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Double Scattering of Polarized X-Rays

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Formulas are derived for calculation of the intensity of twice scattered x-radiation emerging from a scattering body at 90° to an incident beam of arbitrary polarization. Spherical and cylindrical scatterers are treated, and with the latter body the cases of primary radiation incident parallel and normal to the cylinder axis are separately considered. Thomson scattering is assumed but with approximate corrections for energy losses to recoil electrons. Certain aspects of the theory are amenable to experimental testing and have been satisfactorily checked.

I. INTRODUCTION

WHEN irradiated matter gives origin to scattered x-radiation the scattered beams are invariably complex, consisting of singly scattered rays accompanied by radiation which has emerged after two or more successive deviations within the scattering matter. It is evident that the ratio of the power doubly or multiply scattered to that singly scattered will approach zero as the number of electrons in the scatterer approaches unity, but all real observations are conducted far short of this impracticable limit, so a pure singly scattered beam is an idealization.

The effects of multiple scattering upon Compton effect observations have been observed by Hulubei[1] and discussed theoretically by DuMond[2] in a paper to which later reference will be made. Appropriate corrections for such effects were applied to polarization measurements by Compton and Hagenow[3] in 1924 but have been omitted from the discussions of other experimenters in this field both before and since, through oversight or because of the undeveloped state of the theory of such effects.

The present paper presents chiefly the derivation of formulas for calculating the intensity of twice scattered radiation emerging from a scatterer at right angles to an incident beam of any degree of polarization. Spherical and cylindrical scatterers have been treated and the latter case has been carried out for two differing orientations of the cylinder relative to the incident beam. The selection of ninety degree scattering and of these particular geometric forms has been dictated jointly by the necessities for mathematically tractable expressions and for theoretical conditions applicable to polarization experiments. At the same time it is deemed that the specialization of conditions has not been such as to destroy the pertinence of these results to scattering investigations of other types and even to some problems of the medical radiologist.

[1] H. Hulubei, Comptes rendus **195**, 1249 (1932); Ann. de physique **1**, 5 (1934).
[2] J. W. M. DuMond, Phys. Rev. **36**, 1685 (1930).
[3] A. H. Compton and C. F. Hagenow, J. O. S. A. and R. S. I. **8**, 487 (1924).

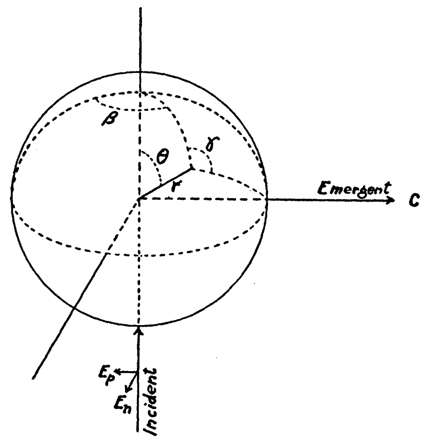

FIG. 1. Coordinate system as related to directions of incident and emergent rays with spherical scatterer.

II. Theory of the Spherical Scatterer

Central primary electron

We consider first a spherical coordinate system with radius r, colatitude θ and longitude β whose origin coincides with the center of a spherical scatterer of radius L and linear absorption coefficient μ. A single electron at the origin is illuminated by plane polarized radiation propagated along the polar axis of the coordinate system in the direction $\theta=0$. The electric vector lies in the origin plane of β, and the electric intensity at the central primary electron will be designated by E_p. Subsequently the complementary polarized component, E_n, will be discussed. In either case we shall be concerned with the radiation emerging from the scatterer after being primarily scattered in a random direction by the original electron and secondarily scattered in a direction parallel to the line $\theta=\pi/2$, $\beta=\pi$ by an electron of the scattering sphere having the random position coordinates r, θ, β. The situation is illustrated in Fig. 1. In an analysis similar to that first presented by DuMond[2] we shall evaluate the electric intensity of the doubly scattered beam from two electrons at a point C distant r_3 from the origin in the designated direction of second scattering, r_3 being large as compared to L. The expression obtained will be integrated over the volume of the spherical scatterer to obtain the total double scattered intensity arising from the primary irradiation of a single electron by the given component of the incident radiation. It will be economical to speak of the electrons which scatter incident primary radiation as primary electrons, and those re-scattering the once scattered radiation as secondary electrons.

