section, thus permitting a check of the consistency of the whole theory.

Note added in proof: Recently published experiments by J. Halpern, I. Estermann, O. C. Simpson and O. Stern, Phys. Rev. **52**, 142 (1937) indicate that the scattering cross section of *ortho*-H_2 for liquid-air neutrons is much larger than the corresponding *para*-H_2 scattering cross section. This proves conclusively that the singlet state of the deuteron is virtual.

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Statistics and Nuclear Reactions

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It is possible to apply statistical methods to the calculation of nuclear processes provided that the energies involved are large in comparison with the lowest excitation energies of nuclei. Expressions are obtained for the emission probability of neutrons or charged particles by highly excited heavy nuclei. These expressions are built up in a similar way to the formula for the probability of evaporation of a particle from a body at low temperatures. In applying it to the impact of high energy neutrons on heavy nuclei, the mean energy loss per impact turns out to be $E[1-2(a/E)^{\frac{1}{2}}]$ where E is the energy of the incident neutrons and a is dependent on the nuclear structure; we can put approximately $a\sim 0.05$–0.2 MV. The energy distribution of the scattered neutrons is approximately a Maxwellian one with a mean energy of $2(aE)^{\frac{1}{2}}$.

I

THE application of quantum mechanics to the calculation of nuclear reactions in heavy nuclei gives rise to extremely complicated expressions which cannot be treated by ordinary methods, since there is no approximate solution for this complex many-body problem on account of the intense interaction between the constituents of atomic nuclei. On the other hand Bohr[1] has pointed out that just the extreme facility of energy exchange gives rise to a characteristic simplification of the course of every nuclear process initiated by a collision between a particle or a light quantum and a nucleus. It consists in the possibility of dividing the process into two well-separated stages. The first is the formation of a compound nucleus in a well-defined state in which the incident energy is shared among all the constituents, the second is the disintegration of that compound system, which can be treated as quite independent of the first stage of the process. This conception has been extremely fruitful in the treatment of all kinds of nuclear reactions.[2] Qualitative statistical conclusions about the energy exchange between the nuclear constituents in the compound state have led to simple explanations of many characteristic features of nuclear reactions. In particular the use of thermodynamical analogies has proved very convenient for describing the general trend of nuclear processes. The energy stored in the compound nucleus can in fact be compared with the heat energy of a solid body or a liquid, and, as first emphasized by Frenkel,[3] the subsequent expulsion of particles is analogous to an evaporation process.[4]

In this note an attempt is made to apply statistical methods in a more quantitative way to nuclear processes in which heavy nuclei are involved and become highly excited as, for example, in the case of the collision of very fast neutrons with heavy nuclei. The individual properties of the separate nuclear quantum states are then of no interest, on account of the extremely small distance between the energy levels of highly excited heavy nuclei; it is thus possible to obtain statistical information on the behavior of these nuclei by averaging over many quantum states

[1] N. Bohr, Nature **137**, 344 (1936); N. Bohr, Science (1937), in print.

[2] A comprehensive account of the application of the ideas in question to nuclear phenomena will be given in a paper by Bohr and Kalckar to appear shortly in the Proceedings of the Danish Academy.

[3] I. Frenkel, Sov. Phys. **9**, 533 (1936).

[4] A general discussion of the application of thermodynamical conceptions to nuclear processes can be found in the paper by Bohr, Science, reference 1.

of approximately the same energy.[5] A general statistical formula is derived for the emission of particles by highly excited heavy nuclei. It can be shown by means of this expression to what extent it is possible to describe the disintegration of the compound nucleus as an ordinary evaporation process. The statistics of systems with comparatively few particles gives rise to characteristic discrepancies from the ordinary result obtained for the case of macroscopic bodies. We cannot, for example, speak unambiguously of an evaporation temperature of an excited nucleus, since after emitting one particle it has lost such a large amount of energy that often a much lower temperature must be ascribed to the remaining nucleus. Furthermore all the other terms and quantities which are neglected in ordinary statistics on account of the great number of degrees of freedom must be taken into consideration.

The general expression contains, however, several quantities which are closely connected with the special structure of the nuclei considered. The most important is the density of levels as a function of the excitation energy. Our knowledge of the structure of heavy nuclei is so far very restricted, so that the quantitative conclusions which can be drawn have only a very low accuracy. Thus most of the deviations from the ordinary evaporation formula are much smaller than the inaccuracy of the quantities inserted. On the other hand the general expression can be used to draw conclusions about the density of levels from the observed energy distribution of particles emitted by highly excited nuclei.

