Collision of Proton and Deuteron

K. OCHIAI*
Physical Institute, Imperial University, Tokyo, Japan
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The present calculation is similar to that of Schiff on neutron-deuteron collisions, and is based on the suggestion of Morse that the function $\sin(kr+\delta)-e^{-\epsilon r}\sin\delta\cos kr$ will be a good approximation to the actual wave function if the parameters are properly adjusted. By joining this function smoothly with the Coulombian field wave function, we obtain the scattering formula for proton-deuteron collisions, while, by extending this function to infinity, we get the neutron-deuteron scattering. The coordinate system, in which the Hamiltonian is separable, was used, and this is the main difference from Schiff's calculation. The cross section for neutron-deuteron scattering was found to be 3.2×10^{-24} cm^2 at vanishing neutron energy, in good agreement with experiments. For proton-deuteron scattering, no good agreement was obtained. In the range of angle of scattering less than about 70°, the calculation gives still a tolerable result, but, for larger angles, the experimental scattering increases very rapidly. Such a rapid increase does not seem to be explained without some other important improvement of the theory.

THE wave equation for a system of a deuteron and a proton,[1] in the rest coordinate system, is

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}}-\frac{\hbar^2}{M}\Delta_{\mathbf{r}_{23}}+V_{12}+V_{13}+V_{23}+\frac{e^2}{r_{12}}-W_D-W\right]\Psi=\left[-\frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}'}-\frac{\hbar^2}{M}\Delta_{\mathbf{r}_{13}}+V_{12}+V_{13}+V_{23}+\frac{e^2}{r_{12}}-W_D-W\right]\Psi=0, \quad (1)$$

where

μ = reduced mass $=\frac{2}{3}M$,
W_D = binding energy of deuteron,
W = collision energy in the rest system $=\frac{2}{3}W_0$,
W_0 = collision energy in the laboratory system (deuteron initially at rest),
$\mathbf{r}_1$, $\mathbf{r}_2$ = space coordinates of protons 1 and 2,
$\mathbf{r}_3$ = space coordinates of neutron 3,

$$\mathbf{r}=\mathbf{r}_1-\frac{\mathbf{r}_2+\mathbf{r}_3}{2},\quad \mathbf{r}'=\mathbf{r}_2-\frac{\mathbf{r}_1+\mathbf{r}_3}{2}=P(\mathbf{r}_1\mathbf{r}_2)\mathbf{r},$$

$$\mathbf{r}_{23}=\mathbf{r}_2-\mathbf{r}_3,\qquad \mathbf{r}_{13}=\mathbf{r}_1-\mathbf{r}_3$$

and P is the interchange operator. We assume a solution of the form

$$\Psi=\psi_D(23)\psi(r,\theta)S(1,2,3)+\psi_D(13)\psi'(r',\theta')S'(1,2,3), \quad (2)$$

$$S'(1,2,3)=S(2,1,3),$$

where S is the spin function,[2] and $\psi_D(ij)$ is the deuteron wave function, satisfying

$$[-(\hbar^2/M)\Delta r_{ij}+V_{ij}-W_D]\psi_D(ij)=0.$$

θ and θ' are the angles between the direction of the initial current and the vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively.

Putting (2) into (1), we have

$$[-(\hbar^2/2\mu)\psi_D(23)\Delta_r\psi+V\psi_D(23)\psi-W\psi_D(23)\psi]S+[-(\hbar^2/2\mu)\psi_D(13)\Delta_{r'}\psi'+V'\psi_D(13)\psi'-W\psi_D(13)\psi']S'=0 \quad (3)$$

with

$$V=V_{12}+V_{23}+(e^2/r_{12}),\quad V'=V_{12}+V_{23}+(e^2/r_{12}).$$

Considering the nature of nuclear forces, these potential energies may be replaced by

$$V=\begin{cases}V_{12}+V_{13}, & r<r_0\\ (e^2/r) & r>r_0\end{cases},\qquad V'=\begin{cases}V_{12}+V_{23}, & r'<r_0\\ (e^2/r'), & r'>r_0\end{cases}, \quad (4)$$

where r_0 is a certain distance of the order of nuclear radius, at which nuclear forces practically vanish.