The assumption of Thomson scattering involves errors at particular angles but these errors go far toward canceling each other in cases where, as at present, integration over all scattering angles is to be performed. The integrated Thomson scattering is in error through failure to take account of the energy of recoil electrons but this defect may be approximately rectified by application of the Breit-Dirac correction factor after rather than before the integration.

E_p may be resolved into a component $E_p \cos\beta$ lying in the plane of θ, the angle of primary scattering, and a component $E_p \sin\beta$ normal to that plane. The primary scattering of these components gives rise to two components of electric intensity in the once scattered ray which at a distance r are, respectively,

$$X=E_p(M/r)\cos\beta\cos\theta,$$
$$Y=E_p(M/r)\sin\beta,$$

where $M\equiv e^2/mc^2$. The electric vector whose components are X and Y is scattered out in the direction of C (Fig. 1) by the electron at r, θ, β, but before evaluating this twice scattered intensity it is desirable to resolve the vector whose components are X and Y in directions conveniently related to the plane of the second scattering angle, θ_2. The resolved components in and normal to the plane of the second scattering angle are, respectively,

$$H=X\cos\gamma+Y\sin\gamma,$$
$$Q=X\sin\gamma-Y\cos\gamma,$$

where γ is now defined as the dihedral angle between the planes of the first and second scattering angles. The scattering of the vectorial components H and Q by the secondary electron produces doubly scattered radiation with electric components

$$R=(HM/r_3)\cos\theta_2,$$
$$S=QM/r_3,$$

and the total *radiation* intensity at C produced by the cooperation of these two electrons is measured by R^2+S^2.

In preparing these expressions for integration we eliminate undesired angles by means of the geometrical relations[4]

$$\cos\theta_2 = -\sin\theta\cos\beta,$$
$$\cos\gamma = \cos\theta\cos\beta/(1-\sin^2\theta\cos^2\beta)^{\frac{1}{2}};$$

and obtain

$$S^2=0,$$
$$R^2=(E_p{}^2M^4/r^2r_3{}^2)\sin^2\theta\cos^2\beta(1-\sin^2\theta\cos^2\beta).$$

If ρ denote the volume density of electrons in the scatterer we have as the total radiation intensity at C due to scattering of E_p by the central electron and re-scattering by all other electrons of the sphere

$$\rho\int_0^L\int_0^\pi\int_0^{2\pi}(R^2+S^2)r^2\sin\theta\, dr\, d\theta\, d\beta = 8\pi\rho LM^4E_p{}^2/15r_3{}^2. \quad (1)$$

The corresponding analysis for the other incident component, E_n, follows the above procedure so closely that it may be summarized without discussion by the following parallel set of equations. The resolved components of E_n lying in and normal to the plane of θ are respectively $E_n\sin\beta$ and $E_n\cos\beta$.

$$X=E_n(M/r)\sin\beta\cos\theta,$$
$$Y=E_n(M/r)\cos\beta,$$
$$H=X\cos\gamma-Y\sin\gamma,$$
$$Q=X\sin\gamma+Y\cos\gamma,$$
$$R=(HM/r_3)\cos\theta_2,$$
$$S=(QM/r_3),$$
$$R^2=(E_n{}^2M^4/r^2r_3{}^2)(\sin^6\theta\sin^2\beta\cos^2\beta/\sec^2\beta-\sin^2\theta),$$
$$S^2=(E_n{}^2M^4/r^2r_3{}^2)\cos^2\theta/(1-\sin^2\theta\cos^2\beta).$$

Total radiation intensity at C due to scattering of E_n by the central electron and rescattering by all other electrons of the sphere is

$$\rho\int_0^L\int_0^\pi\int_0^{2\pi}(R^2+S^2)r^2\sin\theta\, dr\, d\theta\, d\beta = 12\pi\rho LM^4E_n{}^2/5r_3{}^2. \quad (2)$$

The *single* scattering arriving at C is due exclusively to E_n, and a comparison of (1) and (2) shows that the predominance of E_n persists in the *double* scattering also, since this component contributes $4\frac{1}{2}$ times as much as does E_p. The expressions (1) and (2) enable the calculation of the double scattering from an unpolarized or partly polarized beam since in scattering problems such beams may be represented by two mutually incoherent and perpendicular plane polarized components of suitable intensity.[5] This extension is directly applicable only in case either the maximum or minimum electric vector of the partially polarized beam lie in the plane AOC, but in other cases the computation may be achieved with only very slight inaccuracy in the following way.

