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Internal Friction in Solids

I. Theory of Internal Friction in Reeds

CLARENCE ZENER
Wayman Crow Hall of Physics, Washington University, St. Louis, Missouri

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In a vibrating reed opposite sides have dilations of opposite signs. Thus when one side is heated the other is cooled. At low frequencies the vibrations are isothermal. At high frequencies they are adiabatic. At intermediate frequencies they are of a hybrid type accompanied by internal friction. In this paper this internal friction is calculated solely from thermodynamical considerations. It is predicted that the internal friction associated with this hybrid type of vibration is of a larger order of magnitude than that due to all other causes.

§1. INTRODUCTION

IN spite of numerous experimental investigations of the internal friction in solids, no satisfactory theoretical study has been given. Our ignorance as to the mechanism of internal friction is due partly to the fact that no one mechanism is responsible for the energy losses in all cases of vibrating solids. As an example of the complexity of the phenomena, when the strains are increased beyond a critical value the losses increase more rapidly than as the square of the strains.[1]

In searching for possible mechanisms of internal friction in solids, it is natural to seek suggestions from the much studied field of internal friction in gases. Here sound waves are damped most strongly in bands whose periods of vibration are comparable to the relaxation times of the gas.[2] For example, sound waves in N_2 are damped most strongly for those periods of vibration which are comparable to the time required for temperature equilibrium to be established between the vibrational and the translational degrees of freedom of the N_2 molecules. We are thus led to investigate types of thermal nonequilibria within a solid.

The following possibilities present themselves. (1) A close analogy to the above mentioned case of N_2 gas would arise from the unequal cooling during expansion, or the unequal heating during compression, of the Debye normal modes of different frequencies. Thermal equilibrium between the different normal vibrations is, however, established so rapidly that damping would occur only for frequencies much higher than those at present available. (2) A lack of thermal equilibrium exists between the vibrating solid and the surrounding air. In an investigation on springs, Sayre[3] found that the partial lack of thermal equilibrium between a wire and the surrounding air gives rise to the dominant energy losses in certain cases. Since the time required for the establishment of temperature equilibrium between a wire and the air is comparatively long, such losses are confined to very low frequencies. (3) We come finally to the lack of thermal equilibrium between various parts of the vibrating solid itself. In isotropic solids with a positive thermal expansion coefficient, a positive (cubic) dilation lowers the temperature, a negative dilation raises the temperature. However, just as in a gas, the time required for the establishment of thermal equilibrium between regions separated by a half wave-length is much longer than the period of vibration. A solid differs from a gas in that fluctuations in dilation, and hence also in temperature, may exist between regions much closer together than a half wave-length. For example, the two sides of a reed vibrating transversely have dilations of opposite signs. Again, the dilation varies rapidly in the immediate vicinity of imperfections in the solid.

The purpose of this paper is to investigate whether the local fluctuations in temperature within a vibrating solid may be of importance in

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[1] See Voight, Ann. d. Physik **47**, 671 (1892); Wegel and Walther, Physics **6**, 141 (1935).

[2] P. S. H. Henry, Proc. Camb. Phil. Soc. **28**, 249 (1932).

[3] Sayre, Rheology **3**, 206 (1932).

internal friction. After defining the various yardsticks of internal friction (§2), the effect of the temperature fluctuation across a vibrating reed upon its internal friction is calculated (§3). The results are discussed quantitatively in §4. It is predicted that over a wide frequency band the effect here discussed will be the predominant cause of internal friction. This thermodynamic effect disappears at low temperatures.

A direct experimental check of this theory may be obtained by observing whether the internal friction of a reed has a maximum in the frequency range predicted by Eq. (11).

§2. Yardsticks of Internal Friction

The displacement vector $\mathbf{U}(\mathbf{r}, t)$ in a vibrating solid is determined by a differential equation of the form

$$\partial^2\mathbf{U}/\partial t^2 + L\mathbf{U} = \mathbf{F} + \mathbf{f}, \tag{1}$$

and by a set of boundary conditions. Here the applied body force (per unit mass) is represented by $\mathbf{F}$. The term $-L\mathbf{U}$ represents the force acting upon unit mass due to the elasticity of the solid. The elastic coefficients in the operator L are taken to be the isothermal coefficients. The forces, per unit mass, which arise from temperature changes,[4] from viscosity, etc., are denoted by $\mathbf{f}$. In particular, the surface forces due to the air are taken care of by a term in $\mathbf{f}$ which is zero everywhere except near the surface.

