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Molecular Viewpoints in Nuclear Structure

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A study of the transition between the H_2^+ molecule and the corresponding three-body nucleus shows (1) that it is useful to regard the system as composed of two parts (heavy particle; heavy and light particles) between which acts an effective potential; (2) that this potential depends more and more on the relative velocity of the two parts as the masses of the light and heavy particles approach equality; (3) that the wave equation and the potential for the relative motion are obtained in a consistent manner by requiring that a certain form of approximate wave function give the best possible representation of the motion of the three particles, in the sense of the variation principle. This wave function represents a state in which the system resonates between the groupings atom-ion and ion-atom. It is adapted to the treatment of the scattering of neutrons in deuterium and is also used in the text to calculate the binding energy of H^3. Application of the same type of approximate wave function to the description of nuclear structure in general, gives rise to the concept of resonating group structure. This picture regards the constituent neutrons and protons as divided into various groups (such as alpha-particles) which are continually being broken and reformed in new ways. Group theory gives information as to which groupings are most important in describing a particular state of a given nucleus. The interchange of neutrons and protons between the groups is rapid. It is largely responsible for the intergroup forces, but also prevents one from attributing any well-defined individuality to the groups except as follows: If the time required for a particle to diffuse between two parts of the nucleus vibrating in opposite phase (in the language of the liquid droplet model) is large in comparison with the period of the vibration, then the particles of the nucleus may be divided into groups which preserve their identity long enough to make possible a simple description of the nuclear motion in terms of the relative displacements of these clusters. Arguments are given to show that the diffusion condition is satisfied for low excitation energies. When the nuclear energy is higher, the groups have significance only in providing a suitable mathematical scheme to treat the nuclear motion (see following paper). Allowed types of motion and energies for low states of Be^8, C^{12}, and O^{16} are calculated in terms of the relative motion of alpha-particle groups, using the methods familiar in molecular structure. The modes of vibration are closely related to those given by the liquid model of Bohr and Kalckar, but many low levels are excluded on symmetry grounds. The general methods outlined here for the description of nuclear structure are to a large extent independent of the nature of the forces between elementary particles. A discussion of the possible existence of many-body forces is given (i.e., forces which cannot be described by a potential that is a sum of potentials involving two particles at a time). The observed variation of nuclear binding energy with atomic number is found not to give sufficient evidence from which to draw any general conclusion. Electron positron theory indicates that a part of the nuclear forces consists of many-body interactions.

UNDERSTANDING of certain features of nuclear constitution has been gained by applying, with proper limitations, concepts native to atomic structure. To these ideas we wish in the following to add some points of view more closely related to molecular structure. If atomic structure be characterized by a central force dominating the motion of almost independent particles, and nuclear constitution, by those collective types of motion which Bohr and Kalckar[2] liken to the modes of vibration and rotation of a liquid droplet, then the feature which distinguishes molecular structure from these is its division of the constituent particles into more or less well-defined groups, between which it is a good approximation to say that inter- "atomic" forces act. It is the usefulness, and limitations, of this concept of group structure that we wish to study in connection with the mechanical description of the atomic nucleus.

THE THREE-BODY PROBLEM

We consider in illustration first the three-body problem (Fig. 1), because here the differences between the treatments of atom, nucleus, and molecule are not accentuated sufficiently that the similarities escape notice. In both the atomic and the nuclear systems, there is no one particle which provides a dominating center of force; and

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[1] I am indebted to the University of North Carolina for leave of absence, December 15, 1936 to March 15, 1937, to the Institute for Advanced Study, where part of this work was done, for its hospitality, and to the other members of the 1937 Washington Conference on Theoretical Physics for discussion of some of the results.

[2] N. Bohr and F. Kalckar, Kgl. Dansk Viden. Selskab, in press.

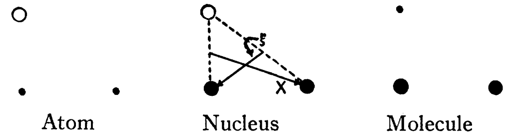

FIG. 1.

introduction of a time average field to represent the action of two particles on the third is not satisfactory because the two speeds involved are comparable. These difficulties, which distinguish the helium atom among other atomic systems, are the same ones experienced in approaching the description of nuclear constitution in general by the Hartree central field approximation.

In the molecular three-body problem, on the other hand, the rapidity of the electronic motion in comparison with the displacement of the heavy particles is known to allow an approximate separation[3] of the system into (1) electronic motion with fixed force centers and (2) slow nuclear motion in the time average field determined by (1). In contrast to the situation in the helium problem, we have here a relatively simple basis from which to approach the nuclear three-body system.

As the mass of the light particle is gradually increased in the transition from molecule to nucleus, two new features reveal themselves. First, the molecular potential acts not from nucleus to nucleus but from the center of gravity of a pair consisting of a heavy and a light particle to the other heavy particle. Second, the velocity of approach of these two parts of the system has an influence on the interaction which grows as the mass of the light particle is increased and its speed becomes more comparable with that of the intermolecular motion.

We shall find it possible to bring both of the above facts into evidence by dividing the original molecular potential into two parts:

V, an ordinary potential; and

K, a velocity dependent potential, represented for quantum-mechanical purposes by an integral operator.[4] In addition, we introduce

$\mathbf{X}$, a vector with components X, Y, Z, representing the separation between one heavy particle (which, not stated!) and the center of gravity of the other two particles;

$\boldsymbol{\xi}=(\xi, \eta, \zeta)$, simply a vector variable of integration, which however is shown below to be interpretable as the separation between the second heavy particle and the center of gravity of the other two particles (see Fig. 1); and

μ_{12}, the reduced mass associated with the relative motion of the two groups of particles composing the system:

$$\mu_{12}^{-1}=M^{-1}+(M+m)^{-1}.$$

Then the equation we use to describe the intergroup motion is

$$(\hbar^2/2\mu_{12})\nabla^2\psi(\mathbf{X})+E\psi(\mathbf{X})=V(\mathbf{X})\psi(\mathbf{X})+\int K(\mathbf{X},\boldsymbol{\xi})\psi(\boldsymbol{\xi})d\boldsymbol{\xi} \quad (1)$$

