change caused by an adiabatic stretching is equal to $(\delta T/\delta V)_Q \delta V$, where δV is the change in volume accompanying the strain.

Now $\quad e_{yy}=e_{zz}=-e_{xx}$
when $\quad Y_y=Z_z=0.$
Hence $\quad \delta V/V=e_{xx}+e_{yy}+e_{zz}=(1-2\sigma)e_{xx},$
and so $\quad (\delta T/\delta e_{xx})_{Q,\,\pi}=(1-2\sigma)(\delta T/\delta \log V)_Q.$

The second factor in (a-2) may obviously be written as $\frac{1}{3}(\delta \log V/\partial T)_p$. Hence (a-2) reduces to

$$-\frac{1-2\sigma}{3}\left(\frac{\partial T}{\partial V}\right)_Q\left(\frac{\partial V}{\partial T}\right)_p. \tag{a-3}$$

This expression may be written in a more illuminating form. If only two of the four variables x_1, x_2, x_3, x_4 are independent, then the following identity is true:

$$\frac{(\partial x_1/\partial x_2)_{x_4}}{(\partial x_1/\partial x_2)_{x_3}}=1-\frac{(\partial x_3/\partial x_2)_{x_4}}{(\partial x_3/\partial x_2)_{x_1}}.$$

Replacing x_1, x_2, x_3, x_4 by V, T, Q, p, respectively, we find that (a-3) becomes

$$\tfrac{1}{3}(1-2\sigma)(C_p-C_v)/C_p.$$

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Collector Theory for Ions with Maxwellian and Drift Velocities

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For a gaseous discharge in which the ions have a drift velocity superimposed on a Maxwellian distribution, the Mott-Smith and Langmuir collector theory deals only with collectors having a high ratio of sheath to collector diameters. The present paper removes this restriction, but points out that except when there is negligible sheath distortion due to low drift velocity or to low collector potential the theory gives only approximate results. It is also shown that for ratios of drift to random current density as low as seven, the space potential is not indicated by a break on the semi-logarithmic plot of the collector characteristic.

An electrical discharge in a gas is conveniently studied by the use of a collector or probe connected to a suitable auxiliary circuit. Mott-Smith and Langmuir[1] have shown that if the logarithm of the collector current be plotted against the potential applied to it, the resulting "characteristic" is a line which departs from linearity at the space potential. This holds for a Maxwellian distribution of the ions (or electrons) of the discharge; but when there is a small drift velocity superimposed on the Maxwellian distribution the characteristic is modified as shown in their paper. Their results refer only to the case where there is a high ratio of sheath[2] to collector diameter. The present paper deals with sheaths of all sizes.

The basic assumptions of this collector theory are discussed by Mott-Smith and Langmuir.[1] Special attention is directed to the two assumptions (1) that the gas pressure is so low that collisions between ions or electrons and gas molecules in the sheath have a negligible effect on the collector current, and (2) that there is no reflection of ions or electrons at the collector surface.

Because of the drift velocity, the cross section of the space charge sheath around a cylindrical collector (with its axis at right angles to the direction of the drift) is distorted from the circular shape. But obviously it still must have bilateral symmetry about the direction of the drift, which is therefore taken as the coordinate axis. The length of the collector is assumed to be so great that end effects may be neglected.

Without the restriction imposed by Mott-Smith and Langmuir that the sheath is circular and by methods analogous to theirs, it may be shown that, as a first approximation,

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[1] H. M. Mott-Smith and I. Langmuir, Phys. Rev. **28**, 727 (1926).

[2] The sheath boundary may be briefly defined as the surface beyond which the collector potential exerts no force on the ions or electrons of the discharge.

$$\frac{i}{2\pi r l I_r} \approx \frac{2}{\pi^{\frac{3}{2}} r} e^{-\alpha^2} \int_0^{\pi} \int_{0,\,(-\eta\tau/\sigma)^{\frac{1}{2}}}^{\infty} \int_{x\chi/\tau-(\sigma x^2+\tau\eta)^{\frac{1}{2}}/\tau}^{x\chi/\tau+(\sigma x^2+\tau\eta)^{\frac{1}{2}}/\tau} \frac{\partial s}{\partial \psi} x \exp\left(-x^2-y^2+2\alpha x\cos\psi+2\alpha y\sin\psi\right) dy\,dx\,d\psi, \quad (1)$$

where i is the current to the collector, r and l are the radius and exposed length, respectively, of the cylindrical collector, I_r is the random current density corresponding to the Maxwellian distribution, s is the circumference of the sheath around the collector, α is $I_d/2\sqrt{\pi}I_r$ (I_d is the drift current density), η is eV/kT (e is the electronic charge, k is Boltzmann's constant, V is the potential of the collector with reference to the space potential, and T is the temperature corresponding to the Maxwellian distribution), σ is $(b/r)^2-1$, τ is $(b\cos\varphi/r)^2-1$, χ is $(b/r)^2\cos\varphi\sin\varphi$ and φ is $\psi-\theta$. b and θ are the coordinates of points on the circumference of the sheath; and the normal at the point b makes the angle ψ with the coordinate axis. See Fig. 1.