II

Let us examine the state of a heavy nucleus A when it is excited to an energy E_A which we suppose to be greater than the binding energy of an elementary particle (proton, neutron, α-particle). This excitation may be produced by absorption of a hard γ-ray or by the collision of a nucleus with a neutron or another particle, whereby the compound nucleus A is created. Before the collision the colliding particle must have a kinetic energy E_A-E_0, E_0 being its binding energy when bound to the nucleus A. The cross section for that collision can in general be very simply determined. The validity of the statistical conclusions requires very high incident energies (at least 3 MV for incident neutrons, see below) so that the wave-length of the incident particles is much smaller than the radius r of the nucleus. The collision can therefore be described classically and the cross section will be of the order πr^2.

We consider now the emission of a neutron by the excited nucleus $A(E_A)$.[6] In order to be able to get statistical information there must be a great number of possibilities for the reaction

$$A=B+n$$

the various possibilities differing according to the various excited states of the remaining nucleus $B(E_B)$. The energy E_A must therefore be much greater than the energy E_0 which binds the neutron to the nucleus A—indeed there must be a great many levels of B having an excitation energy less than E_A-E_0. This condition is fulfilled sufficiently well in heavy nuclei (mass number greater than 100), if

$$E_A-E_0>3\text{MV}. \tag{1}$$

There is of course an upper limit for E_A too; it must be small in comparison with the total binding energy of the nucleus. Otherwise the compound state could not be formed at all.

Let us calculate the probability per unit time $W_n(\epsilon)d\epsilon$ of the nucleus A, excited to the energy E_A, emitting a neutron with kinetic energy between ϵ and $\epsilon+d\epsilon$, thus transforming itself into a nucleus B with an excitation energy $E_B=E_A-E_0-\epsilon$. This probability may possibly depend on some special features of the initial excited level of A. Therefore we understand by $W_n(\epsilon)d\epsilon$ the average of this probability over all excited states of A whose energy lies near E_A. The probability $W_n(\epsilon)d\epsilon$ has, of course, no meaning unless the interval $d\epsilon$ is chosen so big that there are a large number of levels of the nucleus B with energies between E_B and $E_B-d\epsilon$. To get a continuous function $W_n(\epsilon)$ it is necessary to average over a number of final states. We shall always assume that the interval $d\epsilon$ is much bigger than

[5] Similar ideas recently have been put forward by L. Landau, Sov. Phys. **11**, 556 (1937).

[6] We shall use the symbols $A(E)$ or $B(E)$ to designate nuclei A or B which are excited by an energy E *above the ground state.*

the distance between the levels of the nuclei in the excited states considered.

It is possible to give an expression for $W_n(\epsilon)d\epsilon$ as a function of the cross section corresponding to the reverse process: i.e., of the mean cross section $\sigma(E_A, \epsilon)$ for the collision of a neutron with energy ϵ with a nucleus $B(E_A-E_0-\epsilon)$ producing a compound nucleus $A(E_A)$. We can write

$$W_n(\epsilon)d\epsilon=\sigma(E_A, \epsilon)\frac{gm\epsilon}{\pi^2\hbar^3}\frac{\omega_B(E_B)}{\omega_A(E_A)}d\epsilon, \qquad (2)$$

where $\omega_A(E)dE$ and $\omega_B(E)dE$ are the numbers of levels of the nuclei A and B, respectively, between E and $E+dE$, the energies being measured *from the ground state* of the particular nucleus under consideration. m is the mass of the emitted particle, h Planck's constant divided by 2π and g denotes the number of states for the spin of the particle considered. ($g=2$ for neutrons and protons, $g=1$ for α-particles. This factor is left undetermined in (2) so that we can apply (2) to the emission of other particles.)

To derive this relation, let us consider the nucleus and the neutron enclosed in a volume Ω. Then

$$W_c=\sigma(E_A, \epsilon)v/\Omega$$

is the mean probability per unit time of the neutron, with an energy between ϵ and $\epsilon+d\epsilon$ and a velocity $v=(2\epsilon/m)^{\frac{1}{2}}$, being captured by the nucleus $B(E_A-E_0-\epsilon)$, forming nucleus A with an energy between E_A and $E_A+d\epsilon$. We then obtain the probability $W_n(\epsilon)d\epsilon$ for the reverse process by dividing W_c by the number $\omega_A(E_A)d\epsilon$ of states in which the neutron can be captured and multiplying by the number of states into which $A(E_A)$ can decay. There are $\omega_B(E_B)d\epsilon$ states of the nucleus B into which $A(E_A)$ can decay if the energy of the neutron lies between ϵ and $\epsilon+d\epsilon$ and there are

$$\frac{\Omega gm}{2\pi^2\hbar^3}(2m\epsilon)^{\frac{1}{2}}d\epsilon$$

quantum states in the volume Ω in the energy range $d\epsilon$ at the disposal of the neutron. Therefore we get

$$W_n(\epsilon)d\epsilon=W_c\frac{\omega_B(E_B)}{\omega_A(E_A)}\frac{\Omega gm}{2\pi^2\hbar^3}(2m\epsilon)^{\frac{1}{2}}d\epsilon$$

which leads at once to (2).