(an equation which is derived independently in Appendix I). Rapid motion means many nodes in ψ and a diminished contribution from the integral; thus the velocity dependence of the potential. Integration over $\boldsymbol{\xi}$, $\mathbf{X}$ remaining constant, is seen from the figure to represent a type of averaging over the different internal configurations of a pair consisting of a heavy and a light particle. In the limiting case where the mass of the light particle approaches zero, the average difference between $\boldsymbol{\xi}$ and $-\mathbf{X}$ becomes smaller and smaller. Thus, for fixed value of $\mathbf{X}$, the function $K(\mathbf{X}, \boldsymbol{\xi})$ is different from zero in a more and more limited range of values of $\boldsymbol{\xi}$ near the point $\boldsymbol{\xi}=-\mathbf{X}$, and eventually becomes representable as a delta-function:

$$K(\mathbf{X}, \boldsymbol{\xi})\rightarrow W(\mathbf{X})\delta(\mathbf{X}+\boldsymbol{\xi}).$$

In this limit, the molecular potential is now $V(\mathbf{X})+W(\mathbf{X})$; as in the customary treatment of molecular binding, it shows no dependence on velocity. The very deep narrow "trough" presented by the interaction kernel $K(\mathbf{X}, \boldsymbol{\xi})$ in the molecular case, widens out as the mass of the light particle is increased. In the nuclear limit, the half-width of K as a function of $\boldsymbol{\xi}+\mathbf{X}$, so far as this concept is definable, is of the same order as the average distance between the heavy particles.

The relation of molecular ideas to the nuclear three-body problem, brought out by the introduction of velocity dependent forces, is more

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[3] To the order $(m/M)^{\frac{1}{4}}$, m and M being electronic and nuclear masses. M. Born and J. R. Oppenheimer, Ann. d. Physik **84**, 457 (1927).

[4] A discussion of the quantum mechanics of velocity dependent forces is given in Phys. Rev. **50**, 643 (1936). Cf. also J. H. Van Vleck, Phys. Rev. **48**, 367 (1935), and appendix of Ostrofsky, Breit, Johnson, Phys. Rev. **49**, 196 (1936). A distinction must be made between possible velocity dependence of the forces between the fundamental particles themselves, and the variation of intergroup forces with velocity, of which we speak here.

easily visualized with the help of the concept of resonance. Just as the general diatomic molecule is said to resonate between a condition of ionic binding and one of atomic binding, so the three particles in the nucleus may be regarded as spending part of their time in the grouping (13)2 and part in the configuration (23)1. In the corresponding mathematical treatment, the nuclear wave function, $\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, is approximated by a linear combination of terms associated with the two groupings:

$$\Psi=\psi\left(\mathbf{r}_2-\frac{M\mathbf{r}_1+m\mathbf{r}_3}{M+m}\right)\Phi(\mathbf{r}_3-\mathbf{r}_1) \pm\psi\left(\mathbf{r}_1-\frac{M\mathbf{r}_2+m\mathbf{r}_3}{M+m}\right)\Phi(\mathbf{r}_3-\mathbf{r}_2). \quad (2)$$

The first product represents light particle 3 bound to heavy particle 1, and heavy particle 2 interacting with the center of gravity of 1 and 3. The sign joining the two products depends upon the kind of statistics obeyed by the heavy particles. As a starting point for the treatment of the molecular and nuclear three-body problem, Eq. (2) is independent of special assumptions as to the nature of the binding forces.

The concept of resonating group structure leads directly to the introduction of velocity dependent forces. If we require that a wave function of the form (2) give the best possible representation of the nuclear motion, in the sense of the variation principle, then we find indeed exactly the wave equation (1), containing a potential, part of which is expressed in the form of an integral operator. The proof of this fact and the explicit calculation of the interaction between the two groups of the system is carried out in Appendix I, where the relationship to the Heitler-London procedure is also traced.

Three features characterize the method just described for treating nuclear structure. First, it supplies a rational basis for the customary type of variational calculation of nuclear binding energies. The calculation in the appendix, for example, gives -6.4 mMU (milli-mass-units) for the binding energy of H^3, in contrast to the value -5.6 mMU obtained with the same force constants but a two parameter variational function of the form $\exp[-\nu(3-1)^2-\nu(3-2)^2-\mu(2-1)^2]$. Second, the form of the wave function is adapted to the description of excited states of the nucleus as well as its ground level; in particular, collision processes, involving states in the continuous spectrun, fall under this classification. Third, the method has the disadvantage of not explicitly taking into account the effect of polarization of one group by the other. To be sure, we could introduce into the internal group wave function Φ of (2) a dependence on the position of the third particle, but the mathematical treatment would become unduly complicated (see discussion at end of following paper). On the other hand, for certain states of the total system, the two terms of (2) are already correlated in such a way as partially to allow for the action of polarization.

Resonating Group Structure

The method of resonating group structure shows a new characteristic when applied to the nuclear four-body system. In contrast to the preceding example, which shows only a single type of grouping, the alpha-particle may be divided into two deuterons, triton and neutron, and He^3 plus proton. The wave function is therefore expressed as a linear combination of six terms, of which the first corresponds to one possible grouping into deuterons, and the second is its mate, with sign chosen as in (2) so the two form a combination antisymmetrical in neutrons and in protons; the other four terms are divided into two similar pairs, associated with the second and third types of grouping. The relative importance of each term may be regarded as determined by the fraction of the time the whole system remains in the corresponding configuration. Three simultaneous wave equations are now needed to determine the relative motion of the groups in the three different configurations. These equations may be used to describe both disintegration processes and elastic scattering. For the detailed calculations for the nuclear four-body problem, reference is made to the work of Parker.[5]

The general mathematical details of applying the method of resonating group structure to the description of the motion in a nucleus containing

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[5] H. Parker, Phys. Rev. **51**, 683 (1937) and M. A. thesis, University of North Carolina (1937).

many neutrons and protons are given in the following paper. Below, however, we are interested in the question: Which of the unmanageable number of possible types of grouping into which a large nucleus may be divided are most suited to describing approximately the actual state of the system?

Characteristic of the heavy nucleus is the close binding of the constituent particles. Viewed classically, the motion of a typical particle is a rapid zigzag motion superimposed on a general random drift. The velocity of the motion of general drift is so low that it contributes little to the total energy. The rapid motion with respect to nearest neighbors, however, represents the same type of exchange phenomena which is responsible for the binding between the two groups in the three- and four-body problems. The local nature of the binding in a heavy nucleus can for our purposes be best described by a division into groups of contiguous particles, between which act intergroup velocity dependent forces.

