{aligned} J_p = & -\partial F_{HX}' / \partial H, \quad e_{ij}' = -\partial F_{HX}' / \partial X_{ij}; \\ \text{or} \quad \text{(Ni)} \quad J_p = & (J_s / \sqrt{3}) \sum p_i l_i, \quad (26) \\ e_{ij}' = & (2\kappa_2/3 + \kappa_6/9) p_i p_j \quad (j \neq i); \end{aligned}$$

$$\begin{aligned} \text{(Fe)} \quad J_p = & \pm J_s l_r, \quad (26') \\ e_{ii}' = & (\kappa_1 + \kappa_4) (\delta_{ir} - \frac{1}{3}). \end{aligned}$$

Gans and v. Harlem found it necessary to include the quantities denoted here by κ_4 and κ_6 , in order to obtain a good fit at saturation. For our purposes the coefficients of $p_i p_j$ and of $(\delta_{ir} - 1/3)$ are merely constants, and they will therefore be written hereafter as $2\kappa_2/3$ and κ_1 respectively, in conformity with Akulov's notation. They may be determined from the saturation magnetostriction in the [111] and [100] directions, by the relations

$$\kappa_2 = 3\lambda_{111}/2, \quad \kappa_1 = 3\lambda_{100}/2. \quad (27)$$

Assuming uniform strain rather than uniform stress, we need only replace e , $-X$, κ by X , e , k respectively in the analysis to follow. The new κ 's defined by (27) are related to the corresponding k 's by the formulas

$$\begin{aligned} -k_2/\kappa_2 = & c_{44} = 1/s_{44}, \\ -k_1/\kappa_1 = & c_{11} - c_{12} = 1/(s_{11} - s_{12}). \quad (27') \end{aligned}$$

NICKEL CRYSTALS

The calculation is simplified and the symmetry of the equations increased, if the field is assumed to remain constant in direction and to vary only in magnitude. Equivalent results are obtained by treating the three field-components as independent variables; Eq. (18) must then be used instead of (16). We shall present only the simpler derivation. By comparison of Eqs. (25) and (3), we see that for a nickel crystal, assuming uniform stress, we may take $F_{HX'}$ as V' , and write

$$\begin{aligned} y_0 &= -H, & y_1 &= -X_{ij}, & y_2 &= -X_{ik} \quad (k \neq i), \\ Y_0 &= J_p, & Y_1 &= e_{ij}, & Y_2 &= e_{ik}, \end{aligned} \quad (28)$$

$$Y_{0\sigma} = (J_s/\sqrt{3})\sum p_i l_i, \quad Y_{1\sigma} = 2\kappa_2 p_i p_j / 3,$$

$$Y_{2\sigma} = 2\kappa_2 p_i p_k / 3.$$

The first of Eqs. (11) becomes

$$\begin{aligned} S &= \sum_p \exp [(-LHJ_s/\sqrt{3})\sum p_i l_i] \\ &= \sum_p \exp [\eta \sum p_i l_i], \end{aligned}$$

$$\text{where} \quad \eta = -LHJ_s/\sqrt{3}; \quad (29)$$

the outside summation is over the eight combinations of values $(p_1, p_2, p_3) = (1, 1, 1); (1, 1, -1); \dots (-1, -1, -1)$, so that

$$S = 8 \cosh \eta l_1 \cosh \eta l_2 \cosh \eta l_3. \quad (30)$$

The evaluation of the other sums is facilitated by noting that p_i , multiplying the exponential factor, is equivalent to $\partial/\partial(\eta l_i)$, which may be taken outside the summation sign \sum_p ; moreover $[\partial/\partial(\eta l_i)]^n$, operating on S , is equivalent to 1 if n is even and to $\tanh \eta l_i$ if n is odd. Writing for brevity $t_i = \tanh \eta l_i$, $s_i = \text{sech } \eta l_i$, we have