It is convenient to introduce the logarithm of the density of the levels:

$$S_A(E)=\lg\,\omega_A(E), \quad S_B(E)=\lg\,\omega_B(E).$$

$S(E)$ corresponds to the entropy of the nucleus having an energy between E and $E+dE$. We obtain then

$$W_n(\epsilon)d\epsilon=\sigma(E_A, \epsilon)\frac{gm\epsilon}{\pi^2\hbar^3}e^{S_B(E_A-E_0-\epsilon)-S_A(E_A)}d\epsilon. \qquad (3)$$

This is a general expression for the emission of a particle of mass m, from a system having the free energy E_A and a density of levels $\omega_A(E)$, the remaining system having a density of levels equal to $\omega_B(E)$. The usual formula for the probability of evaporation can be derived from (3) by assuming that E_A is much greater than both the binding energy E_0 and the kinetic energy ϵ of the emitted particle ($E_A\gg E_0$, $E_A\gg\epsilon$), and that S_A and S_B are identical functions ($S_A(E)=S_B(E)$). It is then possible to develop

$$S_B(E_A-E_0-\epsilon)=S_A(E_A)-(E_0+\epsilon)(dS_A/dE)_{E_A}.$$

The derivative of S_A can be expressed by means of a temperature, namely

$$dS_A/dE=1/T_A(E), \qquad (4)$$

where $T_A(E)$ is the temperature at which the most probable energy of the body A is equal to E at thermodynamical equilibrium.[7] The temperatures T given here are of the dimensions of an energy, and are equal to the ordinary temperature times the Boltzmann constant k. We can then write

$$W_n(\epsilon)d\epsilon=\sigma(E_A, \epsilon)\frac{gm}{\pi^2\hbar^3}e^{-E_0/T_A(E_A)}\epsilon e^{-\epsilon/T_A(E_A)}d\epsilon. \qquad (5)$$

This is exactly the usual equation for evaporation. The meaning of σ becomes clearer in putting $\sigma=q\cdot\gamma(\epsilon)$ where q is the actual spatial cross section of the vaporizing body and $\gamma(\epsilon)\leqq 1$ gives the ratio of the number of particles captured to the number of incident particles with an energy ϵ. The cross section for elastic reflection is then $q(1-\gamma(\epsilon))$.

If we apply these considerations to the emission of particles by nuclei, we must observe that the assumptions necessary to establish (5) are not realized. E_0 in particular is not small in comparison with E_A. Nevertheless it is possible to

[7] We distinguish here between the most probable energy and the mean energy. Formula (4) is the statistical definition of the former.

transform (3) into an expression similar to (5), by assuming only $\epsilon \ll E_A - E_0$. We can then develop

$$S_B(E_A-E_0-\epsilon)=S_B(E_A-E_0) -\epsilon/(T_B(E_A-E_0))-f(\epsilon), \quad (6)$$

where just as in (4) we define:

$$1/T_B(E)=dS_B/dE. \quad (4a)$$

$T_B(E)$ is the temperature at which E is the most probable energy of the nucleus B in thermodynamical equilibrium. $f(\epsilon)$ contains all further terms of the development. We thus obtain

$$W_n(\epsilon)d\epsilon=\sigma(E_A,\epsilon)\frac{gm}{\pi^2h^3}e^{S_B(E_A-E_0)-S_A(E_A)} \times \epsilon e^{-\epsilon/T_B(E_A-E_0)}e^{-f(\epsilon)}d\epsilon. \quad (7)$$

The total probability of a neutron being emitted is obtained by integrating with respect to ϵ; if multiplied by h, it is identical with the "neutron breadth" Γ_n of the emitting level of the nucleus A (i.e., that part of the level-breadth which arises from the emission of a neutron):

$$\Gamma_n=h\int W(\epsilon)d\epsilon=\bar{\sigma}\frac{gm}{\pi^2h^2}T_B{}^2(E_A-E_0) \times e^{S_B(E_A-E_0)-S_A(E_A)}, \quad (8)$$

$\bar{\sigma}$ is a mean value of $\sigma(E_A,\epsilon)e^{-f(\epsilon)}$ averaged over the Maxwell distribution:

$$\bar{\sigma}=\int \epsilon\sigma(E_A,\epsilon)e^{-\epsilon/T_B-f(\epsilon)}d\epsilon \Big/ \int \epsilon e^{-\epsilon/T_B}d\epsilon.$$