We will try to solve the Eq. (3) by the method of pertubation.[3] For this purpose, it is convenient to introduce a new potential U instead of V, which can be interpreted as the average potential of a proton in the field of a deuteron.

* At present a guest at the Massachusetts Institute of Technology.
[1] Schiff, Phys. Rev. **52**, 149 (1937).
[2] For spin functions, see Schiff, reference 1. We will write S and S' for $S(1, 2, 3)$ and $S'(1, 2, 3)$, respectively.
[3] This may be allowed for a qualitative discussion of lightest nuclei, where statistical treatment is impossible, although the method is in general inapplicable to heavier nuclear collisions (Bethe, Rev. Mod. Phys. **9**, 71 (1937)).

$$U(r)=\begin{cases}\bar{V}_{12}+\bar{V}_{13}, & r<r_0\\ (e^2/r), & r>r_0\end{cases}$$
$$U(r')=\begin{cases}\bar{V}_{12}+\bar{V}_{23}, & r'<r_0\\ (e^2/r'), & r'>r_0\end{cases} \tag{5}$$

Here $\bar{V}$ is some suitable average of V over the deuteron coordinates, the position of the unbound proton being fixed. If a good choice of this U were obtained, we may consider

$$V-U(r)\ll V, \quad V'-U(r')\ll V'.$$

Now we assume that, proton 2 is initially bound to neutron 3. Then the main term in (2) is that which contains $\psi(r,\theta)$. Dividing $\psi(r,\theta)$ into two parts $\psi(r,\theta)=\psi_0(r,\theta)+\psi_1(r,\theta)$, where ψ_0 satisfies

$$[-(\hbar^2/2\mu)\Delta_r+U(r)-W]\psi_0(r,\theta)=0. \tag{6}$$

we may consider solution (2) as consisting of three parts: ψ_0 (first approximation), ψ_1 and ψ' (second approximation).

Multiplying (3) by $S\psi_D^*(23)$ and integrating (spin summation also included), we get

$$[-(\hbar^2/2\mu)\Delta_r+U(r)-W]\psi_1 = -\int S\psi_D^*(23)(V-U)\psi_D(23)\psi_0 Sd\mathbf{r}_{23}. \tag{7}$$

Similarly

$$[-(\hbar^2/2\mu)\Delta_{r'}+U(r')-W]\psi' = -\int S'\psi_D^*(13)(V-U)\psi_D(23)\psi_0 Sd\mathbf{r}_{13}. \tag{8}$$

The problem is now: (i) to find U, (ii) to solve (6), (7), (8). It will be found later that

$$U(r)=\begin{cases}1-De^{-\beta r^2}, & r<r_0\\ (e^2/r), & r>r_0\end{cases} \tag{9}$$

is a good approximation for U, where the constant D must be chosen differently for the two different spin states, quartet and doublet.

For $r>r_0$, where $U(r)$ is Coulombian, the Eq. (6) (if expanded into spherical harmonics) has two linearly independent solutions $F_l(\rho)$ and $G_l(\rho)$, which behave asymptotically like[4]

$$F_l(\rho)\sim\sin(\rho-(l\pi/2)-\eta\log 2\rho+\sigma_l),$$

$$G_l(\rho)\sim\cos(\rho-(l\pi/2)-\eta\log 2\rho+\sigma_l).$$

Hence

$$\psi_0=\sum_{l=0}^{\infty} i^l(2l+1)P_l(\cos\theta)e^{i(\sigma_l+K_l)} \times\{A_lF_l(\rho)+B_lG_l(\rho)\}/\rho, \quad r>r_0 \tag{10}$$

$$\rho=kr,\quad k=(2\mu W)^{\frac{1}{2}}/\hbar=\mu v/\hbar,\quad \eta=e^2/(\hbar v),$$
$$\sigma_l=\arg\Gamma(l+1+i\eta),$$