Equation (1) is approximate instead of exact because in its development the angular momentum of an ion at the sheath surface is equated to its angular momentum on arrival at the collector surface; but this equality does not hold for a noncircular sheath since the force on the ion due to the collector potential is noncentral (i.e., it is not directed always to the collector axis). The exact solution of this problem involves the determination of the shape of the sheath and the distribution of the force-field within it, a problem which is not dealt with in this paper. However, the sign of the error can be deduced by introducing the plausible assumption that for retarding potentials on the collector (η negative) the sheath shape is such that ϕ is positive (as in Fig. 1) and that for accelerating potentials (η positive) ϕ is negative. Eq. (1) is equivalent to the assertion that ions will reach the collector if they arrive at the sheath boundary traveling in a certain direction (or within a certain range of directions) and within a certain range of velocities. Owing to the sheath distortion assumed above, the ions which will reach the collector are those having a direction making an angle with the coordinate axis greater than if the sheath was circular; and since the distribution function shows that there are fewer of such ions it follows that the current given by Eq. (1) is too large whether the potential on the collector is retarding or accelerating.

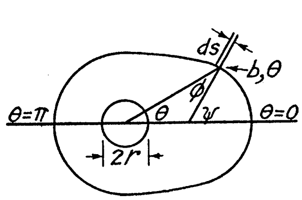

FIG. 1.

For a small sheath, $(b/r=1)$, Eq. (1) becomes exact, since the sheath is now circular. The equation reduces to

$$\mathfrak{J} = \frac{2}{\pi} \int_0^{\pi} \int_{0,\,\sqrt{(-\eta)}}^{\infty} x e^{-(x-\alpha\cos\theta)^2} dx\,d\theta, \quad (2)$$

where $\mathfrak{J}$ is written for $i/2\pi r l I_r$. This further reduces to

$$\mathfrak{J} = e^{-\alpha^2/2}\left[(1+\alpha^2)I_0(\alpha^2/2)+\alpha^2 I_1(\alpha^2/2)\right] \quad \text{for } \eta \geqq 0, \quad (2a)$$

$$\mathfrak{J} = e^{\eta-\alpha^2/2}\Big[(1+\alpha^2)I_0(\alpha^2/2)I_0(t)-\alpha^2 I_0(\alpha^2/2)I_2(t)$$
$$+2(1+\alpha^2)\sum_1^{\infty}(-1)^p I_p(\alpha^2/2)I_{2p}(t)$$
$$-\alpha^2\sum_1^{\infty}(-1)^p I_p(\alpha^2/2)\{I_{2p-2}(t)+I_{2p+2}(t)\}\Big] \quad \text{for } \eta \leqq 0, \quad (2b)$$

where $t=2\alpha\sqrt{(-\eta)}$ and $I_p(x)$ is the modified Bessel function of the first kind and the pth order.

Without any restriction as to sheath size, but considering the limiting case where the sheath is a circle of radius b, Eq. (1) reduces to

$$\mathfrak{J} = \frac{1}{\pi}\frac{b}{r} \int_0^{\pi} \int_{0,\,\sqrt{(-\eta)}}^{\infty} x e^{-(x-\alpha\cos\theta)^2} \times \left[\frac{2}{\sqrt{\pi}} \operatorname{Erf}\{[(x^2+\eta)/\sigma]^{\frac{1}{2}}+\alpha\sin\theta\} + \frac{2}{\sqrt{\pi}} \operatorname{Erf}\{[(x^2+\eta)/\sigma]^{\frac{1}{2}}-\alpha\sin\theta\}\right] dx\,d\theta, \quad (3)$$