The expressions (1) and (2) may be regarded as statements of the double scattered intensity issuing from a single plane polarized vector in the end-on and broadside directions, so to speak. No considerable error can result from an assumption that for intermediate directions the double scattered intensity distribution is elliptical. This assumption takes the place of a complicated and perhaps impossible series of integrations and makes the polarization orientation of the incident beam a matter of indifference since the double scattered contributions resulting from its components may readily be separately computed under the elliptical assumption.

For the special case of unpolarized incident radiation, the case with which DuMond[2] was concerned, we put $E_p=E_n$ and add (1) and (2) obtaining for the total double scattered radiation intensity $44\pi\rho LE_n{}^2M^4/15r_3{}^2$. This may be compared to the *single* scattered intensity, which is entirely due to E_n and has the magnitude $E_n{}^2M^2/r_3{}^2$. The ratio of double to single scattering is then $(44/15)\pi\rho LM^2$, which is identical with DuMond's Eq. (29) when the latter is evaluated for 90° scattering.

Effect of absorption

The ratio of double to single scattering as just given requires correction for the effects of absorption in the scatterer. The proportional effect of absorption of the *entering* ray upon the intensity measured at C is approximately the same whether the final intensity results from single scattering or from double, and this is also true of the emergence segment. In the double scattering case, however, the ray between the

[4] The derivation of relations of this kind is carried out in more detail and with illustrative figures in DuMond's paper, reference 2.

[5] See for example Kirkpatrick, Phys. Rev. **29**, 632 (1927).

two electrons suffers from an absorption which has no counterpart in the single scattering case, and this absorption we now determine. Evidently it suffices to multiply the values of R^2 and S^2 by $e^{-\mu r}$ before integration, an operation which attaches a factor $(1-e^{-\mu L})/\mu L$ to the right members of each of the equations (1) and (2). These amended expressions will be used in the next section.

Eccentric primary electrons

Up to this point only the central electron of the sphere has been considered illuminated by the primary radiation. The irradiation of any other single electron would produce a doubly scattered beam not profoundly different from that just calculated but necessarily somewhat weaker because of the increased distances between primary and secondary electrons. To a good approximation the intensities of the doubly scattered beams resulting from the primary irradiation of differently located electrons within the scatterer will be proportional respectively to $\sum 1/p^2$, where p is the distance from the primarily irradiated electron to any other electron, the summation extending over the entire scatterer. Relative to the central electron we have readily $\sum 1/p^2 \equiv \sum 1/r^2 = 4\pi\rho L$, while for an electron situated at $r=s$ it may be shown that

$$\sum 1/p^2 = 2\pi\rho L + [\pi\rho(L^2-s^2)/s] \log (L+s/L-s).$$

Then if J represents the double scattered intensity resulting from the irradiation of a single central electron (as formulated in Eqs. (1) and (2)) we have for the intensity doubly scattered as a result of the primary irradiation of a volume element dV at r the following expression:

$$J\rho/4L[2L+(L^2+r^2/r) \log (L+r/L-r)]dV.$$

Integration of this expression over the sphere yields $\pi\rho JL^3$ as the total doubly scattered intensity when the entire sphere is bathed in primary radiation. Since the total number of electrons present is $(4/3)\pi\rho L^3$ it appears that the average yield of doubly scattered radiation per primary electron is $\frac{3}{4}J$, a conclusion which was also reached by DuMond[2] by a different argument.

The double scattering which results from irradiation of a single eccentric primary electron is not only weaker than that deriving from the central primary but it possesses a different distribution in space. However, nonmathematical inspection and a certain amount of numerical checking indicates that symmetrically disposed primary electrons tend to balance off each other's individual asymmetries and produce in cooperation spatial distributions of doubly scattered radiation differing but slightly from that of the central primary.