$$\begin{aligned} S_0/S &= (J_s/\sqrt{3})\sum l_i t_i, \\ S_1/S &= 2\kappa_2 t_i t_j / 3, \\ S_2/S &= 2\kappa_2 t_j t_k / 3; \end{aligned} \quad (31)$$

$$\begin{aligned} P_{00} &= J_s^2 \sum l_i^2 s_i^2 / 3, \\ P_{10} &= (2\kappa_2 J_s / 3\sqrt{3})(l_i s_i^2 t_j + l_j s_j^2 t_i), \\ P_{20} &= (2\kappa_2 J_s / 3\sqrt{3})(l_j s_j^2 t_k + l_k s_k^2 t_j), \\ P_{11} &= (4\kappa_2^2 / 9)(1 - t_i^2 t_j^2), \\ P_{22} &= (4\kappa_2^2 / 9)(1 - t_j^2 t_k^2), \\ P_{12} &= (4\kappa_2^2 / 9)t_i s_j^2 t_k. \end{aligned} \quad (32)$$

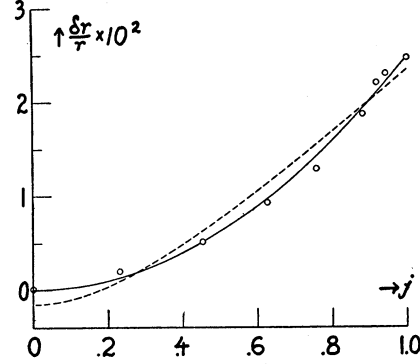


FIG. 1. Variation of longitudinal resistivity with magnetization for nickel crystal magnetized in $[111]$ direction. Points, experimental values of Kaya as plotted by Gans and v. Harlem; Dashed curve, formula of Gans and v. Harlem; solid curve, corrected formula. Each curve has been fitted separately to the data by least squares.

Eqs. (12') and (13') therefore become

$$e_{ij}{}^0 = 2\kappa_2 t_i t_j / 3, \quad j_p^0 = J_p^0 / J_s = (1/\sqrt{3})\sum l_i t_i. \quad (33)$$

Eqs. (16) and (17) become

$$\begin{aligned} \partial e_{ij}{}^0 / \partial X_{ij} &= L_0(P_{11} - P_{10}^2 / P_{00}) + \alpha \chi P_{10}^2 / P_{00}^2, \\ \partial e_{ij}{}^0 / \partial X_{jk} &= L_0(P_{12} - P_{10} P_{20} / P_{00}) \\ &\quad + \alpha \chi P_{10} P_{20} / P_{00}^2, \end{aligned} \quad (34)$$

where $\chi = \partial J_p^0 / \partial H$; $L_0 (= -L)$ is determined as a function of H by (29), the second of Eqs. (33), and the magnetization curve at zero stress; α is 1 if the differentiation is at constant H , 0 if at constant J_p ; and the P_{ij} 's are given by Eqs. (32).

The two Eqs. (33) give in parametric form the functional relation between the magnetostrictive strain components $e_{ij}{}^0$ and the magnetization J_p^0 in the field direction. The parameter η runs from 0 at demagnetization to ∞ at pseudo-saturation ($j_p^0 = \sum |l_i| / \sqrt{3}$). The magnetostrictive elongation in direction (v_1, v_2, v_3) is

$$e_v{}^0 = \sum e_{ik}{}^0 v_i v_k = 2\kappa_2 \sum v_j v_k t_j t_k / 3, \quad (35)$$

and a similar formula holds for the magnetization-dependent part of the electrical resistivity. The magnetization components (see Eq. (4)) are given by

$$j_i^0 = J_i^0 / J_s = t_i / \sqrt{3}. \quad (36)$$

Eq. (35) may therefore be written

$$e_v{}^0 = 2\kappa_2 \sum v_i v_k j_i^0 j_k^0 = 2\kappa_2 j^{02} \sum v_i v_k \sigma_i \sigma_k, \quad (37)$$

where J^0 is the magnitude of \mathbf{J}^0 (not to be confused with the parallel component J_p^0), $j^0 = J^0 / J_s$,