where $$\mathrm{Erf}\,(t)=\int_0^t e^{-y^2}dy.$$

For a very large sheath ($b/r=\infty$), Eq. (1) reduces to

$$\mathfrak{J}\approx\frac{4}{\sqrt{\pi}}e^{-\alpha^2}\int_{0,\,\sqrt{(-\eta)}}^{\infty}x(x^2+\eta)^{\frac{1}{2}}e^{-x^2}I_0(2\alpha x)dx,\quad (4)$$

which is equivalent (changing $\approx$ to $=$) to Eq. (46)[3] of Mott-Smith and Langmuir whose basic equation referred to a circular sheath; but it should be observed that Eq. (4) is approximate, not exact, for noncircular sheaths—in disagreement with the statement by Mott-Smith and Langmuir that "the current for very large sheaths must be independent of the actual shape of the sheath."[4] While they reached this conclusion by an alternative method of calculating the collector current, this method also involved the angular momentum of an ion, and is inaccurate for the same reason as Eq. (4).

Since the error in Eq. (4) caused by the noncircularity of the sheath must approach zero as the sheath becomes less distorted, it is justifiable to use it when α is small; but for large values of α when the sheath must certainly be greatly distorted (unless it is small, because of a low value of η) Eq. (4), or any equations such as Eqs. (MS 54) to (MS 57) developed from its equivalent Eq. (MS 46), should not be used until it is shown that their error is negligible.

For negative values of η, Mott-Smith and Langmuir have reduced their Eq. (46) to Eq. (47). It may further be shown that Eq. (2b) is algebraically equivalent to Eq. (MS 47). When $\alpha/\sqrt{(-\eta)}$ exceeds unity, Eq. (2b) is suitable for computation; for low values of the ratio, Eq. (MS 47) is better.[5]

It was found that computations of Eq. (3) for $b/r=\sqrt{2}$ and 10.05 gave the same collector current as obtained from Eq. (2b), ($b/r=1$) or its equivalent Eq. (MS 47), ($b/r=\infty$). Thus the collector current is independent of the sheath size for the restricted cases where sheath distortion is negligible, i.e., when α or η are small. Mott-Smith and Langmuir have shown this to be true for three simple types of velocity distributions.[6]

For positive values of η, the solution of Eqs. (3) and (4) depends on a knowledge of the size of the

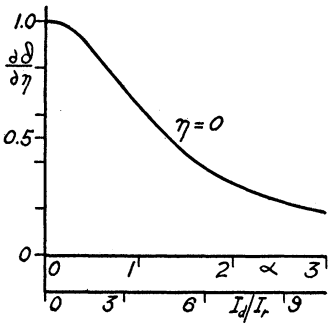

FIG. 2. $\partial\mathfrak{J}/\partial\eta$ as a function of α.

sheath which is not available. Nevertheless, the collector current must lie between the values given by Eqs. (2a) and (4). The latter may be reduced to two equivalent forms,

$$\mathfrak{J}\approx e^{\eta-\alpha^2}\sum_0^\infty\frac{(2p+1)!}{(p!)^2 4^p}\left\{(-1)^{p+1}\left(\frac{\sqrt{\eta}}{\alpha}\right)^{p+\frac{3}{2}}J_{p+3/2}(t)+\left(\frac{\alpha}{\sqrt{\eta}}\right)^p J_p(t)\right\}\quad \eta\geq 0,\quad (5a)$$

$$\mathfrak{J}\approx e^{\eta-\alpha^2}\left(\frac{\sqrt{\eta}}{\alpha}\right)^{\frac{3}{2}}\sum_0^\infty(-1)^{p+1}\frac{(2p+1)!}{(p!)^2 4^p}\times\left(\frac{\sqrt{\eta}}{\alpha}\right)^p J_{p+3/2}(t)+e^{\eta-\alpha^2/2}\Big[(1+\alpha^2)I_0(\alpha^2/2)J_0(t)+\alpha^2 I_0(\alpha^2/2)J_2(t)+2(1+\alpha^2)\sum_1^\infty I_p(\alpha^2/2)J_{2p}(t)+\alpha^2\sum_1^\infty I_p(\alpha^2/2)\{J_{2p-2}(t)+J_{2p+2}(t)\}\Big]\quad \eta\geq 0,\quad (5b)$$

where $J_p(x)$ is the Bessel function of the first kind and the pth order, and $t=2\alpha\sqrt{\eta}$. As α and η become large, these series converge very slowly and the computation becomes more laborious than a graphic integration of Eq. (4).

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[3] It will be shown below that i_∞ may properly be changed to i in Eq. (MS 46) *et seq.*

[4] Reference 1, p. 752.