From the present section and the two preceding we have the conclusion that the ratio of doubly to singly scattered intensity when x-rays of any degree of polarization are scattered through 90° by a spherical scatterer is

$$(\pi\rho M^2/5\mu)(1-e^{-\mu L})(2P+9), \tag{3}$$

where $P=(E_p/E_n)^2$ is a measure of the polarization of the incident radiation.

This conclusion and others above concerning intensity find their best application to cases wherein the scattering is predominantly unmodified. For modified scattering the intensity expressions should be multiplied before integration by the Breit-Dirac factor $(1+(h\nu/mc^2)\text{ vers. }\phi)^{-3}$, where ϕ is the angle of scattering, but such treatment renders the integrands so discouragingly complicated that we have effected a first approximation to this usually small correction by applying the factor to the integrals instead of to the integrands, putting $\phi=90°$. The Breit-Dirac factor applies twice to double scattering expressions and only once in the case of single scattering, so the net effect upon (3) is to multiply it by $(1+\alpha)^{-3}$, where $\alpha=h\nu/mc^2$.

III. Theory of the Cylindrical Scatterer

Side irradiation

Figure 2 sufficiently illustrates the system of cylindrical coordinates appropriate to this case and the orientation of the incident and emergent rays in this system. The axis of the scatterer coincides with the axis of the coordinate system but it is not required that the scatterer be situated symmetrically with respect to the origin or even that it include the origin. Let the radius of the scatterer be L and the y coordinates of its ends be Y_2 and Y_1. Considering first E_p, the

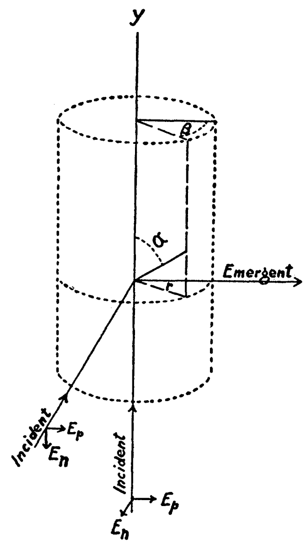

FIG. 2. Coordinate system as related to directions of incident and emergent rays with cylindrical scatterer.

electric intensity of that component of the incident beam which lies in a plane containing the emergent ray, we regard it as being scattered by a single electron at the origin of the coordinate system and the resulting radiation as being rescattered in the emergent direction by an electron at r, y, β. The distance separating these electrons is $r_2=(r^2+y^2)^{\frac{1}{2}}$. The plane containing the y axis and the second electron will be called, for brevity, the β plane. θ_2 is the second scattering angle and γ is the dihedral angle between the β plane and the plane of θ_2. The meaning of α is apparent from Fig. 2.

E_p is decomposed into a component $E_p \cos\beta$ parallel to the β plane and a component $E_p \sin\beta$ perpendicular to the β plane. That the plane containing these components is not normal to the direction of propagation need cause no concern.

Proceeding as in the case of the sphere but neglecting absorption we have the following results.

Electric intensity at secondary electron:

Component in β plane $\equiv X$
$$=E_p(M/r_2)\cos\beta\cos\alpha,$$
Component normal to β plane $\equiv Y$
$$=E_p(M/r_2)\sin\beta,$$
Component in second scattering plane $\equiv H$
$$=Y\sin\gamma-X\cos\gamma,$$
Component normal to 2nd scat plane $\equiv Q$
$$=Y\cos\gamma+X\sin\gamma.$$

Angular relations:

$$\cos\theta_2=\sin\alpha\cos\beta,$$
$$\cos\gamma=(-\cos\alpha\cos\beta)/(1-\sin^2\alpha\cos^2\beta)^{\frac{1}{2}}.$$

Electric intensity components at distance r_3:

In second scattering plane; $R=(HM/r_3)\cos\theta_2$
Normal to second scattering plane;
$$S=(QM/r_3)=0.$$

We have therefore

$$R^2=(M^4E_p^2/r_2^2r_3^2)\sin^2\alpha\cos^2\beta(1-\sin^2\alpha\cos^2\beta),$$

in which r_2 may be replaced through a relation given above and $\sin^2\alpha=r^2/(r^2+y^2)$. Using these substitutions we obtain the total doubly scattered radiation intensity resulting from the incidence of E_p at a distance r_3 by integrating ρR^2 over the volume of the cylinder.