Let us pick out of all the particles in the nucleus those neutrons whose spins at a certain instant[5a] are all $+\frac{1}{2}$. The Pauli principle assures us that the wave function of the nucleus is antisymmetrical in these particles and hence has a node wherever two of these neutrons approach each other. To make the kinetic energy of the system as low as possible, the various nodes space themselves as far apart as is consistent with considerations of potential energy. Close to each of the neutrons just mentioned, a neutron with opposite spin finds a position of lowest potential energy, and similarly protons of both spins tend to follow the structural plan of the original neutrons. This follows most directly from our present information on nuclear forces: Unlike particles, and like particles of opposite spin, attract each other when the wave function is symmetric with respect to interchange of their coordinates, but not when it is antisymmetric. The configuration of the nucleus which we have just described, represents a grouping into alpha-particles, but alpha-particles of continually changing individuality. The interchange of neutrons and protons between the groupings, in fact, furnishes the bonds that tie the groups together.

The usefulness of the group structure picture for a description of nuclear vibration and rotation, along molecular lines, depends on the extent to which the groups remain well defined during the period of one vibration. For this reason, application of the method of resonating group structure to the nuclear three-body problem, although as suitable there as the idea of resonance is in the case of molecular structure, is not to be justified on the basis of any clear-cut physical division of the system into deuteron and neutron, any more than the molecular treatment can be founded on a sharp distinction between atomic and ionic binding. On the other hand, division of a large nucleus into a few groups, each containing many particles, represents an extreme case where the identities of the clusters remain relatively sharp even after several periods of vibration. It is just this fact that Bohr has recognized in his division of nuclear excitation into rotational energy and energy of volume and surface tension vibrations. We shall see that many of these modes of excitation find a simple representation in terms of the motion of alpha-particle groups.

Alpha-Particle Groups

Before describing any mode of nuclear motion in terms of the displacements of alpha-particle groupings, we require that the following two conditions be satisfied: (a) the half wave-length of the disturbance must be larger than the group, and (b), the period, τ, of vibration must be shorter than the average interval, t, required for a moving alpha-particle to exchange a neutron or proton with a part of its surroundings moving in the opposite phase. The quantity t is a property of the mode of vibration in question, and may be termed "characteristic diffusion time:" it is in order of magnitude the time required for a particle to diffuse a distance, ρ, of one-half wavelength: $\rho=\lambda/2$.

There appears to be no method by which to calculate the characteristic diffusion time quantum mechanically without entering into the details of the problem. An estimate on the basis of the classical diffusion theory is limited in its validity to energies much higher than those in

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[5a] In mathematical terms, we suppose the nuclear wave function to be known as a function of spins and coordinates; we give the spins of the various particles fixed values and then investigate the dependence of the wave function on coordinates.

which we are interested; namely, the following conditions must be satisfied: (1) the uncertainty in the position of the particle due to its finite wave-length must be less than the effective mean free path, λ_d, for diffusion; (2) the average momentum transfer per collision must be larger than that uncertainty in the momentum of the particle which is conditioned by our knowledge of its position; (3) potential barriers surrounding the given particle, due to interaction with its neighbors, must be low in comparison with the available energy of the particle. In spite of its limitations, such an estimate is instructive. The classical equation gives

$$\rho^2=(\lambda_d v/3)t \tag{3}$$

for the mean square distance of diffusion in terms of the average speed, v, of the particle. We insert $\rho=\lambda/2$, where λ is of the order $2\pi R/n$ for the nth characteristic mode of surface tension vibration (R=nuclear radius), and using $\lambda_d\sim 2\times 10^{-13}$ cm, $M_p v^2/2\sim 25$ Mev, $R\sim 2\times 10^{-13}A^{\frac{1}{3}}$ cm, we obtain

$$t\sim n^{-2}A^{\frac{2}{3}}\times 10^{-21}\text{ sec.} \tag{4}$$

The period of the surface oscillation, on the other hand, is given by[6]

$$\begin{aligned}\tau&=2\pi/\omega,\\ \omega^2&=n(n-1)(n+2)(G/S)(4\pi/3M_pA),\end{aligned} \tag{5}$$

where G/S is the surface tension and A the atomic number of the nucleus. Using for G/S the value estimated[7] from the empirical variation of binding energy with atomic number, we find

$$\tau\sim n^{-\frac{3}{2}}A^{\frac{1}{2}}2\times 10^{-21}\text{ sec.}, \tag{6}$$

which is in general considerably larger than the classical estimate of the characteristic diffusion time.

In the above classical estimate of the diffusion time the interaction between the particle in question and its neighbors was brought into evidence only through the assumption of an effective mean free path between collisions. But when the average energy of the particle is small in comparison with the height of the surrounding potential barriers due to nearby particle, there is only a small probability that the particle will, by fluctuations in the distribution of energy, acquire the amount required to surmount the ridge. Eyring[8] has shown that diffusion coefficients can be calculated by the same methods which are used in treating other absolute reaction rates. The probability that the required amount of energy be concentrated on one particle is represented in Eyring's formula by the factor $e^{-\epsilon/kT}$, where ϵ is an activation energy associated with the height of the barrier. Allowance must be made also for the possibility that the particle will penetrate through the potential hill. Eyring has shown that, even in ordinary liquids (diffusion of D_2O in H_2O, for example) the value of the activation energy has an important influence on the rate of the diffusion; and it is dominating, of course, for a solid body.

In the nucleus, the local potential barriers encountered by a typical particle may be estimated very roughly to be of the order of 30 Mev. The typical particle has a zero-point energy of possibly 20 Mev and in addition a fluctuating fraction of the nuclear excitation energy. It is essentially only this latter energy which is available for concentration, by chance, on the particle in whose diffusion we are interested. We would estimate, then an activation energy of roughly 10 Mev. A more plausible way of stating this is as follows: the binding energy per particle is 10 Mev; the saturation character of nuclear forces means that the given particle is bound as tightly to its close neighbors as to the whole of the nucleus; hence an energy of the above order of magnitude is required to overcome their attraction.