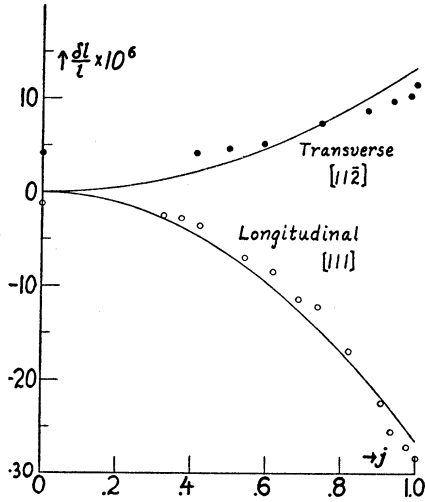


FIG. 2. Variation of longitudinal and transverse ([112]) magnetostriction with magnetization for nickel crystal magnetized in [111] direction. Points, experimental values of Mashiyama as plotted by Gans and v. Harlem. Curves, corrected formula. The curves have been fitted to the data by least squares with a single value of κ_2 , 39.9×10^{-6} .

and the σ_i 's are the direction cosines of the magnetization. This formula is useful only when it is known from symmetry considerations that the magnetization direction $(\sigma_1, \sigma_2, \sigma_3)$ coincides with the field direction (l_1, l_2, l_3) . This is the case when the field is in one of the directions [100], [110], [111]; for these three cases, the right member of (37) reduces to

$$[100] 0, [110] \kappa_2 j^2 v_1 v_2, [111] 2\kappa_2 j^2 \sum v_i v_k / 3. \quad (38)$$

The first two of these agree with the formulas of Gans and v. Harlem, but in their [111] formulas¹⁰ our factor j^2 is replaced by $\frac{1}{2}[(8j^2+1)^{\frac{1}{2}}-1]$. The discrepancy is traceable to their erroneous assumption¹¹ that when the resultant magnetization is in the [111] direction, the same number of domains are magnetized in each of the directions $[\bar{1}\bar{1}1]$, $[1\bar{1}\bar{1}]$, $[11\bar{1}]$ on the one hand, and in each of the directions $[1\bar{1}\bar{1}]$, $[\bar{1}\bar{1}1]$ on the other.

Figure 1 shows the improved agreement between theory and experiment obtained by using the corrected formula for the longitudinal resistivity.

¹⁰ See Eq. (6) of their first paper and Eqs. (11), (12) of their second (reference 2). R. H. Fowler, *Statistical Mechanics*, second edition (1936), p. 520, gives correct formulas and curves for the longitudinal magnetostriction in these three special directions.

¹¹ See the third of Eqs. (5), first paper, reference 2.

The agreement with the data on longitudinal and transverse magnetostriction is less satisfactory (see Fig. 2), but still constitutes an improvement over the original formula.

The distribution functions n_σ do not occur explicitly in any of the present formulas; they may, however, be found from Eq. (9). Thus the fractional number of domains with magnetization along $(p_1/\sqrt{3}, p_2/\sqrt{3}, p_3/\sqrt{3})$ when the stresses are zero is given by

$$\begin{aligned} n^0_{p_1 p_2 p_3} &= [\exp(\eta \sum l_i p_i)] / S \\ &= (1/S) \prod_i (\cosh \eta l_i + p_i \sinh \eta l_i) \\ &= [1 + \sum p_i l_i + \sum p_i p_k l_i l_k \\ &\quad + p_1 p_2 p_3 l_1 l_2 l_3] / 8 \quad (39) \\ &= [1 + \sqrt{3} j^0 \sum p_i \sigma_i + 3 j^0 \sum p_i p_k \sigma_i \sigma_k \\ &\quad + 3\sqrt{3} j^0 p_1 p_2 p_3 \sigma_1 \sigma_2 \sigma_3] / 8. \end{aligned}$$

This reduces to formulas identical with those of Gans and v. Harlem¹² for the special case $\mathbf{J} \parallel [110]$, and it is identical with the formulas obtained from those of Akulov and Kondorsky¹³ by setting the tension equal to zero. The present derivation shows however that the formula thus obtained is rigorous to all orders in the magnetization, and not merely a third-order approximation as might be expected from Akulov and Kondorsky's derivation.