$$\rho\int_0^L\int_{Y_1}^{Y_2}\int_0^{2\pi}R^2r\,dr\,dy\,d\beta$$
$$=\frac{\pi\rho E_p^2M^4}{8r_3^2}\left[Y_2\log\left(1+\frac{L^2}{Y_2^2}\right)-Y_1\log\left(1+\frac{L^2}{Y_1^2}\right)\right.$$
$$+\frac{7}{4}L\tan^{-1}\frac{Y_2}{L}-\frac{7}{4}L\tan^{-1}\frac{Y_1}{L}$$
$$\left.+\frac{3L^2Y_2}{4(L^2+Y_2^2)}-\frac{3L^2Y_1}{4(L^2+Y_1^2)}\right].\quad(4)$$

Inasmuch as the application of formulas deduced in this paper must often be to scatterers which do not precisely conform to the ideal geometry assumed in theory it is comforting to note that the predicted double scattering for a sphere and for an equivalent cylinder are only very slightly different. If (4) be evaluated for a cylinder whose length is equal to its diameter with a centrally located primary electron the resulting doubly scattered intensity differs by less than one percent from the prediction of (1) for a sphere of the same volume. It seems safe to suppose that the double scattering from a cube of the same volume would not differ importantly from these results.

The component E_n of the primary beam lies in the β plane and has an electric intensity $E_n(M/r_2)\sin\alpha$. Continuing the use of symbols employed above we have

$$H=E_n(M/r_2)\sin\alpha\cos\gamma,\qquad R=(HM/r_3)\cos\theta_2,$$
$$Q=E_n(M/r_2)\sin\alpha\sin\gamma,\qquad S=QM/r_3,$$

$$\rho\int(R^2+S^2)dV=(M^4E_n{}^2\rho/r_3{}^2)\int_0^L\int_{Y_1}^{Y_2}\int_0^{2\pi}\sin^2\alpha$$
$$\times(1-\cos^2\alpha\cos^2\beta)(r/r_2{}^2)dr\,d\beta\,dy$$
$$=\frac{\pi\rho M^4E_n{}^2}{r_3{}^2}\bigg[Y_2\log\left(1+\frac{L^2}{Y_2{}^2}\right)-Y_1\log\left(1+\frac{L^2}{Y_1{}^2}\right)$$
$$+\frac{7}{8}L\tan^{-1}\frac{Y_2}{L}-\frac{7}{8}L\tan^{-1}\frac{Y_1}{L}-\frac{L^2Y_2}{8(L^2+Y_2{}^2)}+\frac{L^2Y_1}{8(L^2+Y_1{}^2)}\bigg].\quad(5)$$

It has not been found possible to extend (4) and (5) to the practical case of primary irradiation of all electrons in the cylinder by the devices which served in the treatment of the spherical scatterer. However, extension to an axial column of primaries is achieved without difficulty by integration of (4) and (5) with respect to y and yields for radiation of arbitrary polarization the result:

$$\frac{\pi^2\rho^2(\Delta r)^2M^4E_n{}^2}{16r_3{}^2}\Bigg\{\bigg[(D-b)^2\log\left(1+\frac{L^2}{(D-b)^2}\right)-(U-b)^2\log\left(1+\frac{L^2}{(U-b)^2}\right)$$
$$-(D-a)^2\log\left(1+\frac{L^2}{(D-a)^2}\right)+(U-a)^2\log\left(1+\frac{L^2}{(U-a)^2}\right)\bigg](8+P)\quad(6)$$
$$+\frac{7L}{2}\bigg[(D-b)\tan^{-1}\frac{D-b}{L}-(U-b)\tan^{-1}\frac{U-b}{L}-(D-a)\tan^{-1}\frac{D-a}{L}+(U-a)\tan^{-1}\frac{U-a}{L}\bigg](4+P)\Bigg\}.$$

This expresses the doubly scattered radiation intensity at a point distant r_3 from the scatterer for 90° deviation of a beam of radiation of polarization $P=(E_p/E_n)^2$. The primary electrons occupy a cylinder of radius Δr and axial length from $y=a$ to $y=b$. The secondary electrons occupy a cylinder of radius L and axial length from $y=D$ to $y=U$.