The nuclear excitation in ordinary transmutation experiments is ordinarily never enough to raise the nuclear temperature, kT, above 2 Mev. As the activation energy appears to be considerably larger than this amount, we have to expect a very strong dependence of the rate of diffusion upon excitation energy. When the nuclear excitation is low the simple classical treatment for this reason undoubtedly overestimates the rate of diffusion (cf. Eq. (4)) very considerably. It seems safe to say that actually the characteristic time of diffusion, t, is then definitely smaller that the period of nuclear vibration, τ, (Eq. (6)) and that the condition is satisfied for applying the simple

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[6] Following Lord Rayleigh, *Scientific Papers*, Vol. I (Cambridge, 1899), Eq. (40), p. 401. I am indebted to Professor Bohr for reference to this paper.

[7] Cf. for example H. Bethe, Rev. Mod. Phys. **9**, 69 (1937), Eq. (314).

[8] H. Eyring, J. Chem. Phys. **4**, 283 (1936).

group structure picture of the nucleus mentioned above. With higher and higher nuclear excitation, on the other hand, the influence of diffusion becomes progressively stronger, and predictions based on a simple group structure treatment gradually lose close correspondence with the actual properties of the nucleus.

In accordance with the preceding considerations on the special stability of alpha-particle groupings and on the influence of diffusion, we shall use a simple alpha-particle model on molecular lines to describe low states of excitation of the nucleus. It is necessary to distinguish between such a model, which must assume the forces between alpha-particles to be given beforehand, and a direct application of the method of resonating group structure, which starts out from first principles, and only introduces the alpha-particle concept in setting up a suitable approximate wave function for the nucleus. The two treatments are, of course, closely related.

The wave function of the whole system has the proper antisymmetry when written in the form

$$\Psi(1,\cdots N)=\sum_{\text{dist}} F\left(\frac{1+2+3+4}{4},\ \frac{5+6+7+8}{4},\ \cdots\right)\Phi(1234)\Phi(5678)\cdots \quad (7)$$

Here Φ is the wave function for a single alpha-particle, completely antisymmetrical[9] with respect to interchange of any two particles; $F(\mathbf{X}_{\text{I}}, \mathbf{X}_{\text{II}}, \cdots)$ is a symmetric function of the coordinates of the centers of gravity of the various α-particles $(4\mathbf{X}_{\text{I}}=\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4=1+2+3+4)$; and the sum runs over all those even permutations of the N particles which give really different distributions into alpha-particles. It is in F that we are interested; it describes the rotation and vibration of the framework of alpha-particle groupings. We write F as the product $\psi_R\psi_V$ of a rotational and vibrational wave function, and apply the results of molecular theory to determine the allowed rotational and vibrational quantum numbers for the nuclei Be^8, C^{12}, and O^{16}.

TABLE I. *Number of allowed vibration-rotation levels for three alpha-particles, determined from symmetry characters of* ψ_R *and* ψ_V.

$n_1=0,1,2,\cdots;\ n_2=$			0	1	2	3	4
J	K	Character	α	γ, δ	α, γ, δ	$\alpha, \beta, \gamma, \delta$	$\alpha, 2(\gamma, \delta)$
0	0	α	1	0	1	1	1
1	1	γ, δ	0	1	1	1	2
1	0	β	0	0	0	1	0
2	2	γ, δ	0	1	1	1	2
2	1	γ, δ	0	1	1	1	2
2	0	α	1	0	1	1	1
3	3	α, β	1	0	1	2	1
3	2	γ, δ	0	1	1	1	2
3	1	γ, δ	0	1	1	1	2
3	0	β	0	0	0	1	0

LOW LEVELS OF BERYLLIUM, CARBON AND OXYGEN[9a]

A. Allowed states

a. Two alpha-particles.—The rotational energy is given by the well-known expression

$$E_R=(\hbar^2/2A)J(J+1). \quad (8)$$

Here the moment of inertia, A, is somewhat greater than the moment, I, of a spherical nucleus of the same density and mass: $A=I/g$ defines a dimensionless number $g<1$. ψ_R is symmetric for even J, antisymmetric for odd J; ψ_V is symmetric for all vibrational quantum menbers; hence only levels with $J=0, 2, 4, \cdots$ are allowed. As a matter of fact, owing to the stability of Be^8, only the lowest vibrational level has a life long enough to be observable. It seems likely that rotational levels, too, above $J=4$ are too much widened by dissociation to be of interest. We therefore consider only the three levels

$$E=E_0,\ E_0+6g(\hbar^2/2I),\ E_0+20g(\hbar^2/2I). \quad (J=0, 2, 4). \quad (9)$$

b. Three alpha-particles.—Neglecting for the moment the influence of vibration on rotation, we consider the centers of gravity of the alpha-particles located at the vertices of an equilateral triangle. The moments of inertia about the axis of symmetry and about a line in the plane of the triangle we write, respectively, $C=I/g_2$ and $A=I/g_1$, where I is the moment of inertia which

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[9] For simplicity, we treat neutrons and protons as different spin states of the same particle. See Heisenberg, *Rapport du VII^me Congres Solvay* (Paris, 1934), for proof of equivalence to the treatment in which two different types of particles are used.

[9a] Considerations on the influence of alpha-particle symmetry upon allowed nuclear levels had occurred to Professor E. Teller independently, and were mentioned to the author in conversation.

TABLE II. *Symmetry characters of wave functions belonging to the vibrations ω_2 and ω_3.*

n_2 \ n_3	0	1	2
0	A_1 (1)	F_2 (3)	A_1+E+F_2 (6)
1	E (2)	F_1+F_2 (6)	$A_1+A_2+2E+F_1+F_2$ (12)
2	A_1+E (3)	F_1+2F_2 (9)	$2A_1+A_2+3E+F_1+2F_2$ (18)
3	A_1+A_2+E (4)	$2F_1+2F_2$ (12)	$2A_1+2A_2+4E+2F_1+2F_2$ (24)

would belong to a spherical nucleus of the same density and mass. g_2 and g_1 depend on the distribution of mass; for an ellipsoid of revolution with major axis ρ times the minor axis, $g_2=\rho^{-2/3}$, $g_1=2/(\rho^{2/3}+\rho^{-4/3})$. The rotational energy of the system is

$$E_R=(\hbar^2/2I)[g_1J(J+1)-(g_1-g_2)K^2], \quad (10)$$

where K is the quantized projection of the total angular momentum, J, along the symmetry axis.