Equations (34) give the variation of the effective elastic moduli s_{ij}' of the crystal with magnetization. We see that the tension-elongation moduli are unaffected by the magnetization, but that the relations between shearing stress and shearing strain are altered: the field destroys the cubic symmetry and leads to relations of the more general type

$$e_{12} = s_{66}' X_{12} + s_{64}' X_{23} + s_{65}' X_{31}.$$

In (34), α is to be set equal to 1 if the moduli are measured by a static method. The same equations with $\alpha=0$ give the values when the direction of the field and the component of magnetization (and hence approximately of flux density) in this direction remain constant during the strain. This will be the case, in a dynamic measurement of the elastic moduli, only if the

¹² First paper, Eqs. (5), reference 2.

¹³ Akulov and Kondorsky, *Zeits. f. Physik* **85**, 661 (1933), Eq. (22).

field-direction and the stress system have such symmetry with respect to the crystal axes that the magnetization and the field are in the same direction even during the strain. This condition can be satisfied by utilizing the longitudinal vibrations of a thin crystalline cylinder with cylinder axis in one of the orientations [100], [110], [111]. In general, however, the values at constant *vector* magnetization must be obtained by means of Eq. (20).

Since experimental data are not available, it is not yet possible to test these theoretical conclusions in regard to the elastic properties of ferromagnetic crystals. When experiments of this nature are undertaken, it should be remembered that the theory takes no account of magnetic hysteresis, and that therefore the use of the actual magnetization curve may lead to erroneous results. Since the stress-strain relations are themselves nearly reversible,⁷ it seems plausible that so far as these are concerned the actual specimen may be equivalent to a reversible specimen whose differential susceptibility, at any magnetization, is equal to the *reversible* susceptibility¹⁴ of the actual specimen. This, as well as the usual magnetization data, should therefore be determined for the specimens used in the elastic measurements.

The theory should apply rigorously to data obtained anhysteretically: that is, by shaking the specimen into its most stable state before each reading. This is usually accomplished by applying an alternating magnetic field and gradually reducing it to zero. The writer has found, however, that in obtaining anhysteretic magnetization curves it is safer not to rely on a single reduction of this sort, but to vary the alternating field rapidly between zero and its maximum value, several times, in order to insure the penetration of the field to the interior of the specimen. In order that the data may be truly anhysteretic, it is of course necessary that this procedure be followed whenever either field or stress is changed; *vibrations* at a point on the anhysteretic magnetization curve will not give an anhysteretic value of Young's modulus.

¹⁴ R. Gans, *Physik. Zeits.* **12**, 1053 (1911); *Ann. d. Physik* **61**, 379 (1920); P. Debye, *Handbuch der Radiologie* (1925), Vol. 6, 721.

We shall now use the results already obtained to derive formulas corresponding to those of Akulov and Kondorsky for the ΔE effect at low magnetizations.

THIRD-ORDER APPROXIMATION

We assume that the magnetization components J_i^0 under zero stress are known as functions of the field components H_i , to the fourth order. The assumed functional relations are subject to three types of restriction: (a) those imposed by thermodynamics; (b) those imposed by the cubic symmetry; (c) the restriction, already mentioned, that Eqs. (36) with $i=1, 2, 3$ must yield a unique value of η . The thermodynamic conditions may be satisfied by deriving the J_i 's from the thermodynamic potential $F=F_H, x=0$, which must be expressed to the fifth order in the H_i 's. The cubic symmetry then requires that this be in the form

$$-F = \frac{1}{2}\chi_0 H^2 + \frac{1}{4}\chi_1 H^4 + \frac{1}{4}\chi_2 \sum H_i^4,$$

where the χ 's are constants. Hence

$$J_i^0 = -\partial F / \partial H_i = H_i [\chi_0 + (\chi_1 H^2 + \chi_2 H_i^2)] \\ = H l_i [\chi_0 + (\chi_1 + \chi_2 l_i^2) H^2]. \quad (40)$$