For the important special case in which the cylinder of primary electrons is axially coextensive with the secondary column (6) becomes

$$\frac{\pi^2\rho^2(\Delta r)^2M^4E_n{}^2}{16r_3{}^2}\bigg[2h^2(8+P)\log\left(1+\frac{L^2}{h^2}\right)+7hL(4+P)\tan^{-1}\frac{h}{L}\bigg],\quad(7)$$

in which $h=U-D$, the length of the cylinder.

If we put $\Delta r=L$ in (6) and (7) these equations express upper limit values of double scattering when the entire cross section of the cylinder receives primary radiation. The errors of these values, consequent upon this drastic extension of a differential quantity, depend greatly upon the dimensions of the cylinder but are typically of the order of several percent.

In some applications it is desired to know the intensity of 90° double scattering from two bodies, one of which alone is illuminated by the primary radiation. The singly scattered radiation from this body impinges upon the other, which thereby becomes a source of doubly scattered radiation. A point of interest concerning this kind of double scattering is that it can be observed and measured, whereas in the other cases discussed there is an overwhelming adulteration of singly scattered radiation.

Let the two scattering bodies be two short coaxial cylinders or disks separated by a distance y and having a common radius L and a common thickness Δy. That part of the double scattering which results from the primary action of a single central electron of the first disk is given by (4) and (5). Following the argument advanced in Section II we now make the approximately correct assertion that the ratio of the total double scattered intensity to the intensity given by (4) and/or (5) is the ratio of $\sum\sum 1/p^2$ to $\sum 1/p_0{}^2$. The single summation signifies the sum of the inverse squares of distances from the central primary electron to the several secondary electrons, and has the value $\pi\rho \log(1+b)\Delta y$, where $b=L^2/y^2$. The double summation sums the inverse squares of *all* interelectronic distances separating a primary from a secondary electron.

Except when $b\gg 1$ it is found that

$$\Sigma\Sigma\frac{1}{p^2}\Big/\Sigma\frac{1}{p_0{}^2}=\pi\rho y^2\,dy\,f(b)/\log(1+b), \quad (8)$$

where

$$f(b)=b\log(1+b)+\frac{b(2b+1)}{(b+1)^2}\log\frac{1+(4b+1)^{\frac{1}{2}}}{2}-\frac{b+1}{4}+\frac{(5b+2)(4b+1)^{\frac{1}{2}}}{6(b+1)^2}-\frac{1+8b+5b^2}{12(b+1)}-\frac{b^2}{6(b+1)^2}.$$

A calculation of doubly scattered intensity by Eqs. (6), (7), and (8) is required in Section IV.

Axial irradiation

Consider the scatterer of Fig. 2 to receive radiation propagated in a direction parallel to the axis of the cylinder. That component of electric intensity (E_p) which lies in the direction of the emergent scattered rays will, according to the Thomson scattering formula, produce scattering effects identical with those produced by the E_p component of a primary beam incident normally to the axis of the cylinder. This case has been considered and the results, for a single axial primary electron, embodied in Eq. (4).

The other component of the axial incident beam (E_n in Fig. 2) may be resolved into a component $E_n \sin\beta$ in the plane of the first scattering angle and a component $E_n \cos\beta$ normal to that plane. Succeeding steps yield the following results:

Electric intensity components at secondary electron:

In first scattering plane,
$$X=E_n(M/r_2)\sin\beta\cos\theta,$$
Normal to first scattering plane,
$$Y=E_n(M/r_2)\cos\beta,$$
In second scattering plane,
$$H=X\cos\gamma+Y\sin\gamma,$$
Normal to second scattering plane,
$$Q=X\sin\gamma-Y\cos\gamma.$$

Angular relations for eliminating θ_2 and γ:

$$\cos\theta_2=\sin\theta\cos\beta,$$
$$\cos\gamma=-\cos\theta\cos\beta/(1-\sin^2\theta\cos^2\beta)^{\frac{1}{2}}.$$

Electric intensity components at distance r_3:

In second scattering plane,
$$R=(HM/r_3)\cos\theta_2,$$
Normal to second scattering plane,
$$S=(M/r_3)Q.$$