We shall examine first the solutions of (1) when F is a periodic function of time. We eliminate time from this equation by the usual artifice of complex symbols. Using German characters to denote vectors which are functions of the coordinates, but not of time, we set

$$\mathbf{F}(\mathbf{r}, t) = \mathfrak{F}(\mathbf{r})e^{i\omega t},$$
$$\mathbf{f}(\mathbf{r}, t) = \mathfrak{f}(\mathbf{r})e^{i\omega t},$$
$$\mathbf{U}(\mathbf{r}, t) = \mathfrak{U}(\mathbf{r})e^{i\omega t}.$$

Equation (1) now becomes

$$(L-\omega^2)\mathfrak{U} = \mathfrak{F} + \mathfrak{f}. \tag{2}$$

In general $\mathbf{U}$ and $\mathbf{f}$ are not in phase with each other. In fact, it is just this difference in phase between $\mathbf{U}$ and $\mathbf{f}$ which gives rise to damping. If we choose the origin of time so that $\mathfrak{U}$ is real, $\mathfrak{f}$ will be complex. Let $\mathfrak{f}_1$, $\mathfrak{f}_2$ be its real and imaginary parts, respectively, i.e.,

$$\mathfrak{f} = \mathfrak{f}_1 + i\mathfrak{f}_2.$$

Further let $\mathfrak{U}_0$ be the solution of (2) when $\mathfrak{f}$ is neglected. Since the force $\mathfrak{f}$ has only a small effect upon the solution of (2), it may be calculated assuming the displacement of the solid to be given by $\mathfrak{U}_0$. Then the ratio

$$Q^{-1} = \frac{|\int \mathfrak{U}_0 \cdot \mathfrak{f}_2 dv|}{|\int \mathfrak{U}_0 \cdot L\mathfrak{U}_0 dv|} \tag{3}$$

is a measure of the internal friction of a solid, the integration being over the entire solid. The definition (3) is analogous to the definition of the Q of an electrical circuit. Here, as in the electrical analogy, Q is in general a function of frequency.

When ω is nearly equal to one and to only one natural angular frequency, ω_k, the *form* of the vector function $\mathfrak{U}$ will be nearly identical to that of a vibration vector $\mathfrak{U}_k$ of the free undamped oscillation. This vector satisfies the equation

$$(L-\omega_k^2)\mathfrak{U}_k = 0.$$

The *amplitude* of $\mathfrak{U}$ will depend upon the impressed frequency $\omega/2\pi$. We shall take $\mathfrak{U}_k$ to be normalized, i.e.,

$$\int \mathfrak{U}_k \cdot \mathfrak{U}_k dv = 1,$$

and set $\mathfrak{U} = C\mathfrak{U}_k,$

where C is a function of the impressed angular frequency ω. The force $\mathfrak{f}$ will then also be proportional to C. We shall denote by $\mathfrak{f}_k$ that force which is associated with the normalized function U_k. Then we may set

$$\mathfrak{f} = C\mathfrak{f}_k.$$

We substitute these expressions into (2), multiply to the left by $\mathfrak{U}_k$, and integrate over the solid. We obtain

$$C(\omega) = \frac{F_k}{(\omega_k^2-\omega^2)+i\eta_k},$$

where $F_k = \int \mathfrak{U}_k \cdot \mathfrak{F} dv,$

and $\eta_k = \int \mathfrak{U}_k \cdot \mathfrak{f}_{k2} dv.$

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[4] Love, *The Mathematical Theory of Elasticity* (Cambridge Press, 1920), p. 106.