v = initial relative velocity, $\tan K_l=B_l/A_l$. This becomes asymptotically

$$\psi_0\sim\left[1-\frac{\eta^2}{2ikr\sin^2\theta/2}\right]e^{ikr\cos\theta+i\eta\log 2kr+i\eta\log\sin^2(\theta/2)}-\frac{e^{ikr-i\eta\log 2kr-i\eta\log\sin^2(\theta/2)+2i\sigma_0}}{r}$$
$$\times\left[\frac{\eta}{2k\sin^2\theta/2}+\frac{i}{2k}\sum_{l=0}^{\infty}(2l+1)(e^{2iK_l}-1)P_l(\cos\theta)e^{i\eta\log\sin^2(\theta/2)+2i(\sigma_l-\sigma_0)}\right].$$

The important quantity K_l (or the ratio B_l/A_l) for the calculation of the cross section can be detrmined by requiring that the solution of (6) for $r<r_0$, should join smoothly with (10) at $r=r_0$. Namely, if we put for $r<r_0$, instead of (10),

$$\psi_0=\sum_{l=0}^{\infty} i^l(2l+1)P_l(\cos\theta)e^{i(\sigma_l+K_l)}R_l(\rho)/\rho \quad (r<r_0), \tag{11}$$

then the conditions

$$\left.\begin{aligned}R_l(\rho)&=A_lF_l(\rho)+B_lG_l(\rho)\\ \frac{dR_l(\rho)}{d\rho}&=A_l\frac{dF_l(\rho)}{d\rho}+B_l\frac{dG_l(\rho)}{d\rho}\end{aligned}\right\}\text{at } r=r_0 \ (\text{or } \rho=kr_0=\rho_0)$$

determine $\tan K_l=B_l/A_l$. Assuming that only the partial wave $1=0$ is appreciably affected by the internuclear forces, while all other partial waves $1\geqq 1$ remain practically the same as in the

[4] Breit, Condon and Present, Phys. Rev. **50**, 825 (1936); Yost, Wheeler and Breit, Phys. Rev. **49**, 174 (1936).

case of purely Coulombian field (i.e., $K_l=0$ for $l\geqq 1$), we need only to calculate K_0 by

$$\frac{\rho}{R_0(\rho)}\cdot\frac{dR_0(\rho)}{d\rho}=\frac{\Phi_0^*(\rho)}{\Phi_0(\rho)}-\frac{1}{\Phi_0(\rho)\Theta_0(\rho)+C_0^2\rho\Phi_0^2(\rho)\cot K_0}\quad\text{at }\rho=\rho_0$$

Φ_0, Φ_0^*, Θ_0 and C_0 being defined in the papers mentioned above. Following the procedure of Morse,[5] we set

$$R_0=A[\sin(kr+\delta)-e^{-\epsilon r}\sin\delta\cos kr],\tag{12}$$

where A takes care of the possible change of amplitude due to the Coulombian field, and ϵ is obviously of the order of magnitude of the constant $(\beta)^{\frac{1}{2}}$ in the potential function (9). We put simply $\epsilon/(\beta)^{\frac{1}{2}}=2$, since this choice of the ratio seems to provide a good correspondence of the potential $e^{-\beta r^2}$ to that of Morse type $-2e^{\epsilon r}+e^{-2\epsilon r}$. The parameter δ will be determined later for each specified value of k.

We will now proceed to solve the equations of second approximation (7) and (8). These equations differ from the equation of first approximation (6) only by the existence of their right-hand side integrals, and these integrals vanish as r or r' becomes greater than r_0, the Coulombian parts in U and V canceling each other out. So that, for r or $r'>r_0$, the solutions ψ_1 and ψ' are again expressible by the Coulombian field wave functions F_l and G_l. Further we want such a solution which asymptotically represents only a divergent spherical wave. Such a solution is obtainable, when the solution of the homogeneous equation is already known. The method is given by Mott and Massey,[6] which applies also to our case with a slight modification due to the existence of Coulombian potential.