By expansion of the hyperbolic tangent to the fourth order in η , we get from Eq. (36)

$$J_i^0 = (J_s / \sqrt{3}) \eta l_i (1 - \eta^2 l_i^2 / 3). \quad (41)$$

If we equate the right members of (40) and (41) and solve for η to the fourth order in H , the result, by restriction (c), must be the same for $i=1, 2$, and 3. This requires

$$\chi_2 = -\chi_0^3 / J_s^2; \quad (42)$$

then

$$\eta = (\sqrt{3}\chi_0 H / J_s) [1 + (\chi_1 / \chi_0) H^2], \\ J_i^0 = H l_i [\chi_0 + \chi_1 H^2 - (\chi_0^3 / J_s^2) H_i^2], \\ J_p^0 = \sum J_i^0 l_i = H [\chi_0 + (\chi_1 - \chi_0^3 \sum l_i^4 / J_s^2) H^2]. \quad (43)$$

In terms of j_p^0 we have

$$\eta = \sqrt{3} j_p^0 [1 + (\sum l_i^4) j_p^{0^2}], \\ L_0 = (3\chi_0 / J_s^2) [1 + (\chi_1 J_s^2 / \chi_0^3) j_p^{0^2}]. \quad (44)$$

From (33) and (34) we now get, to the third order in j_p^0 ,

$$\begin{aligned}
 e_{ij}'^0 &= 2\kappa_2 l_i l_j j_p^0, \\
 \left(\frac{\partial e_{ij}'}{\partial X_{ij}}\right)_H &= \frac{4\kappa_2^2 \chi_0}{3J_s^2} \left\{ 1 + \frac{\chi_1 J_s^2}{\chi_0^3} j_p^0 \right\}, \\
 \left(\frac{\partial e_{ij}'}{\partial X_{jk}}\right)_H &= \frac{4\kappa_2^2 \chi_0}{J_s^2} l_i l_k j_p^0, \\
 \left(\frac{\partial e_{ij}'}{\partial X_{ij}}\right)_{J_p} &= \frac{4\kappa_2^2 \chi_0}{3J_s^2} \left\{ 1 - \left(12l_i^2 l_j^2 - \frac{\chi_1 J_s^2}{\chi_0^3} \right) j_p^0 \right\}, \\
 \left(\frac{\partial e_{ij}'}{\partial X_{jk}}\right)_{J_p} &= \frac{4\kappa_2^2 \chi_0}{J_s^2} l_i l_k (1 - 4l_j^2) j_p^0.
 \end{aligned} \tag{45}$$

Although these results have been written for convenience in terms of j_p^0 , they can be expressed in terms of H by replacing j_p^0 by $\chi_0 H/J_s$. The corresponding formulas when the magnetization components are the independent variables are

$$\begin{aligned}
 e_{ij}'^0 &= 2\kappa_2 \sigma_i \sigma_j j^2 \quad (j = J/J_s), \\
 \left(\frac{\partial e_{ij}'}{\partial X_{ij}}\right)_J &= \frac{4\kappa_2^2 \chi_0}{3J_s^2} \left\{ 1 - \left[3(\sigma_i^2 + \sigma_j^2) - \frac{\chi_1 J_s^2}{\chi_0^3} \right] j^2 \right\}, \\
 \left(\frac{\partial e_{ij}'}{\partial X_{jk}}\right)_J &= 0.
 \end{aligned} \tag{46}$$