A plot of $|C_\omega|^2$ against ω has a sharp maximum at $\omega=\omega_k$. The width of this resonance curve at half maximum is a second measure of internal friction. Denoting this width by $\Delta\omega_k$, we have

$$\Delta\omega_k=\eta_k/\omega_k.$$

Since
$$\int \mathfrak{U}_k\cdot L\mathfrak{U}_k dv=\omega_k^2,$$

we have, setting $\omega_k=2\pi\nu_k$,

$$\Delta\nu_k/\nu_k=\eta_k/\omega_k^2=Q^{-1}(\nu_k).$$

One favorite method of measuring internal friction is to observe the logarithmic decrement of free oscillation. In order to obtain the relation between this yardstick of internal friction, and the two previous yardsticks, $\Delta\nu_k/\nu_k$ and Q^{-1}, we must solve (1) when $\mathbf{F}$ is zero. Taking the solid to be initially vibrating in the kth normal mode, we set

$$\mathbf{U}(\mathbf{r},\,t)=C_k(t)\mathfrak{U}_k(\mathbf{r})e^{i\omega_k t},$$
$$\mathbf{f}(\mathbf{r},\,t)=C_k(t)\mathfrak{f}_k(\mathbf{r})e^{i\omega_k t}.$$

Neglecting $(d/dt)\log C_k$ in comparison to ω_k, we obtain, after multiplying (1) to the left by $\mathfrak{U}_k$ and integrating over the solid,

$$dC_k/dt=-(\eta_k/2\omega_k)C_k.$$

Comparing the solution of this equation,

$$C_k(t)=C_k(0)\exp\{-(\eta_k/2\omega_k)t\},$$

with the definition of the logarithmic decrement, Δ_k,

$$C_k(t)=C_k(0)\exp\{-\Delta_k\nu_k t\},$$

we obtain

$$\Delta_k=\eta_k/(2\nu_k\omega_k)=\pi\Delta\nu_k/\nu_k=\pi Q^{-1}(\nu_k). \tag{4}$$

Summary

One yardstick of internal friction is defined by (3). For each type of applied force, torsional or tensile, Q is a continuous function of frequency. Two other yardsticks are defined, $\Delta\nu_k/\nu_k$ and Δ_k. These have a meaning only with reference to the natural modes of vibration. They are related to each other, and to the value of Q at the natural frequencies, by Eq. (4).

§3. Calculation of Internal Friction

The equation which governs the transverse vibrations in a reed may be written in a simple form, provided the wave-lengths are long compared to the thickness of the reed. Let the reed lie along the x axis, with its face normal to the y axis. Further let $y(x,\,t)$ denote the displacement function of the reed with respect to its position of rest. Then y is governed by the well-known equation[5]

$$\rho\frac{\partial^2 y}{\delta t^2}+\frac{\delta^2 M}{\delta x^2}=f.$$

Here ρ is the mass per unit length, $M(x,\,t)$ is the bending moment, and $f(x,\,t)$ is the normal component of the applied force per unit length. The bending moment may be calculated directly from the function y, and from a knowledge of the distribution of temperature across the reed. This is done as follows.

We separate M into two parts, M_1 and M_2. The first part is the moment which would arise from the bending of the reed if the temperature gradient across the reed were zero. The second part is the bending moment which arises from the variation of temperature across the reed. The first part is commonly written as

$$M_1=EI\delta^2 y/\partial x^2,$$

where E is Young's modulus, and

$$I=ha^3/12.$$

Here h is the width, and a the thickness, of the reed.

If we let s denote the y coordinate inside the reed, with origin at the center of the reed, and T_0 the average temperature, then

$$M_2=\alpha h\int_{-a/2}^{a/2}(T-T_0)ds. \tag{5}$$

Here
$$\alpha=(\delta X_x/\delta T)_\pi,$$

the suffix π denoting that the normal stresses across the xy and the zx plane are kept constant in the partial differentiation.

In order to calculate M_2 explicitly, we must find the temperature fluctuation $T-T_0$ across

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[5] Southwell, *Theory of Elasticity* (Oxford Press, 1936), pp. 172–174.