In order to get a closer analogy to the evaporation formula (5) we put

$$S_B(E_A-E_0)-S_A(E_A)=-E_0{}^*/T_A(E_A), \quad (9)$$

where $E_0{}^*$ is an energy which is only equal to E_0 if $E_0 \ll E_A$ and $S_A(E)=S_B(E)$. When (9) is inserted in (7) the difference between (7) and the evaporation-formula (5) consists in the following: the energy $E_0{}^*$ is not identical with the binding energy E_0 and the energy-distribution of the neutrons is not of the Maxwell type, but differs from it by the factor $e^{-f(\epsilon)}$. Furthermore the temperature $T_B(E_A-E_0)$ is by no means the temperature of the evaporating compound-nucleus $A(E_A)$ (its temperature is defined by $T_A(E_A)$), but it is the temperature of the remaining nucleus $B(E_A-E_0)$. This difference can also be explained as follows: on the liberation of the neutron a considerable cooling down takes place. The energy-distribution of the neutrons corresponds to the temperature of the remaining nucleus.

The corrections $f(\epsilon)$ and $E_0{}^*$ can be determined still more exactly. It follows from (4) and (4a) that dS/dE decreases monotonically with increasing E, since this expression is equal to $1/T(E)$. Therefore $S(E)$ as plotted against E is a curve which is concave downwards; from this fact and from (6) we conclude that

$$f(0)=0, \quad f(\epsilon)\geqq 0$$

and finally that $f(\epsilon)$ increases monotonically. Hence the relative energy-distribution of the emitted neutrons

$$W_n(\epsilon)=\text{const. } \sigma(E_A,\epsilon)\epsilon e^{-\epsilon/T_B}e^{-f(\epsilon)} \quad (10)$$

possesses a maximum which, as compared with the distribution (5), is shifted towards higher energies and has a smaller breadth.

We can apply the same considerations to estimate the difference (9). Assuming $S_A(E) \sim S_B(E)$ we get

$$S_A(E_A)-S_B(E_A-E_0)>E_0/T_A(E_A). \quad (11)$$

This implies that $E_0{}^*>E_0$.

It may be noticed that the observed neutron energy distribution for a highly excited nucleus $A(E_A)$ can be different from (10) because the remaining nucleus $B(E_B)$ may possibly emit neutrons again. Such double emissions of neutrons have been found by Heyn.[8] If, however, E_A is less than the energy necessary for removing two neutrons the observed distribution should always be represented by (10).

Eq. (7) leads to the following qualitative conclusions concerning collisions between fast neutrons and nuclei: if a neutron with an energy E collides with the nonexcited nucleus B, and gives rise to the nucleus A with energy $E+E_0$, it will be reemitted on an average with a much smaller energy $\sim 2T_B(E)$ and therefore give a great part

[8] F. A. Heyn, Nature **138**, 723 (1936); Physica **4**, 160 (1937).

of its energy to the nucleus B. $2T$ is the mean energy of particles *emitted* with a Maxwell distribution corresponding to a temperature T. Here it is assumed that the cross section $\sigma(E_A, \epsilon)$ does not depend appreciably on ϵ as is shown in the following section. The energy $T_B(E)$ is generally approximately equal to E divided by the number of degrees of freedom of the nucleus B which are excited at this energy. We see that this represents a very inelastic collision.

In applying formula (7) to excitations made by colliding particles or light quanta, we must take into account the fact, that (7) represents an average over all levels of A having an energy of about E_A. But not all of those levels can be excited by the process mentioned above. Excitation is only possible to levels whose angular momentum differs not too much from the momentum of the original nucleus, since a particle or a light-quantum with the momentum p hitting a nucleus cannot transfer a bigger amount of angular momentum than $\sim pr+s$ (r is the radius of the nucleus and s is the spin angular momentum of the incident particle, which is $h/2$ for neutrons and h for photons). We meet here a characteristic difficulty in nuclear statistics. The conservation of angular momentum, which is found in nuclear processes, is not usually taken into consideration in ordinary statistics. It is easy to see that the emission probability from a level with small angular momentum will in general be bigger than the average value (7) over all angular momenta since there are among the levels with energy E_A several which cannot emit at all. The latter are those with an angular momentum $l>L_B+l_\epsilon$. L_B is the maximum angular momentum of the levels in $B(E_B)$ and l_ϵ is the maximal angular momentum the outgoing neutrons can have ($l_\epsilon\sim r(2m\epsilon)^{\frac{1}{2}}+s\ll L_B$). We get a rough estimate of the emission probability $W_n'(\epsilon)d\epsilon$ for levels with low angular momentum by assuming that all the levels with $l<L_B$ emit with the same probability and that the levels with $l>L_B$ do not emit at all. The only change in the formula (7) for $W_n(\epsilon)$ caused by this assumption consists in putting $\omega_A'(E_A)$ instead of $\omega_A(E_A)$. $\omega_A'(E_A)$ being the density of levels with angular momentum less than L_B. In what follows we shall neglect this change, since up to the present it is impossible to say anything about the angular momentum distribution among nuclear levels. All estimates of $\omega_A(E)$ are far too inexact to distinguish between ω_A' and ω_A.