These may be obtained by use of Eq. (20); or they may be derived from Eqs. (45) by first writing the complete expression for e_{ij}' to the third order in the H_i 's and to the first order in the stresses, and then substituting for H_i its value to the second order in the J_i 's and to the first in the stresses. This is obtained by inversion of the formula

$$J_i = \chi_0 H_i + c(H_i X_{ij} + H_k X_{ik}),$$

where

$$c = \frac{1}{H_j} \left(\frac{\partial J_i}{\partial X_{ij}} \right)_H = \frac{1}{H_j} \frac{\partial e_{ij}'^0}{\partial H_i} = \frac{2\kappa_2}{J_s^2} \chi_0^2. \tag{47}$$

From the formulas just given, we can derive the following formulas for the elongation e_v' in direction (v_1, v_2, v_3) when the stress-system consists merely of a tension T in the same direction $(X_{ij} = v_i v_j T)$:

$$\begin{aligned}
 e_v'^0 &= 2\kappa_2 j_p^0 \sum l_i l_k v_i v_k, \\
 \left(\frac{\partial e_v'}{\partial T}\right)_H &= \frac{4\kappa_2^2 \chi_0}{3J_s^2} \sum v_i^2 v_k^2 + \frac{4\kappa_2^2 \chi_0}{J_s^2} j_p^0 \\
 &\quad \times \left[2 \sum v_i^2 v_j v_k l_i l_k + \frac{1}{3} \frac{\chi_1 J_s^2}{\chi_0^3} \sum v_i^2 v_k^2 \right].
 \end{aligned} \tag{48}$$

In $(\partial e_v'/\partial T)_{J_p}$, the quantity in square brackets is replaced by

$$\begin{aligned}
 &[2 \sum v_i^2 v_j v_k l_i l_k - 4 \sum l_i^2 l_k^2 v_i^2 v_k^2 - 8 \sum l_i^2 l_j l_k v_i^2 v_j v_k \\
 &\quad + \frac{1}{3} (\chi_1 J_s^2 / \chi_0^3) \sum v_i^2 v_k^2].
 \end{aligned} \tag{49}$$

If the magnetization components are taken as independent variables, we have

$$\begin{aligned}
 e_v'^0 &= 2\kappa_2 j^2 \sum \sigma_i \sigma_k v_i v_k, \\
 \left(\frac{\partial e_v'}{\partial T}\right)_J &= \frac{4\kappa_2^2 \chi_0}{3J_s^2} \sum v_i^2 v_k^2 - \frac{4\kappa_2^2 \chi_0}{J_s^2} j^2 \\
 &\quad \times \left[\left(1 - \frac{1}{3} \frac{\chi_1 J_s^2}{\chi_0^3} \right) \sum v_i^2 v_k^2 - \sum \sigma_i^2 v_j^2 v_k^2 \right].
 \end{aligned} \tag{50}$$

The last formula should agree with that of Akulov and Kondorsky. In their formula, however, the quantity $-\sum \sigma_i^2 v_j^2 v_k^2$ is replaced by $+2 \sum v_i^2 v_j v_k \sigma_i \sigma_k$. The difference appears to be due to the omission of terms in the tension, in certain of their formulas in which such terms should be present: particularly in the assumed relation between the J_i 's and the H_i 's, where no account is taken of the effect of the tension upon the magnetization curve.

The formulas to be used when the direction of the tension coincides with that of the field or of the magnetization are obtained by setting $v_i = l_i$ in (48-49) or $v_i = \sigma_i$ in (50). It is easily verified that (49) and (50) then lead to identical results when this direction is [100], [110], or [111].

POLYCRYSTALLINE NICKEL

Before averaging for polycrystalline material, we must decide which quantities are to be assumed constant during the stress, and also which quantities are to be assumed uniform from crystal to crystal. In the dynamical measurement of Young's modulus, it is the value at constant B that is measured; this however differs inappreciably from the value at constant J . This con-

stancy of flux density is due to macroscopic eddy currents, whose field neutralizes the tendency of the stress to change the flux density. It is evident from the relation $\partial J/\partial T = \partial e_v/\partial H$ and from (50) that this neutralization cannot be exact in each crystal: only the *average* magnetization remains constant. We therefore average (48) at uniform H and then use, for the polycrystalline material as a whole, the relation

$$\left(\frac{\partial e_v}{\partial T}\right)_J = \left(\frac{\partial e_v}{\partial T}\right)_H - \left(\frac{\partial e_v}{\partial H}\right)_T / \chi. \quad (51)$$