the reed. Now the time derivative $\delta T/\delta t$ may be written as the sum of two terms. The first is due to the flow of heat by diffusion, and is given by $D\delta^2T/\delta s^2$, where D is the thermal diffusion constant. The second term is due to the generation and absorption of heat by the dilation. Since the stresses Y_y and Z_z are both zero,[6] the strains e_{yy} and e_{zz} are completely determined by e_{xx}. The second part of M_2 may thus be written as $\beta\delta e_{xx}/\delta t$, where

$$\beta=(\delta T/\delta e_{xx})_{Q,\,\pi}.$$

The suffix π has the same significance as before; the suffix Q refers to an adiabatic change of e_{xx}. Now the e_{xx} in a reed is given by $-s\delta^2y/\delta x^2$. Since we need only consider the steady state of forced oscillation, we set

$$y(x,\,t)=Y(x)e^{i\omega t}.$$

We thus obtain the following equation for T:

$$\frac{\delta T}{\delta t}=D\frac{\delta^2 T}{\delta s^2}-\left(i\omega\beta\frac{\delta^2 Y}{\delta x^2}\right)se^{i\omega t}. \tag{6}$$

The boundary conditions for this equation are obtained by observing that the establishment of temperature equilibrium in the reed is due almost entirely to a flow of heat across the reed, and not to and from the air. Now a zero flow of heat across the surface of the reed implies that the temperature gradient is there zero. Our boundary condition is thus

$$\delta T/\delta s=0 \quad \text{at} \quad s=\pm a/2. \tag{6a}$$

That solution of Eq. (6) which satisfies the condition (6a) may readily be obtained as a trigonometric series. This series will automatically satisfy the boundary condition (6a) if each term satisfies this condition. Such a series is given by

$$T=T_0+\Sigma_k g_k(t)\sin\{(2k+1)\pi s/a\}. \tag{7}$$

The functions $g_k(t)$ are obtained by substitution of this series into (6), multiplying by $\sin\{(2k+1)\pi s/a\}$, and integrating across the reed. We obtain

$$g_k=-\frac{(i\omega\beta\delta^2Y/\delta s^2)s_k e^{i\omega t}}{i\omega+\mu_k}.$$

Here

$$s_k=(2/a)\int_{-a/2}^{a/2} s\sin\{(2k+1)\pi s/a\}ds. \tag{8}$$

and $\mu_k=\{(2k+1)\pi/a\}^2D.$

As our final step in obtaining M_2, we substitute T from (7) into (5). Setting

$$M_2(x,\,t)=\mathfrak{M}_2(x)e^{i\omega t}.$$

and separating $\mathfrak{M}_2$ into its real and imaginary parts, we obtain

$$\Re\mathfrak{M}_2=\tfrac{1}{2}\alpha\beta ad\frac{\partial^2 Y}{\partial x^2}\Sigma_k\frac{\omega^2}{\omega^2+\mu_k^2}s_k^2,$$

$$\Im\mathfrak{M}_2=\tfrac{1}{2}\alpha\beta ad\frac{\partial^2 Y}{\partial x^2}\Sigma_k\frac{\omega\mu_k}{\omega^2+\mu_k^2}s_k^2.$$

The Q of the reed is now, by definition,

$$Q^{-1}=\Im\frac{\partial^2}{\partial x^2}\mathfrak{M}\bigg/\Re\frac{\delta^2}{\partial x^2}\mathfrak{M}.$$

This may be written explicitly as

$$Q^{-1}=\frac{(\alpha\beta/E)\Sigma_k\dfrac{\omega\mu_k}{\omega^2+\mu_k^2}(6s_k^2/a)}{1+(\alpha\beta/E)\Sigma_k\dfrac{\omega^2}{\omega^2+\mu_k^2}(6s_k^2/a)}. \tag{9}$$

From (8) we obtain

$$6s_k^2/a=96\pi^{-4}(2k+1)^{-4}.$$

For $k=0,\ 1,\ 2$ the right member has the values 0.986, 0.012, 0.0016. Hence very little error is made in replacing $6s_k^2/a$ by unity when $k=0$, by zero for all other values of k.

In the appendix we find that

$$\alpha\beta/E=\tfrac{1}{3}(1-2\sigma)(C_p-C_v)/C_v, \tag{10}$$

where σ is Poisson's ratio. Since the right member of (10) is always less than 0.01, we make only a slight error by neglecting the second term in the denominator of (9). We finally obtain as a very good approximation

$$Q_{\text{th}}^{-1}=\frac{1-2\sigma}{3}\cdot\frac{C_p-C_v}{C_v}\cdot\frac{\omega\mu}{\omega^2+\mu^2}, \tag{11}$$

where $\mu=(\pi/a)^2D, \tag{12}$

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[6] Reference 5, p. 161.

and where the suffix th is a reminder that the right side of (11) gives only that part of Q^{-1} which is due to the thermodynamical effects above discussed.