Two types of vibration are possible: a doubly degenerate vibration (ω_2), in which one side of the triangle shrinks and at the same time moves away from the third alpha-particle; and a dilatation vibration (ω_1) in which the triangle expands and contracts isotropically. The latter vibration is symmetric in the three alpha-particles. The symmetry character of the wave function of the system therefore does not depend on the vibrational quantum number n_1. On the other hand, associated with any value of n_2, there are n_2+1 different states having the vibrational energy $(n_2+1)\hbar\omega_2$. For even values of n_2, one of these states[10] is symmetric (α) and the others divide up into $n_2/2$ pairs of states having doubly degenerate symmetry character (γ, δ). For odd values of n_2 greater than 2, we have in addition a completely antisymmetric state (symmetry character β).

We combine a given vibrational state with a rotational state of the symmetry character necessary to make the total wave function completely symmetrical in the three particles.[11] This has been done in Table I. Example: For any value of n_1, and for $n_2=4$, the rotational state $J=1$, $K=0$ is not allowed; but two states $J=1$, $K=1$ exist. (Each splits into $2J+1=7$ states in a magnetic field!) These two states have the same energy until we allow for the coupling of vibration and rotation. Neglecting this and similar refinements, we have

$$E=E_0+E_R(J,K)+n_1\hbar\omega_1+n_2\hbar\omega_2 \quad (11)$$

for the energy of C^{12}.

c. Four alpha-particles.—The system has tetrahedral symmetry and the following types of vibration occur: a single vibration (ω_1), corresponding to isotropic dilation of the nucleus; a doubly degenerate vibration (ω_2) in which the alpha-particles are paired into two dumbbells twisting with respect to each other; and a triply degenerate vibration (ω_3) in which one dumbbell shortens, the other lengthens, but no tipping occurs. In addition there exists the possibility that one alpha-particle will push through or around the other three and invert the pyramid. This motion is not associated with a new degree of freedom but there is a finite probability for it to occur when the amplitudes of the above nine vibrations are as large as they are in the nucleus. Splittings, $\Delta E=2\epsilon$, will occur in the energy level system proportional to the frequency with which such inversions occur.

The wave function representing a particular one of the three modes of vibration belongs to one of the five representations[12] of the tetrahedral rotation reflection group:

A_1; function unchanged in sign by any rotation or reflection.

A_2; only reflection at a symmetry plane changes the sign of the function.

E; the function is paired with another; neither is altered by rotation about a twofold axis; other operations transform the two functions among each other.

F_1; function paired with two others; transform among each other for all operations of the group; character of reflection in a plane drawn through two alpha-particles is -1.

F_2; like F_1, except the reflection has character $+1$.

TABLE III. *Allowed states for four alpha-particles.*

$n_1=0,1,\cdots;\ (n_2,n_3)=$		(0,0)		(0,1)		(0,2)		(1,0)		(1,1)		(2,0)		(2,1)		(3,0)	
J	Character R	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
0	A	0	1	0	0	0	1	0	0	0	0	0	1	0	0	1	1
1	F	0	0	0	1	0	1	0	0	1	1	0	0	1	2	0	0
2	$E+F$	0	0	0	1	1	2	1	1	1	1	1	1	1	2	1	1
3	$A+2F$	0	1	0	2	0	3	0	0	2	2	0	1	2	4	1	1
4	$A+E+2F$	0	1	0	2	1	4	1	1	2	2	1	2	2	4	2	2
5	$E+3F$	0	0	0	3	1	4	1	1	3	3	1	1	3	6	1	1

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[10] Here and below we follow the terminology and use the results of Hund, Zeits. f. Physik **43**, 805 (1927), presented in convenient form by Dennison, Rev. Mod. Phys. **3**, 340 (1931).

[11] The combination rules are given in simple form by Dennison, reference 10.

[12] We use here the notation and results of L. Tisza on the symmetry of the vibrations of polyatomic molecules, Zeits. f. Physik **82**, 67 (1932).

The character of the vibration ω_1 is A_1 (symmetric) for all values of n_1, and thus has no influence on the symmetry of the total wave function. Table II gives the types of vibrational wave functions occurring for any given state of excitation of ω_2 and ω_3; thus, for $n_1=n_1$, $n_2=0$, $n_3=2$, we have a single symmetric vibrational state, a doubly degenerate state, and a triply degenerate state of type F_2, making 6 different wave functions altogether, provided we consider only small amplitudes of displacement. But for large amplitudes, inversion occurs, and a wave function will or will not change sign according as it belongs to the representation A_1 or A_2 of the inversion operation. A function of the latter type possesses one additional node and therefore has an energy slightly higher than one of the type A_2. Recalling that $A_2A_1=A_2$, $A_2E=E$, $A_2F_2=F_1$, we therefore actually obtain for $n_1=n_1$, $n_2=0$, $n_3=2$ six energy levels, consisting of three close pairs: A_1, A_2; E, E; F_2, F_1, the first member of the pair having in each case the lowest energy.

We now combine the vibrational wave function with the rotational function to obtain a state completely symmetric in the four alpha-particles. Table III gives the number of such allowed states for any value of n_1 and the tabulated values of n_2, n_3, and the rotational quantum number J. Example: For $n_1=n_1$, $n_2=0$, $n_3=1$, $J=2$, we have one allowed level (which splits into $2J+1$ in a magnetic field); this level is symmetric with respect to inversion of the tetrahedron, and therefore has an energy slightly less (cf. minus sign in table) than it would othereise possess. Neglecting interaction of rotation with vibration, we have for the energy of 0^{16} the following expression:

$$E=E_0+(\hbar^2/2I)J(J+1)+n_1\hbar\omega_1+n_2\hbar\omega_2+n_3\hbar\omega_3\pm\epsilon. \quad (12)$$

If, in the example, we had had $n_2=1$, $n_3=0$, we should have had the plus sign as well as the minus sign occurring in the energy expression (ϵ proportional to frequency of inversion).

B. Relation to the treatment by Bohr and Kalckar

a. Types of motion.—Bohr and Kalckar[13] divide the nuclear motion into rotation and volume and surface vibrations of the nuclear material. They calculate the frequency of these vibrations as for a fluid substance. The legitimacy of this treatment depends upon the extent to which diffusion can be neglected during the period of one vibration, which condition is also essential to the considerations in the preceding section.