R^2 and S^2 have identically the forms given in Eq. (1) for the sphere, but the integrals are entirely different since the formulation of the volume element introduces a different function of the variables. Upon integrating we have

$$\rho\int_0^L\int_{Y_1}^{Y_2}\int_0^{2\pi}(R^2+S^2)r\,dr\,dy\,d\beta$$
$$=\frac{\pi\rho E_n{}^2M^4}{8r_3{}^2}\bigg[3Y_2\log\left(1+\frac{L^2}{Y_2{}^2}\right)-3Y_1\log\left(1+\frac{L^2}{Y_1{}^2}\right)+\frac{45}{4}L\tan^{-1}\frac{Y_2}{L}-\frac{45}{4}L\tan^{-1}\frac{Y_1}{L}+\frac{L^2Y_2}{4(L^2+Y_2{}^2)}-\frac{L^2Y_1}{4(L^2+Y_1{}^2)}\bigg]. \quad (9)$$

This is the total doubly scattered radiation intensity at a distance r_3 from the scatterer resulting from the irradiation of a single electron at the origin by the primary component E_n. As in (4) the longitudinal extent of the scatterer is from $y=Y_1$ to $y=Y_2$.

When Eqs. (4) and (9) are each multiplied by $\pi\rho(\Delta r)^2dy$ and integrated with respect to y between the limits given, the combination of the two results gives the radiation intensity of 90° double scattering of axially incident radiation whose polarization is $P=E_p{}^2/E_n{}^2$. The primary electrons in this case occupy a cylinder of radius Δr and length from $y=a$ to $y=b$ and the secondary electrons occupy a cylinder of radius L and length from $y=D$ to $y=U$. This intensity is

$$\frac{\pi^2\rho^2(\Delta r)^2M^4E_n{}^2}{8r_3{}^2}\{\tfrac{1}{2}(P+3)\varphi_1+(L/4)(7P+45)\varphi_2+4L^2\log\varphi_3\},$$

where φ_1, φ_2, and φ_3 are functions of the dimensions of the scattering volumes defined by

$$\varphi_1=(D-b)^2\log\left(1+\frac{L^2}{(D-b)^2}\right)-(U-b)^2\log\left(1+\frac{L^2}{(U-b)^2}\right)$$
$$-(D-a)^2\log\left(1+\frac{L^2}{(D-a)^2}\right)+(U-a)^2\log\left(1+\frac{L^2}{(U-a)^2}\right),$$
$$\varphi_2=(D-b)\tan^{-1}\frac{D-b}{L}-(U-b)\tan^{-1}\frac{U-b}{L}-(D-a)\tan^{-1}\frac{D-a}{L}+(U-a)\tan^{-1}\frac{U-a}{L},$$
$$\varphi_3=\frac{(L^2+(U-b)^2)(L^2+(D-a)^2)}{(L^2+(U-a)^2)(L^2+(D-b)^2)}.$$

This general result of course includes the simple case in which the primary and secondary volumes are coextensive.

Previous remarks about the use of the approximation $\Delta r=L$ are in effect here.

IV. Measurement of Double Scattering in a Cylinder

In view of the considerable number of approximations used in the foregoing derivations, any possible experimental confirmation will be valuable. Satisfactory general methods for making such measurements are not known but important parts of the foregoing theory have been checked by observing the double scattering in a paraffin cylinder 5.5 cm long and 1.75 cm in radius, using the arrangement shown in Fig. 3. Effectively unpolarized x-rays from a tungsten target with cathode stream at 45° to the plane of scattering passed through an absorbing grid which permitted the irradiation of uniformly spaced layers of the cylinder. A similar grid interposed between the scatterer and an ionization chamber passed to the chamber only scattered rays which had originated in *unirradiated* layers of the scatterer and which must therefore have arisen from plural scattering. Intensities thus scattered were compared with the total singly and multiply scattered intensities observed upon removal of the grids.