[5] Present tables of $I_p(x)$ are very limited. More extensive tables will soon be available. Report B.A.A.S. (1935), p. 304.

[6] Reference 1, p. 751.

When α is zero or very small, the graph of the collector current or its logarithm against collector voltage (i.e., $\eta\times$ a constant) indicates clearly the space potential. But it should be observed that as α increases, the value of $\partial\mathfrak{J}/\partial\eta$ (at η equals zero) quickly decreases, (Fig. 2) and therefore the curve of $\mathfrak{J}$ or its logarithm shows no distinctive change at this point for values of α as low as two. (Fig. 3). Thus this method of determining the space potential cannot be applied in these cases. It should be emphasized that this does not preclude a break at the space potential caused by factors specifically excluded by the basic assumptions of this theory, e.g., reflection of ions when there is a retarding potential on the collector.

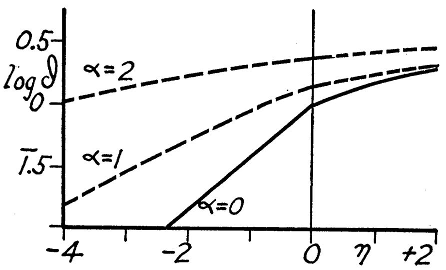

FIG. 3. Log $\mathfrak{J}$ as a function of η.

The equation for $\partial\mathfrak{J}/\partial\eta$ can be given in two equivalent forms and the limiting forms at η equals zero are included for completeness.

$$\partial\mathfrak{J}/\partial\eta=e^{\eta-\alpha^2/2}\left[I_0(\alpha^2/2)I_0(t)+2\sum_1^{\infty}(-1)^p I_p(\alpha^2/2)I_{2p}(t)\right]\quad \text{for } \eta\leq 0, \qquad (6a)$$

$$\partial\mathfrak{J}/\partial\eta=e^{\eta-\alpha^2}\sum_0^{\infty}\frac{(2p)!}{(p!)^2 4^p}\left(\frac{\alpha}{\sqrt{(-\eta)}}\right)^p I_p(t)\quad \text{for } \eta\leq 0, \qquad (6b)$$

where $t=2\alpha\sqrt{(-\eta)}$.

$$\partial\mathfrak{J}/\partial\eta=e^{-\alpha^2/2}I_0(\alpha^2/2)\quad \text{for } \eta=0, \qquad (6c)$$

$$\partial\log\mathfrak{J}/\partial\eta=1/\left[1+\alpha^2\{1+I_1(\alpha^2/2)/I_0(\alpha^2/2)\}\right]\quad \text{for } \eta=0. \qquad (6d)$$

The limit of Eqs. (2) or (4) for η equals zero, as α increases, is $2\alpha/\sqrt{\pi}$, so that the collector current can be expressed in terms of the drift current instead of the random current.

$$\frac{i}{2rlI_d}=\frac{\pi I_r}{I_d}\cdot\mathfrak{J}\approx\frac{\sqrt{\pi}}{2\alpha}\cdot\frac{2\alpha}{\sqrt{\pi}}=1\ (\text{for } \eta=0). \qquad (7)$$

The exact values are 1.282 for $\alpha=1$, 1.062 for $\alpha=2$, 1.028 for $\alpha=3$ (values of $i/2\pi rlI_r$ from Eq. (2a)); so that knowing the drift current density and the collector dimensions we can calculate the minimum value of collector current at the space potential regardless of α.

Attention should be drawn to an improper use of this collector theory by Bramhall,[7] who used a probe in a copper arc. The space potential was determined by a break in the semi-logarithmic plot of the probe characteristic, and the probe current density was there taken to be I_r, the random current density, which is true only when there is no drift current. The drift current density, I_d, was calculated from the arc area and current, giving I_d/I_r roughly 100, or α equals 28. But this high value of α is inconsistent with a break in the characteristic at space potential. Also the probe current calculated from the probe dimensions and the drift current density, Eq. (7), should have a minimum value of 0.4 ampere, whereas the observed value was 0.04 ampere. Further, the part of the theory used by Bramhall in Fig. 9 is based on the Mott-Smith and Langmuir Eq. (54) which is restricted to large values of α, with $\sqrt{-\eta}\approx\alpha$. The value of η in his experiments certainly lay between zero and -25, so that α should not have exceeded 5, a value much too small to warrant the use of Eq. (MS 54)—quite apart from any discussion in this paper as to the validity of that equation.

[7] E. H. Bramhall, Phil. Mag. **13**, 682 (1932).