Retaining only the partial wave $l=0$, we get

$$\psi_1\sim\frac{e^{i(kr-\eta\log 2kr+2\sigma_0+2K_0)}}{r}g,$$

$$\Psi'\sim\frac{e^{i(kr'-\eta\log 2kr'+2\sigma_0+2K_0)}}{r'}g',$$

[5] Morse, Phys. Rev. **51**, 1003 (1937); Morse, Fisk and Schiff, Phys. Rev. **51**, 706 (1937).

[6] Mott and Massey, *The Theory of Atomic Collisions*, p. 101.

$$g=-\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int S\frac{R_0(kr)}{kr}\psi_D^*(23)(V-U)\times\psi_D(23)\frac{R_0(kr)}{kr}Sd\mathbf{r}_{23}d\mathbf{r},\tag{13}$$

$$g'=-\frac{1}{4\pi}\cdot\frac{2\mu}{\hbar^2}\int S'\frac{R_0(kr')}{kr}\psi_D^*(13)(V-U)\times\psi_D(23)\frac{R_0(kr)}{kr}Sd\mathbf{r}_{13}d\mathbf{r}'.\tag{14}$$

After antisymmetrizing the final wave function with respect to the two protons 1 and 2, the cross section will be given by the square of the absolute value of the coefficient of e^{ikr}/r in the asymptotic expression of $\psi_0(r,\theta)+\psi_1(r,\theta)-\psi'(r,\theta)$: namely by

$$\left|-\frac{\eta}{2k\sin^2(\theta/2)}+\frac{e^{iK_0}\sin K_0}{k}e^{i\eta\log\sin^2(\theta/2)}+(g-g')e^{2iK_0+i\eta\log\sin^2(\theta/2)}\right|^2.\tag{15}$$

Results

As a first step to approximation, we have calculated $U(r)$ by

$$U(r)=\int\psi_D^*(23)S(V_{12}+V_{13})\psi_D(23)Sd\mathbf{r}_{23}$$

with

$$V_{ij}=-e^{-\alpha r_{ij}^2}[C_MP(\mathbf{r}_i\mathbf{r}_j)+C_HP(\mathbf{r}_i\mathbf{r}_j)P(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j)]$$

$$\psi_D(ij)=(N_D)^{\frac{1}{2}}e^{-(\nu_D/2)r_{ij}^2}.$$

The values of the constants are[7]

$$C_M+C_H=60,\quad C_H=12.7,\quad\alpha=11.5,\quad\nu_D/\alpha=0.89$$

in nuclear units (length $\hbar/c(nM)^{\frac{1}{2}}$, energy mc^2).

The integration is to be performed under fixed r, and the space exchange operator was omitted for the moment, keeping only the spin exchange operator. It was found that $U(r)$ is of the form (9), with the constants: $D=D_q=82.6$ for quartet, $D=D_d=56.4$ for doublet, and $\beta=9.0$ (always in nuclear units unless otherwise stated). Thus the

[7] These values are roughly in agreement with those of Morse and Schiff, if appropriate correspondence is established between both types of potential.

TABLE I. *Values of* δ.

$W_0(mc^2)$	QUARTET	DOUBLET
0	$\pi(\Delta_q=-0.462)$	$\pi(\Delta_d=-0.500)$
1	$\pi-24°.3$	$\pi-26°.2$
2	$\pi-33°.7$	$\pi-36°.1$

constant ϵ in the function R_0 was fixed: $\epsilon=2(\beta)^{\frac{1}{2}}=6.0$. Another constant δ for the case $k=0$, or the constant Δ defined by $\Delta=-\lim_{k\to 0}(\sin\delta/k)$ is found to be $\Delta_q=-0.412$ for quartet and $\Delta_d=-0.451$ for doublet.