The modes of vibration of the fluid model bear a close relation to those discussed in A from the alpha-particle point of view, but characteristic differences are present due to the clustering which we have attributed to the nucleus. For example, the first surface tension oscillation, $n=2$, represents a motion in which the nucleus shrinks at the equator and expands at top and bottom. The degeneracy is $2n+1$ or 5-fold, corresponding to the different possible orientations of the pole of the motion. On transition to the alpha-particle picture, say for O^{16}, three of these vibrations go over into the first excited level ($n_2=1$) of ω_3, in which the tetrahedron shrinks about a line which is normal to both a twofold and a threefold symmetry axis, and at the same time expands along this line. The other two surface tension oscillations go over to the level $n_1=1$ of ω_2, in which the flattening of the nucleus is brought about by the twisting of the two dumbbells towards the same plane (see Fig. 3). Similar considerations hold for other levels. It is to be expected that the actual position of the low nuclear levels which are observed will lie between these two extremes.

Once the connection is established between the types of motion characteristic of the liquid and the alpha-particle models, it becomes clear that a great many of the levels predicted by the former treatment have been excluded on symmetry grounds in the latter. For example, when the nucleus O^{16} is in its normal vibrational state, all values of the rotational quantum number, J, are allowed according to the homogeneous drop model, whilst we find that the permitted levels are limited to $J=0, 3, 4, 6, \cdots$. Even those states that occur have a weight much lower than predicted by the semi-classical model ($2J+1=7$ for $J=3$ as compared with $(2J+1)^2=49$ for the same value of J). The physical reality of the restrictions which symmetry puts on the energy level scheme depends upon the extent to which

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[13] N. Bohr and F. Kalckar, reference 2. I am indebted to Professor Bohr and Mr. Kalckar for valuable discussions of the liquid model of the nucleus.

the identities of the alpha-particles are preserved during the course of the motion. The higher the energy level, the greater is the amount of diffusion which occurs, during one period of vibration, between parts of the system moving in opposite phase, and the more the symmetry restrictions break down. The influence of symmetry considerations on the energy level system, it now becomes clear, is in actual fact only one of degree rather than one of kind.[14] The types of low levels which were excluded on the alpha-particle picture are to be found in the energy spectrum, but pushed upwards by some millions of volts from the position predicted by the treatment in terms of a homogeneous classical liquid. Those modes of motion which are forbidden on the alpha-particle picture can attain the required antisymmetry in neutrons and protons by modifications in the motion equivalent to excitation of one of the alpha-particles or to other divisions of the neutrons and protons into clusters (i.e., 5+3+4+4 instead of 4+4+4+4). The excitation energy associated with such changes is large in proportion as the alpha-particle type of grouping is well defined. On the other hand, for volume and surface tension oscillations of progressively shorter wave-length, the definition of the group structure in the nucleus becomes less and less distinct under the influence of diffusion of neutrons and protons between parts of the nucleus moving in opposite phase.

To summarize, we may divide the nuclear energy level spectrum into two parts. In the first, extending from the ground level up to an excitation of perhaps four to eight million volts, the effect of the nonhomogeneous structure of the nucleus is to modify the types of motion predicted by the liquid model and in addition to push to higher energies many of the levels which on the classical model would be low lying,[15] but which cannot be described quantum mechanically as being symmetric in terms of alpha-particles or similar stable structures. In the second region of the spectrum, the group structure becomes more and more washed out and the liquid model should give a progressively more correct picture of the nuclear excitation. We are, in fact, approaching the region of high quantum numbers where, according to the correspondence principle, the quantum-mechanical description approaches the classical treatment, and the predictions of the liquid model become valid to the extent that any system of classical particles can be replaced by a continuum.

TABLE IV. *Normal vibrations of* Be^8, C^{12}, *and* O^{16}, *and constants of rotational energy, using "large radius" for nucleus.* (*Energies in thousands of a mass unit.*)

NUCLEUS	$\hbar\omega_1$	$\hbar\omega_2$	$\hbar\omega_3$	$\hbar\omega_s$	$\hbar^2/2I$
Be^8	(5.2)	—	—	1.9	0.41
C^{12}	2.9_2	2.0_6	—	1.5_4	0.21
O^{16}	2.3_0	1.1_5	1.6_3	1.3_3	0.13

b. Vibration frequencies and other constants.—Comparison with the theory of Bohr and Kalckar now gives a control with which to check the simplest assumption we can make regarding the various vibrations of the alpha-particle model. In default of a more satisfactory treatment (cf., however, Section E below) we assume, namely, that a reasonable approximation is obtained by calculating the characteristic frequencies as if quasi-elastic central forces acted between the alpha-particles. (In accordance with the saturation character of nuclear binding, we do not assume that these "equivalent forces" between alpha-particles are the same within different nuclei.) Using $V(\mathbf{r})=V(\mathbf{r}_0)+k(\mathbf{r}-\mathbf{r}_0)^2/2$ for the effective potential between alpha-particles, we readily find for the vibrations of C^{12},

$$\begin{aligned} \omega_1'(\text{dilatation}) &= 3^{\frac{1}{2}}(k'/4M), \\ \omega_2'(\text{tipping}) &= (\tfrac{3}{2})^{\frac{1}{2}}(k'/4M)^{\frac{1}{2}}; \end{aligned} \tag{13}$$

and for O^{16},

$$\begin{aligned} \omega_1''(\text{dilatation}) &= 2(k''/4M)^{\frac{1}{2}}, \\ \omega_2''(\text{twisting}) &= (k''/4M)^{\frac{1}{2}}, \\ \omega_3''(\text{flattening}) &= 2^{\frac{1}{2}}(k''/4M)^{\frac{1}{2}}, \end{aligned} \tag{14}$$

where M is the proton mass. We obtain an order of magnitude estimate for k by equating $(k/2)(R/2)^2$(R=nuclear radius) to the energy required to break an alpha-alpha bond. Allowing for the share of the zero-point kinetic energy[16]

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[14] This is not true for the statistical weights, however.

[15] It is the intention in a later paper with E. Teller to extend these considerations to the low energy levels of heavy nuclei, in connection with the problem of nuclei having the same atomic number and atomic weight but different properties (such as distinct half-lives).