The formula derived above can be applied to the emission of charged particles (protons, α-particles). We need only allow for the fact that the Coulomb field has a strong influence on the cross section $\sigma(E_A, \epsilon)$ for the reverse process, because it repels and deflects the incident charged particles. If the cross section for the reverse process can be roughly determined by the classical conception that any particle hitting the nucleus is absorbed, we get the expression

$$\sigma(E_A, \epsilon)\begin{cases}=\sigma_0(1-V/\epsilon) & \text{for } \epsilon>V\\ =0 & \text{“ } \epsilon<V\end{cases}$$

$$\sigma_0=\pi r^2, \qquad V=ZZ'e^2/r.$$

where Ze and r are the charge and the radius of the nucleus and $Z'e$ the charge of the particle. For $Z=50$ we get approximately $V=7\cdot Z'$ MV. It is better here to use the following expansion instead of (6):

$$S_B(E_A-E_0-\epsilon)=S_B(E_A-E_0-V)-(\epsilon-V)/T_B(E_A-E_0-V)-f(\epsilon-V).$$

Instead of (7) we then get

$$W_P(\epsilon)\,d\epsilon=\sigma_0\frac{gm}{\pi^2h^3}e^{S_B(E_A-E_0-V)-S_A(E_A)}(\epsilon-V)\times e^{-(\epsilon-V)/T_B(E_A-E_0-V)}\cdot e^{-f(\epsilon-V)}d\epsilon$$

for the emission probability of charged particles, and for the total emission-probability (proton breadth, α-particle breadth)

$$\Gamma_P=\sigma_0\frac{gm}{\pi^2h^2}T_B{}^2(E_A-E_0-V)e^{S_B(E_A-E_0-V)-S_A(E_A)}. \tag{12}$$

Comparing this with (7) we note that the binding energy E_0 is replaced by E_0+V and the energy distribution is shifted to higher energies by an amount V without any other change in form. This is easily understood because an additional energy V is given to the emitted particle on account of the Coulomb field of the nucleus. For the same reason the binding energy E_0 is increased, since the energy necessary to liberate the particle must appear in our formula without the energy gained when the particle is accelerated in the Coulomb field, being taken into account. The formula has no longer a meaning when $\epsilon<V$, since then, according to the classical considerations made in deriving the expression, the particle cannot leave the nucleus. But if we take into account the quantum-mechanical possibility of passing through the potential wall, one may expect to find for $\epsilon<V$ an emission probability different from zero, though very small.

The general consequences of the electrostatic repulsion in nuclear reactions are discussed in both papers of Bohr.[1] We restrict ourselves therefore to the following two general conclusions:

The binding energy E_0 is generally as large for protons as for neutrons, in so far as we have to deal with isotopes which are stable with regard to β-emission. Thus, when the Coulomb potential V in the exponential of (12) is comparable with E_0^*, we may expect the emission of protons to be in general much less probable than that of neutrons. In fact, no emission of protons from heavy nuclei $Z>30$ is found when excited by neutrons[9] or γ-rays.[10]

The emission probability for α-particles will be much smaller than that of neutrons as long as the nucleus is stable in respect to α-decay since V is then certainly much bigger than the binding energy of neutrons. The latter is about 8 MV, whereas V is at least 14 MV. For radioactive nuclei however the binding energy of α-particles is negative. It can then happen that E_0^*+V is nearly equal for α-particles, to the binding energy of a neutron. The excited nucleus will then be able to emit neutrons and α-particles with probabilities of the same order of magnitude. Thus we should expect an emission of α-particles by heavy nuclei after the capture of a neutron only in nuclei belonging to the radioactive group. In fact a process of this kind has only been found in thorium.[11, 12]

The nuclear reactions resulting from collisions between charged particles and heavy nuclei are outside the scope of the present paper, because the velocity of charged particles is reduced in the Coulomb field, and (in the experiments which have so far been carried out) they do not therefore possess sufficient energy to excite the nuclei strongly enough. The energy of protons or deuterons should be considerably above 7 MV and that of α-particles above 14 MV.