These values must be corrected so as to include to some extent the exchange operator $P(\mathbf{r}_i\mathbf{r}_j)$ in V_{ij}. For this we will require that the function R_0 should represent the true wave function as nearly as possible apart from the exchange integral g', or in other words, g should be zero. This was done only for vanishing energy $k\to 0$. Namely we have adjusted D and Δ in such a way that $g\to 0$ as $k\to 0$. In carrying out the integration g, we must deal with the integral of the form

$$\int\frac{R_0(kr)}{kr}\exp\left[-ar^2-b\rho^2-c(\mathbf{r}\cdot\rho)\right]\frac{R_0(kr')}{kr'}d\mathbf{r}d\boldsymbol{\rho},$$

$$r'=\text{const.}\left[(\mathbf{r}-\boldsymbol{\rho})^2\right]^{\frac{1}{2}}$$

and $R_0(kr)/kr$ is of the form $\text{const}\cdot[1+(\Delta/r)\times(1-e^{-\epsilon r})]$ for $k\to 0$. The expansion[8]

$$\frac{e^{-\lambda[(r^2+\rho^2-2r\rho\cos\varphi)]^{\frac{1}{2}}}}{[(r^2+\rho^2-2r\rho\cos\varphi)]^{\frac{1}{2}}}=\sum_{n=0}^{\infty}(2n+1)$$

$$\times\frac{K_{n+\frac{1}{2}}(\lambda\rho)}{(\rho)^{\frac{1}{2}}}\frac{I_{n+\frac{1}{2}}(\lambda r)}{(r)^{\frac{1}{2}}}P_n(\cos\varphi)\quad(\rho>r)$$

and similar ones for $e^{\lambda\cos\varphi}$ and

$$1/[(r^2+\rho^2-2r\rho\cos\varphi)]^{\frac{1}{2}}$$

were used. However, these are already very complicated. We have, therefore, averaged the integrand with respect to $\cos(\widehat{\mathbf{r}\boldsymbol{\rho}})$ before integration, i.e., we have taken only the first terms of the expansions into account. Then the integration over $\boldsymbol{\rho}$ produces terms expressed in error functions. The last step was done numerically.

The corrected values are

$$D_q=53,\quad \Delta_q=-0.462,\quad D_d=44,\quad \Delta_d=-0.500,$$

for which $g=0$ for both cases. We adopted these values for $U(r)$, and calculated δ for each assigned value[9] of k (Table I).

The exchange integral g' can be carried out in the same manner as g. This was calculated only for vanishing k, and the values obtained are: $g_q'=+0.151A^2$, $g_d'=-0.047A^2$ for quartet and doublet, respectively.

Before entering into the calculation of K_0, we will briefly discuss neutron-deuteron collisions. If we omit all the terms arising from the Coulombian interaction and extend the function R_0 to infinity, therefore, if we put $\eta=\sigma_0=0$, $A=1$, $K_0=\delta$ in (15), it becomes the cross section formula for neutron-deuteron scattering. For the values given above, the mean cross section is 3.23×10^{-24} cm^2 at vanishing neutron energy. If the exchange integral g' is omitted it is only 2.28×10^{-24} cm^2. This is about one-half of Schiff's result, and the agreement with experiments is satisfactory. The cross section decreases to 1.82×10^{-24} cm^2 at the neutron energy of 1 Mev (without exchange; exchange effect would probably be small). We believe that this improvement of agreement (although the calculation is still a qualitative one) is due to the different use of the coordinate system from that used by Schiff, rather than the different forms of the potential functions. The exchange integral makes the quartet-scattering larger than the doublet-scattering by a factor about 2 (omitting different statistical weights). This effect is much more remarkable in Schiff's result (factor 6 instead of 2). We have now to join the function R_0 with the Coulombian field wave function at some suitable distance $r=r_0$. For this r_0, we have taken (i) the distance at which the combined potential Coulombian plus $-De^{-\beta r^2}$ attains its maximum, (ii) the distance at which these potentials just cancel each other. The results are only slightly different in both

TABLE II. *Values of* K_0.