§4. Quantitative Discussion

That part of the internal friction which is due to the fluctuation of temperature across the reed is given by Eq. (11). Considered as a function of the angular frequency, ω, Q_{th}^{-1} has a maximum at $\omega=\mu$, where μ^{-1} is the relaxation time for the establishment of temperature equilibrium across the reed, and is given by (12). When $\omega\ll\mu$, the vibration proceeds isothermally, when $\omega\gg\omega$, it proceeds adiabatically. In the transitional region between isothermal and adiabatic vibration, $\omega\sim\mu$, stress and strain are no longer in phase. The hysteresis loop caused by this phase shift gives rise to a so-called "internal friction."

The maximum value of Q_{th}^{-1} is given in Table I for several metals. The last column gives the thickness of the reed, a_{1000}, for which the maximum of Q_{th}^{-1} occurs at a frequency of 1000 cycles sec.$^{-1}$ (expressed in cm).

The *actual* Q^{-1} of a reed will be equal to the sum of the Q_{th}^{-1} given by Eq. (11), and the Q^{-1} which arises from all other causes of internal friction. The *measured* Q^{-1} of a reed will also contain a contribution resulting from energy losses at the supports of the reed, and from acoustical losses. The very careful experiments of Wegel and Walther[1] indicate that the Q^{-1} of metals measured by nearly all previous investigators consisted principally of terms arising from apparatus losses. They found that the Q^{-1} of a metal was a structure sensitive property. Thus (see Table II of reference 1) they obtained values of Q, at 10,000 cycles per second, ranging from 9000 to 52,000 for aluminum, from 500 to 10,000 for copper, from 3800 to 40,000 for steel. These values were obtained for the longitudinal vibrations in cylindrical rods, in which the thermodynamical damping discussed in this paper was not present. If the same materials had been used in the form of a reed, the damping due to causes other than that here discussed would probably have been of the same order of magnitude. Comparing the minimum values of Q^{-1} found by Wegel and Walther for, say, copper and aluminum, 10^{-4} and 0.2×10^{-4}, respectively, with the maximum values of Q_{th}^{-1} in Table I, we find that samples may be found in which Q_{th}^{-1} is at least 10 times as large as the Q^{-1} due to all other causes over a broad frequency band.

Table I. *Values of Q_{th}^{-1} for various metals at 20°C and the thickness (a) of the reed for which the maximum value of Q_{th} occurs at 1000 cycles per sec.*

Metal	$(C_p-C_v)/C_v$*	σ	Max. of $Q_{\text{th}}^{-1}\times10^3$	$a_{1000}\times10^2$
W	0.006	0.17	0.66	3.5
Pl	.020	.387	.76	2.0
Au	.038	.42	.81	4.3
Pb	.067	.45	1.0	2.0
Ni	.021	.31	1.3	1.5
Cu	.028	.34	1.4	4.2
Ag	.040	.38	1.6	5.1
Sn	.040	.33	2.3	2.5
Zn	.052	.33	3.0	2.5
Cd	.060	.30	4.0	2.7

* Eucken, *Handbuch der Experimental Physik* Vol. 8, p. 211.

The formula (11) may be checked both by a variation of frequency, and by a variation of temperature. Temperature has a two-fold effect upon Q_{th}^{-1}. (1) A lowering of the temperature displaces the maximum value of Q_{max}^{-1} to higher frequencies by increasing the diffusion constant D. (2) From the empirical formula[7]

$$(C_p-C_v)/C_v=\text{const}\times TC_p$$

we see that a lowering of temperature decreases the maximum of Q_{th}^{-1}.