III

Let us now investigate the order of magnitude of the functions $S(E)$ and $\sigma(E_A, \epsilon)$ which appear in (7). It is obvious that this can only be done roughly, owing to the small knowledge we have of nuclei so that the expressions to be given now are much less exact than those of the preceding section. An attempt has been made by Bethe[13] to calculate the density of levels in nuclear spectra. According to his result (32) we get

$$S(E)=(M\cdot E/2.2)^{\frac{1}{2}}-(5/4)\lg E, \qquad (13)$$

where E is measured in MV and M signifies the mass number. On account of the extreme inaccuracy of any nuclear model we shall try to derive expressions for S by means of two other methods which are less exact but perhaps more general than Bethe's method.

A proper way to calculate $S(E)$ is to find $T(E)$ first. If we know the dependence of the most probable energy E of the nucleus upon the temperature in thermodynamical equilibrium, we can put

$$S(E)=\int dE/T(E), \qquad (14)$$

where the constant of integration does not concern in our problem, since we always have to deal with the difference between two values of S. The energy E can in general be represented over a large range as proportional to a power of T:

$$E=T^n/a,$$

where a is a constant. If all the degrees of freedom are excited (classical limiting case), n must be put equal to unity; in the reverse case (i.e., for relatively low temperatures) $n=4$ for solid bodies, and $n=2$ for degenerate gases. In the present problem we can check the power n and the constant a in two almost independent ways:

(1) As an approximation we represent the degrees of freedom of the nucleus under consideration by proper oscillations with frequencies $\nu_1, \nu_2, \cdots, \nu_i \cdots$.[14] Then $E(T)$ can be evaluated from $Z(\nu)$, $Z(\nu)$ being the number of proper frequencies $\nu_i<\nu$ as function of ν.[15] The most probable energy or the mean energy (the approximation is so rough, that this difference has no essential effect) is in the fact approximately

$$E\sim T\cdot Z(T/h).$$

[9] E. Fermi and collaborators, Proc. Roy. Soc. **A146**, 483 (1934).
[10] W. Bothe and W. Gentner, Naturwiss. **25**, 30, 126 (1937).
[11] I. Curie, H. v. Halban, and P. Preiswerk, J. d. phys. **7**, 6, 361 (1935).
[12] O. Hahn and L. Meitner, Naturwiss. **23**, 320 (1935).
[13] H. A. Bethe, Phys. Rev. **50**, 332 (1936).
[14] I am indebted to Professor Bohr for the following method of estimating the temperature of an excited nucleus.
[15] In this model the proper oscillations and the quantum states must be distinguished. The number $Z(\nu)$ of the oscillations is different from that of the quantum states for which $E<h\nu$: the former $Z(\nu)$, is smaller than the latter.

This is equivalent to the assumption, that the mean energy of a proper oscillation whose frequency is smaller than T/h, is equal to the classical value T and that the proper frequencies $\nu_i > T/h$ are regarded as nonexcited. For energies E small compared with the binding energy of the particles, $Z(E/h)$ is more or less proportional to the energy E, since the proper frequencies in this energy-domain are still statistically distributed. We then obtain $Z(E/h)=E/a$ and

$$E=T^2/a. \tag{16}$$

Thus we find $n=2$ where a is an energy which can be considered as equal to the mean energy distance between the lowest proper frequencies $h\nu_i$. The value of a cannot be easily estimated owing to our restricted knowledge of the structure of heavy nuclei. Only for radioactive nuclei does experimental evidence on the spacing of the lowest levels exist. This spacing can roughly be set equal to the frequencies of the lowest proper oscillations. The experiments on α-ray fine structure and on γ-ray spectra lead to nuclear levels the spacing of which is of the order of 0.1 MV. Since no information on nuclear spectra of heavy nonradioactive nuclei is available, we assume that the value of a is of the same order of magnitude for nuclei with mass numbers greater than 100 and put for these nuclei $a\sim 0.1-0.2$ MV. This check on a is very uncertain; a dependance on the weight of the nuclei and a variation due to the existence of different kinds of nuclei (odd or even) must be expected.