$W_0(mc^2)$	QUARTET	DOUBLET
1	$\pi-14°.2$	$\pi-15°.6$
2	$\pi-24°.1$	$\pi-26°.1$

[8] Watson, *Theory of Bessel Functions* (1922), p. 366.

[9] Professor Morse has kindly provided me with the table of integrals of the form $\int_0^\infty e^{-\frac{1}{2}r^2-\alpha r}\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\beta r dr$ necessary for this calculation.

cases. We give here only the result based on the latter choice. The values of K_0 are given in Table II. The amplitude A in (12) was found to be increasing slightly with increasing energy. It is about 0.90 at $W_0=1$ mc^2, and is about 0.94 at $W_0=2\ mc^2$.

The cross sections for quartet and doublet states were calculated by (15) and, by adding them in the ratio 2 : 1, the mean cross section was obtained.

The ratio of actual cross section to that of Rutherford is given in Table III. Here Θ is the scattering angle in the laboratory coordinate system. In the fourth column, we have attempted to estimate the effect of exchange. It was assumed that the exchange integral is decreasing with increasing energy, and at $W_0=1\ mc^2$ it is about one-half of its value at vanishing energy. At $W_0=2\ mc^2$, the exchange effect would probably be unimportant.

According to the experiment of Tuve, Heydenburg and Hafstad[10] at the energy of 0.83 Mev, this ratio of cross sections does not show a smooth increase with increasing scattering angle. In the range $60° < \Theta < 75°$, the experimental points are found between about 10 to 20. It increases suddenly at about $\Theta=80°$, reaching a value more than 120 at 90°, and increasing still more rapidly with increasing Θ. It is impossible to get such a high value by taking only K_0 into account. We also tried to see whether it is possible to check the experimental curve by choosing K_0 and K_1 suitably. But no appreciable improvement in the agreement was obtained. The assumption[11] that many more K_l values are playing an important rôle in the scattering formula, does not seem to be probable in the case of short range interaction.

TABLE III. *Ratio of cross sections actual/Rutherford.*

θ	Θ	$W_0=1\ mc^2$ WITHOUT EXCHANGE	$W_0=1\ mc^2$ WITH EXCHANGE	$W_0=2\ mc^2$ WITHOUT EXCHANGE
20°	13°.4	1.076	1.082	1.21
40°	26°.9	1.47	1.53	2.30
60°	40°.9	2.30	2.49	5.0
80°	55°.6	3.6	4.0	9.7
100°	71°.7	5.3	6.1	16.4
120°	90°	7.2	8.4	24
140°	112°.5	9.0	10.5	32
160°	142°.1	10.3	12.1	38
180°	180°	10.7	12.6	40

Finally, the effect of polarization of the deuteron due to the Coulombian interaction with the incoming proton was estimated as follows. We have assumed, as the wave function of the polarized deuteron,

$$\psi_D=\frac{1}{[1+f^2(r)/3]^{\frac{1}{2}}}\phi_0(23)[1+f(r)\cos\varphi],$$

where r has the same meaning as before, ϕ_0 is the unpolarized deuteron wave function, and φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}_{23}$.

$f(r)$ was determined by the method of variation, so as to minimize the energy of the polarized deuteron, for each assigned value of r. It was found that $|f(r)|<0.04$, and the effect seems to be quite small.

Therefore, it does not seem likely that such a large backward scattering as is found in the experiments of Tuve, Heydenburg and Hafstad can be explained satisfactorily by calculations of the sort discussed here.

In conclusion, the writer wishes to express his cordial thanks to Professor Philip M. Morse and Dr. L. I. Schiff for much helpful advice.

[10] Tuve, Heydenburg and Hafstad, Phys. Rev. **50**, 806 (1936).

[11] *Note added in proof:* It is possible that a resonance effect, such as is discussed by Primakoff, Phys. Rev. **52**, 1000 (1937), will account for part of the discrepancy with experiment, though not with all of it.