Appendix

The left member of Eq. (10) may be written as

$$\left(\frac{\delta T}{\delta e_{xx}}\right)_{Q,\,\pi}\left(\frac{\delta X_x}{\delta T}\right)_{e_{xx},\,\pi}\Big/\left(\frac{\delta X_x}{\delta e_{xx}}\right)_{T,\,\pi}. \tag{a-1}$$

Here Young's modulus for constant temperature, E, has been written explicitly as $(\delta X_x/\delta e_{xx})_{T,\,\pi}$. A standard transformation in partial differentiation reduces (a-1) to

$$-\left(\frac{\delta T}{\delta e_{xx}}\right)_{Q,\,\pi}\left(\frac{\delta e_{xx}}{\delta T}\right)_{X_x,\,\pi}. \tag{a-2}$$

At this stage it is convenient to replace the strain derivatives by more familiar forms. The temperature

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[7] Eucken, *Handbuch der Experimental Physik* Vol. 8, p. 209.

change caused by an adiabatic stretching is equal to $(\delta T/\delta V)_Q \delta V$, where δV is the change in volume accompanying the strain.

Now $\qquad e_{yy}=e_{zz}=-e_{xx}$
when $\qquad Y_y=Z_z=0.$
Hence $\qquad \delta V/V=e_{xx}+e_{yy}+e_{zz}=(1-2\sigma)e_{xx},$
and so $\qquad (\delta T/\delta e_{xx})_{Q,\,\pi}=(1-2\sigma)(\delta T/\delta \log V)_Q.$

The second factor in (a-2) may obviously be written as $\frac{1}{3}(\delta \log V/\partial T)_p$. Hence (a-2) reduces to

$$-\frac{1-2\sigma}{3}\left(\frac{\partial T}{\partial V}\right)_Q\left(\frac{\partial V}{\partial T}\right)_p. \qquad \text{(a-3)}$$

This expression may be written in a more illuminating form. If only two of the four variables x_1, x_2, x_3, x_4 are independent, then the following identity is true:

$$\frac{(\partial x_1/\partial x_2)_{x_4}}{(\partial x_1/\partial x_2)_{x_3}}=1-\frac{(\partial x_3/\partial x_2)_{x_4}}{(\partial x_3/\partial x_2)_{x_1}}.$$

Replacing x_1, x_2, x_3, x_4 by V, T, Q, p, respectively, we find that (a-3) becomes

$$\tfrac{1}{3}(1-2\sigma)(C_p-C_v)/C_p.$$

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Collector Theory for Ions with Maxwellian and Drift Velocities

A. H. HEATLEY
Electrochemical Laboratory, University of Toronto, Toronto, Canada

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For a gaseous discharge in which the ions have a drift velocity superimposed on a Maxwellian distribution, the Mott-Smith and Langmuir collector theory deals only with collectors having a high ratio of sheath to collector diameters. The present paper removes this restriction, but points out that except when there is negligible sheath distortion due to low drift velocity or to low collector potential the theory gives only approximate results. It is also shown that for ratios of drift to random current density as low as seven, the space potential is not indicated by a break on the semi-logarithmic plot of the collector characteristic.

An electrical discharge in a gas is conveniently studied by the use of a collector or probe connected to a suitable auxiliary circuit. Mott-Smith and Langmuir[1] have shown that if the logarithm of the collector current be plotted against the potential applied to it, the resulting "characteristic" is a line which departs from linearity at the space potential. This holds for a Maxwellian distribution of the ions (or electrons) of the discharge; but when there is a small drift velocity superimposed on the Maxwellian distribution the characteristic is modified as shown in their paper. Their results refer only to the case where there is a high ratio of sheath[2] to collector diameter. The present paper deals with sheaths of all sizes.

The basic assumptions of this collector theory are discussed by Mott-Smith and Langmuir.[1] Special attention is directed to the two assumptions (1) that the gas pressure is so low that collisions between ions or electrons and gas molecules in the sheath have a negligible effect on the collector current, and (2) that there is no reflection of ions or electrons at the collector surface.

Because of the drift velocity, the cross section of the space charge sheath around a cylindrical collector (with its axis at right angles to the direction of the drift) is distorted from the circular shape. But obviously it still must have bilateral symmetry about the direction of the drift, which is therefore taken as the coordinate axis. The length of the collector is assumed to be so great that end effects may be neglected.

Without the restriction imposed by Mott-Smith and Langmuir that the sheath is circular and by methods analogous to theirs, it may be shown that, as a first approximation,

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[1] H. M. Mott-Smith and I. Langmuir, Phys. Rev. **28**, 727 (1926).

[2] The sheath boundary may be briefly defined as the surface beyond which the collector potential exerts no force on the ions or electrons of the discharge.