(2) The other way to check n and a avoids the representation of the dynamics of the nuclei by means of proper oscillations. We try to approximate to the problem from another side, and consider the nucleus as a degenerate gas of particles, with strong interaction forces between them. The energy of a degenerate gas of particles obeying the Fermi-statistics is also given by (16). Since we know from experience of metallic electrons that this law holds in spite of very strong interactions between electrons, we may assume $n=2$ for nuclei also. The constant a can be checked by means of the expression for the density of levels which follows from (14) and (16):

$$\begin{aligned} S(E) &= 2(E/a)^{\frac{1}{2}}+\text{const.},\\ \omega(E) &= \text{const.}\cdot \exp\left[2(E/a)^{\frac{1}{2}}\right]. \end{aligned} \tag{17}$$

Thus the value of a can be calculated if we know the density of levels $\omega(E)$ for two values E_1, E_2:

$$a=4\left(\frac{E_1^{\frac{1}{2}}-E_2^{\frac{1}{2}}}{\lg \omega(E_1)-\lg \omega(E_2)}\right)^2. \tag{18}$$

We have a rough knowledge of the density at an energy $E\sim 8$ MV from the experiments on the capture of slow neutrons. The capturing levels of the compound nucleus have an energy of about 8 MV and their distance apart is about $1-10\cdot 10^{-6}$ MV for heavy nuclei (weight $M\sim 100$) according to Bethe and Placzek.[16] On the other hand we can assume the distance between the lowest nuclear levels to be of the order 0.1 MV. from experiments on γ-ray spectra as stated above. Inserting these two values in (18) we get $a\sim 0.2$ MV. This value may be too high because the actual density of levels at 8 MV is certainly higher than the value inserted, since the experiments on capture of slow neutrons only give information about levels with two fixed successive values of angular momentum.[17] In any case the order of magnitude can hardly be very different since the level densities only appear logarithmically in the expression (18).

The value of a from Bethe's formula (13) is slightly lower than the above results. Neglecting the logarithmic term we get $a=8.8/M$ expressed in MV. The interpretation of a as the distance between the lowest proper oscillation frequencies may however indicate a weaker dependence on M.

From a comparison of all the results obtained, we may take the following expression for $T(E)$ for nuclei of mass number $M>100$

$$T(E)=(aE)^{\frac{1}{2}}, \quad 0.05 \text{ MV} < a < 0.2 \text{ MV}.$$

a is certainly nearer to the upper limit than the lower for the lighter nuclei and decreases with increasing mass number.

The cross section $\sigma(E_A, \epsilon)$ can be estimated in the following way: when the energy ϵ of the neutron is high enough for the corresponding magnitude $\lambda\!\!\!^{-}=\lambda/2\pi=h/(2m\epsilon)^{\frac{1}{2}}$ to be smaller than the range of nuclear forces $d=2\cdot 10^{-13}$ cm, then we can consider the reverse collision as classical, and put $\sigma=\pi r^2$ independently of E_A and ϵ (r is the radius of the nucleus). This is only fulfilled if

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[17] The capture of slow neutrons can only give rise to levels whose angular momentum differs by $h/2$ from the angular momentum of the capturing nucleus.

$\epsilon > 5$ MV. When the magnitude λ cannot be considered as small compared to d, but is still smaller than $r/2 \sim 0.5 \cdot 10^{-12}$ (i.e., $\epsilon > 0.8$ MV), the area πr^2 can be regarded as an upper limit for the cross section. How much σ is less than this upper limit depends on E_A. When the levels of the compound nucleus $A(E_A)$ are separated by large distances, the capture is very selective, which reduces the cross section. But in the case considered here E_A must be very high for $E_A > E_0 + 3$ MV, on account of condition (1). It may generally be expected—and we shall verify it by calculation—that the breadth of the levels is then larger than their distance apart, so that selectivity is no longer possible. No considerable error can then be made in putting

$$\sigma(E_A, \epsilon) = \pi r^2 \quad \text{for} \quad \epsilon > 0.8 \text{ MV}.$$

Since, according to (7) the bulk of the emitted neutrons has an energy $\epsilon \sim 2T_B$, the only interesting value of $\sigma(E_A, \epsilon)$ is that corresponding to this energy. Now the temperatures occurring in nuclear processes which can be handled by statistical methods are generally higher than 0.5 MV. This follows from the fact that a temperature $T \sim 0.55$ MV corresponds to the minimum value of $E_A - E_0$ (see (1)). Thus the mean energy of neutrons will be high enough for putting $\sigma(E_A, \epsilon) = \pi r^2$.