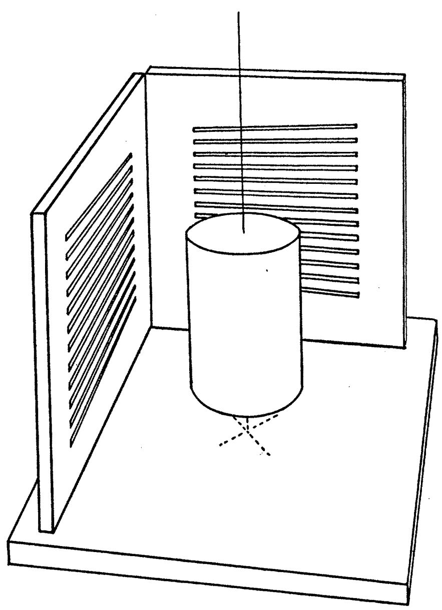

Fig. 3. Cylindrical scatterer in position relative to incidence and emergence grids for measurement of double scattering.

Theory

The primary scattering at 90° is due solely to the E_n component of the incident rays and has, to a first approximation, the Thomsonian intensity $E_n^2M^2\rho V/r_3^2$, where V is the volume of the scatterer. This requires correction for absorption which is roughly achieved by multiplication by $e^{-\mu L}$, and correction for energy lost to recoil electrons, accomplished by application of the factor $J+1/[J+(1+\alpha)^3]$ where J is the ratio of unmodified to modified scattered intensity for the adopted wave-length, scattering material and angle of scattering.

Since the absorption and recoil corrections vary oppositely with variation of incident wave-length their product is not very sensitive to such variation and so neither monochromatization nor precise knowledge of wave-lengths employed is necessary. The wave-length 0.3A has been assumed and the linear absorption coefficient in paraffin taken to be 0.19 per cm.

The radiation observed without the grids contains also a doubly scattered ingredient which may be calculated from Eq. (7) by putting $r=L$, $P=1$, and $h/L=3.17$. Correction factors for absorption and recoil losses are obtained by squaring those applied in the single scattering case. This is equivalent to assuming that the effective path in the scatterer is $2L$ and that 90° is a proper effective scattering angle to represent both primary and secondary scattering for the purposes of this correction.

The radiation observed with the grids is calculated as explained in Section III for the case of double scattering by co-axial disks. For each primarily irradiated section as many as six emergence slots pass appreciable doubly scattered radiation. All such calculated contributions, corrected as in the preceding paragraph, are combined to give the total measurable radiation. Traces of higher order scattering will be present in the observed radiation both with and without the grids but no calculation of such intensities has been made.

Results

The calculated ratio of observable intensities without and with the grids, carried through as outlined above, is 47. The ratio as observed was 44 at an x-ray tube potential of 40 kv, 46 to 50 kv, and 44 at 60 kv, the agreement being somewhat closer than might reasonably have been expected.

The methods of calculation outlined in this paper will subsequently be utilized for the correction of conclusions now in print concerning the polarization of primary x-rays, and for the interpretation of experiments in progress. The author is greatly indebted to Mr. Keith Harworth for assistance in the experimental part of this investigation.

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Preliminary Analysis of the First Spark Spectrum of Cerium—Ce II

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By applying the spectral interval sorter and a newly designed interval recorder to new data obtained with the automatic recording spectrum comparator on the M.I.T.-W.P.A. wavelength program, a preliminary term array for Ce II has been set up in which 584 lines have been accounted for as transitions between 31 lower and 51 upper states. The term diagram is found to be the most complex yet observed for a three-electron spectrum. Both configurations $4f5d6s$ and $4f5d^2$ appear to be low in Ce II, contrary to the analysis of Haspas. Most of the differences between observed wave numbers and those computed from the term array are less than 0.02 cm^{-1}, and 60 percent of the lines are found to be consistent to within 0.002A. Several of the term assignments have been checked with partially resolved Zeeman patterns recorded by King and Albertson, and the absolute J values have been determined by this means. An inclusive description of the spectra of the cerium atom is being undertaken in the range 10,000 to 1000A.

THAT the spectra emitted by rare earth atoms would be unusually difficult to analyze has been expected by spectroscopists for some time, but fortunately a number of these atoms whose spectra are so complex emit outstanding groups of lines. By attacking such lines the beginnings of term arrays have been constructed in a number of cases.[1] Cerium (58), the first element of the rare earth group, which is of unusual interest spectroscopically because of its position in the periodic table, presents no such suggestive features for attack. When we made

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