This gives, when the tension is in the field direction,

$$e_v'^0 = 2\kappa_2 j^{02}/5, \quad (52)$$

$$\left(\frac{\partial e_v'}{\partial T}\right)_H = \frac{4}{15} \frac{\kappa_2^2 \chi_0}{J_s^2} \left[1 + 3j^{02} \left(\frac{2}{7} + \frac{1}{3} \frac{\chi_1 J_s^2}{\chi_0^3} \right) \right], \quad (53)$$

$$\left(\frac{\partial e_v'}{\partial T}\right)_J = \frac{4}{15} \frac{\kappa_2^2 \chi_0}{J_s^2} \left[1 - 3j^{02} \left(\frac{18}{35} - \frac{1}{3} \frac{\chi_1 J_s^2}{\chi_0^3} \right) \right]. \quad (54)$$

(It is no longer necessary to distinguish between j_p^0 and j^0 .) The value of the derivative at constant B may be found by a similar method; it differs from the last expression only in that the fraction 18/35 is multiplied by $(1 - 5\epsilon/9)/(1 + \epsilon)$, where $\epsilon = 1/4\pi\chi_0 \doteq 0.004$. This justifies the statement made above, that these two values differ inappreciably.

Equation (54) agrees with the formula of Akulov and Kondorsky in the constant term, and therefore leads to the same formula for the total change of Young's modulus between demagnetization and saturation. The term in j^{02} , however, has 18/35 where their formula has 9/7. Neglecting the term in χ_1 , we get, in terms of the saturation magnetostriction λ_{111} in the [111] direction,

$$\begin{aligned} (E_\infty - E_0)/E_0 &= a/(1-a), \\ (E - E_0)/E_0 &= 54aj^{02}/35, \end{aligned} \quad (55)$$

where E_0 , E , E_∞ are, respectively, the values of Young's modulus when $J=0$, J , J_s , and

$$a = 3\lambda_{111}^2 \chi_0 E_0 / 5J_s^2. \quad (56)$$

We see that the ratio of $(E_\infty - E_0)/E_0$ to the coefficient of j^{02} in $(E - E_0)/E_0$ should be approximately equal to 35/54, or 0.65, instead of the value $7/27 = 0.26$ predicted by the original theory. The observed ratio in Siegel's experiments⁶ was 0.67. The new formula therefore removes satisfactorily the discrepancy between theory and experiment.

It is to be noted that the term in j^{02} , which in (54) is opposite in sign to the constant term, in (53) has the same sign. This means that, while Young's modulus at constant J increases steadily from demagnetization to saturation, the value at constant H should *decrease* when small fields are applied. This explains the apparent discrepancy between Siegel's results, in which a steady increase of E is observed, and the older results of Honda and Terada¹⁵ (obtained by a static method), in which E first decreases, and subsequently increases. It may be remarked (see Eqs. (45)) that this initial decrease results from the appearance of effective elastic constants s_{45}' , s_{56}' , and s_{64}' in the (elastically) very anisotropic state ensuing when the crystal is slightly magnetized. As saturation is approached, all such effects disappear, and the crystal returns to cubic symmetry, with a higher value of Young's modulus than in the demagnetized state.

The writer plans in subsequent papers to apply the theory to iron crystals, and to polycrystalline material in the manner indicated in the introduction; also to suggest a thermodynamic interpretation of the theory, and to show its relation to Gans's theory¹⁴ of reversible susceptibility. He wishes to acknowledge his indebtedness to Professor S. L. Quimby for helpful suggestions and criticisms made after reading the first draft of this paper.

¹⁵ K. Honda and T. Terada, *Phil. Mag.* **13**, 36 (1907).