[16] In view of the roughness of potential approximation, it does not seem justifiable to attempt to make a more detailed estimate, allowing for the difference in the zero point energies of the nuclei A and $A-4$ and for the different state of internal binding of a free and a bound alpha-particle.

belonging to one alpha-particle, we obtain

$$\begin{aligned}&2(k'R'^2/8)-\hbar(\omega_1'+2\omega_2')/6\\&\qquad\sim 8.0080+4.0040-12.0043(\text{MU}),\\&3(k''R''^2/8)-\hbar(\omega_1''+2\omega_2''+3\omega_3'')/8\\&\qquad\sim 12.0043+4.0040-16.0000(\text{MU}),\end{aligned}\tag{15}$$

whence, with $R'=4.1\times 10^{-13}$, $R''=5.2\times 10^{-13}$,[17] we find the vibrational excitation energies given in Table IV. For comparison, we give the energy, $\hbar\omega_s$, of the lowest surface-tension vibration ($n=2$) of the liquid model, as calculated from Eq. (5), using for the surface tension the value estimated from the empirical variation of nuclear binding energy with atomic number (Bethe, reference 7, Eq. (314)). As was to be expected from our considerations on the transition between the liquid and the alpha-particle model, ω_s for oxygen lies between ω_2 and ω_3. The favorable comparison indicates a general agreement between two quite different methods of estimating the nuclear surface tension.

We obtain the moment of inertia of the nucleus of atomic weight A from $I=(2/5)MAR^2$. Using units $e^2/mc^2=2.80\times 10^{-13}$ cm for length and mMU for energy, we have

$$\hbar^2/M=5.56,$$

and

$$\begin{aligned}\hbar/2I&=(5/4)5.56\,(2.80\times 10^{-13}/2.05\times 10^{-13})^2A^{-5/3}\\&=13.0A^{-5/3}\ \text{mMU (milli-mass-units)}\end{aligned}\tag{16}$$

as given in Table IV. A rough estimate of the flattening of C^{12} is long diameter = 1.5 times short axis, whence the form factors $g_1=1.06$, $g_2=0.76$ (Eq. (10)). Similarly, we estimate for Be^8 long axis = 1.5 times short diameter, $g=0.80$ (Eq. (9)). The uncertainty in the g's is unimportant compared to that of I: a 30 percent decrease of the nuclear radius from $R=2.05\times 10^{-13}A^{1/3}$ increases $\hbar^2/2I$ by a factor of two.

c. Dissociation.—The mean square displacement of a harmonic oscillator of circular frequency ω and equivalent mass m, when in the nth state of vibration, is known to be given by

$$(x^2)_{\text{Av}}=(n+\tfrac{1}{2})(\hbar^2/m)(\hbar\omega)^{-1}.\tag{17}$$

If we assume a quasi-elastic character for the alpha-particle oscillations of C^{12}, we find for the mean square displacement of one alpha-particle from equilibrium the expression

$$(X^2)_{\text{Av}}=(2n_1+1)(0.282)^2+(2n_2+1)(0.336)^2\tag{18}$$

(units $e^2/mc^2=2.80\times 10^{-13}$ cm for distance). The frequencies ω_1 and ω_2 are incommensurable, however, and when the two vibrations happen to be in phase with respect to the radial motion of one alpha-particle, we shall have

$$(\delta R^2)^{\frac{1}{2}}{}_{\text{Av}}=(2n_1+1)^{\frac{1}{2}}0.282+(2n_2+1)^{\frac{1}{2}}0.336.\tag{19}$$

Energy enough for disintegration being available in the nucleus, the probability for the emission of an alpha-particle will be of the order of magnitude of the number of times per second that δR exceeds the range of nuclear attraction, provided that the energy is still not so high that diffusion completely destroys the identity of the alpha-particle during the period of vibration. In particular, this condition requires that the available energy of the system be small in comparison with the amount required for the release of a neutron or proton.

Although the range of validity of the simple alpha-particle description of disintegration is much limited for nuclei of light and medium atomic weight by the energy restrictions above, nevertheless the resemblance to the evaporation concept used in the liquid model is apparent. Thus, on both pictures, emission of a particle from the nucleus occurs when by chance the amounts of kinetic energy associated with the various modes of collective motion of the nucleus have become concentrated on one particle to the extent required to break its binding to the nucleus.

For heavy nuclei, the conditions are improved with respect to the applicability of the alpha-particle picture of disintegration, above all when spontaneous disintegration can occur. In the latter case, each alpha-particle already has the energy required for release, and penetration through the potential barrier is the factor which determines the nuclear lifetime. Whether observed disintegration constants can be used in conjunction with the Gamow penetration formula to calculate nuclear radii is, of course, another question, in view of the ambiguity due to the size of the alpha-particle itself. Application of the method of resonating group structure should

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[17] In default of any certain way of deciding between alternative theories of nuclear radii, we use throughout this paper for definiteness the values given by Bethe, Rev. Mod. Phys. 9, 166 (1937). It appears that the most reliable estimate of nuclear radii will be obtained from observations on the spacing of low rotational energy levels.

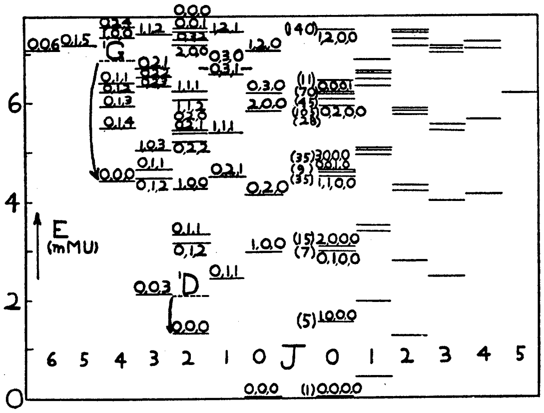

FIG. 2. Low levels predicted for the nucleus C^{12}. Excitation energy in thousands of a mass unit. Jh gives total nuclear angular momentum. At left, the states given by simple alpha-particle model. Labels refer, respectively, to quantum numbers n_1 and n_2 for vibration and K for rotation about symmetry axis of the triangle of alpha-particles. Weight of levels is $2J+1$. Arrows show correspondence with dotted levels calculated by Hartree-Fock procedure (Feenberg, Wigner, Phillips). At right, levels given by the liquid droplet model. Weight given by number in parenthesis times $(2J+1)^2$. Labels refer to degree of excitation of the surface tension vibrations $n=2$, 3, 4, 5, respectively.

make it possible to eliminate this ambiguity by a mathematical treatment which goes beyond the Gamow model and treats the disintegration from first principles (cf. in this connection a following paper on the interaction between two alpha-particles).