The relative energy distribution of the emitted neutrons is then according to (7):

$$W_n(\epsilon)d\epsilon = \text{const. } \epsilon e^{-\epsilon/T_B(E_A-E_0)} e^{-f(\epsilon)} d\epsilon.$$

This represents a Maxwell distribution with a correction $f(\epsilon)$ in the exponent. According to (6) $f(\epsilon)$ is given by

$$f(\epsilon) = -\frac{\epsilon^2}{2}\left(\frac{d^2 S_B}{dE^2}\right)_{E_A - E_0},$$

the higher derivatives being of no importance. With assumption (17) for S this becomes

$$f(\epsilon) = \epsilon^2/4(E_A - E_0) \cdot T_B(E_A - E_0)$$

and the correction $f(\epsilon)$ is therefore smaller than the Maxwell exponent ϵ/T_B itself in the ratio $\epsilon/4(E_A - E_0)$. The Maxwellian mean value of ϵ is given by $\bar{\epsilon} = 2T_B(E_A - E_0)$ so that

$$\frac{\bar{\epsilon}}{4(E_A - E_0)} = \frac{1}{2}\left(\frac{a}{E_A - E_0}\right)^{\frac{1}{2}} \ll 1$$

in virtue of (1). Consequently we see that the correction $f(\epsilon)$ has no great influence. The energy distribution of particles emitted by highly excited nuclei will therefore be nearly a Maxwellian one. When neutrons with energy E fall on a nucleus B they will be reemitted with a mean energy of $2(aE)^{\frac{1}{2}}$. Hence the mean energy loss per impact is $E[1 - 2(a/E)^{\frac{1}{2}}]$. Neutrons of 10 MV will for example be reemitted with a mean energy of only 1.4 − 2.8 MV.

Let us now calculate from (9) the total probability for the emission of neutrons from the excited nucleus $A(E_A)$. This probability expressed in energy-units, is equal to the neutron-breadth of the emitting levels. The fact must be allowed for that the statistical expressions are only valid when $E_A - E_0$ is large, i.e., when the nucleus B can remain excited after the emission of the neutron in many different ways. Consequently the formula cannot be applied to levels resulting from the capture of slow neutrons. From (8) and (17) it follows that

$$\Gamma_n = \frac{a(E_A - E_0)}{\pi E_K} \exp\{2[(E_A - E_0)/a]^{\frac{1}{2}} - [E_A/a]^{\frac{1}{2}}\},$$

where $E_K = h^2/2mr^2 \sim 0.2$ MV. To get an idea of the dependence on E_A and on the assumed value of a we calculate Γ_n for several values of E_A and a, putting $E_0 = 8$ MV. The values of Γ_n in electron volts (not in MV) are given in Table I.

TABLE I. *Values of probability of emission of neutrons given in electron volts (breadth of emitting levels).*

a \ E_A	11 MV	15 MV	20 MV	25 MV
0.05 MV	0.2	18	120	680
0.1 MV	22	450	4800	10000
0.2 MV	860	10000	38000	97000

For increasing mass number E_0 as well as a gets smaller and consequently the values of Γ_n will not depend essentially upon the mass number within the low accuracy of this calculation. It should be noticed from this result that the neutron breadths of the level of $A(E_A)$ are much larger than the mean distances of the levels at the corresponding energy.

The probability of emitting a neutron is much higher than that of emitting a γ-ray. We know the latter, however, only for comparatively low

values of E_A from experiments on capture of slow neutrons. The γ-ray breadth is according to Bethe and Placzek[13] about 10^{-7} MV for an excitation energy of ~ 8 MV. If it is permissible to assume this value valid also for higher excitation, the capture of high energy neutrons ($E > 3$ MV) will be extremely improbable. Even a transition of the compound nucleus to a less energetic unstable state by emission of a γ-ray will be an improbable process since according to the above table the probability of neutron emission is in general several orders of magnitude greater than that for radiation.

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The Beta-Ray Spectrum of Radium E

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The beta-ray spectrum of radium E was examined with a magnetic spectrometer, by using coincidence counting, under various conditions of source strength, mounting and aperture of defining slits. It is concluded that the experimental high energy end point depends on the source strength used, as well as the width of the defining slits. Data from all but very strong sources gave a K.U. plot which was linear within the limits of error set by statistical fluctuations and finite slit widths. Extrapolation of the K.U. plot gave 1.25 ± 0.03 Mev as the high energy end point.

SINCE the original work of Schmidt[1] in 1907 more than a score of workers[2–24] have made measurements on the beta-ray spectrum of radium E with none too concordant results. The disagreement between observers may be appreciated by consulting Table I wherein one finds values for the high energy end point which range from 4500 $H\rho$ up to 12,000 $H\rho$.

The importance of obtaining the true energy distribution curve for the disintegration electrons from radioactive bodies has been recently enhanced by the tentative success of the Konopinski-Uhlenbeck[25] modification of the Fermi[26] theory of beta-disintegration, especially as applied to the lighter artificially radioactivated elements. The present investigation was undertaken with the purpose of securing data from one of the naturally radioactive elements which could be used after the manner proposed by Kurie, Richardson and Paxton[27] as a further criterion in determining more conclusively the validity of the theory.

Fermi's original formula (44), reference[26] gives the probability of disintegration with the emis-

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