It does not seem possible, however, to justify the so-called "many-body" picture[17a] of spontaneous alpha-particle disintegration. In this treatment, the probability of penetration through the potential barrier is multiplied by a factor which is termed "probability for concentration of energy on an alpha-particle," and is identified in order of magnitude with the similar factor encountered in the theory of the disintegration of a nucleus which is normally stable but has been excited. The two cases are actually quite different. In the latter, it is essential that the remainder of the nucleus lose energy if one particle is to have energy enough to escape. We must allow for the time required for the various degrees of freedom to concentrate the energy on the particle in question, by a favorable fluctuation. In the former case, the vibrations associated with the various modes of collective motion in the nucleus are all in their lowest state; the zero point energy is distributed almost uniformly among the various particles; and we do not have to wait for the concentration of energy to occur.

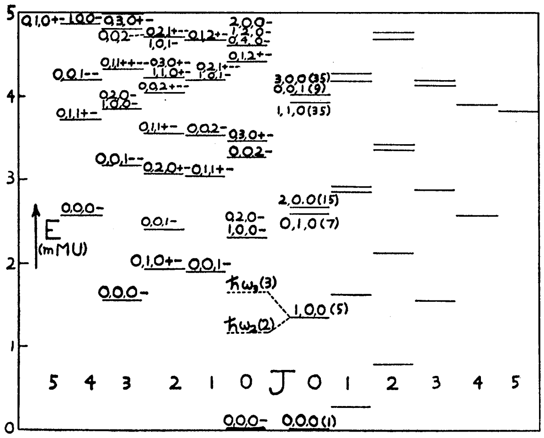

FIG. 3. Low levels of O^{16}. At left, according to simple alpha-particle model; at right, on liquid droplet model. Numbers at left are vibrational quantum numbers for dilatation (ω_1), twisting (ω_2), and flattening (ω_3) of the tetrahedron, respectively. Levels marked with a minus sign are slightly lowered with respect to those marked with a plus sign because the wave function for the minus levels does not change sign on inversion of the tetrahedron. For numbers at right, see Fig. 2. The dotted lines exemplify how a state of the liquid model becomes modified under the influence of a tendency to form alpha-particle groups. In the example, however, the two dotted levels are excluded in the alpha-particle treatment on symmetry grounds, and therefore would also be missing in a refined liquid drop theory. The left portion of the diagram shows similar examples of how profoundly symmetry considerations influence the appearance of the lower part of the nuclear energy level spectrum.

C. Energy level spectrum

a. Beryllium.—With the values of the constants given above, we have $E_R=0.33J(J+1)$ for Be^8. The alpha-particle treatment therefore predicts a normal S level ($J=0$), a D state ($J=2$) at an excitation of 2.0 mMU (milli-mass-units), and a G level ($J=4$) at 6.7 mMU above the ground level. The first excited level has been observed[18] to lie at 3.0 mMU, and to have a width of 0.8 mMU. There also exists evidence[19] for an even broader level at about 10 mMU, which it seems reasonable to attribute to $J=4$. The same normal and excited levels are predicted by Feenberg and Wigner[20] and their energies are calcu-

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[17a] For details, cf. H. Bethe, Phys. Rev. **50**, 977 (1936).

[18] Oliphant, Kempton and Rutherford, Proc. Roy. Soc. **150**, 241 (1935).

[19] See paper on the scattering of alpha-particles in helium, to be published in the *Physical Review.*

[20] E. Feenberg and E. Wigner, Phys. Rev. **51**, 101 (1937); cf. also E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).

lated on the basis of a perturbation treatment using $1s$ and $2p$ harmonic oscillator functions for individual neutrons and protons: $E(G)-E(S)=6.9$ mMU, $E(D)-E(S)=2.1$ mMU. The two theoretical treatments fail by about the same amount. To obtain agreement of the alpha-particle model with experiment, it is necessary to decrease the nuclear radius from $2.05\times10^{-13}A^{\frac{1}{3}}$, used above, to $1.67\times10^{-13}A^{\frac{1}{3}}$ cm.

b. Carbon.—The alpha-particle model in the form used above gives

$$E(\mathrm{C}^{12})=2.92n_1+2.06n_2+0.22J(J+1)-0.06K^2(\mathrm{mMU}). \quad (20)$$

Fig. 2 presents the low allowed energy levels. and shows for comparison the states predicted by the semi-classical droplet model. The effect of symmetry considerations is excluding low levels is very marked. The high lying levels given by the alpha-particle model will certainly be strongly modified by the growing effect of diffusion discussed in *B*. More confidence can be placed in the general feqtures of the spectrum predicted at lower energies. It should be stated that the smaller nuclear radius suggested by the case of Be^8 would increase the rotational energies of C^{12} states by a factor 1.5 and vibrational energies by a factor ~1.2.

In view of the number of excited levels, it appears that a comparison of the spectrum with the three gamma-rays if C^{12} known at present would not yield any useful information. Observations on the fine structure of low levels would be of interest; the separation of levels differing only in the quantum number K depends upon the extent to which the nucleus is flattened by grouping into the alpha-particles.

The dotted levels on the diagram are those predicted by Feenberg and Wigner. Their approximate calculations place the levels at the same position in C^{12} as in Be^8.

c. Oxygen.—The model which we have discussed gives for four alpha-particles

$$E(\mathrm{O}^{16})=2.03n_1+1.15n_2+1.63n_3+0.13J(J+1), \ (\mathrm{mMU}) \quad (21)$$

with uncertainties in the constants similar to those mentioned in the case of carbon. The low lying allowed terms are shown in the left-hand portion of Fig. 3. The dotted lines for $J=0$ indicate the two levels $n_1=0$, $n_2=1$, $n_3=0$ and $n_1=0$, $n_2=0$, $n_3=1$, which are forbidden on symmetry grounds. If symmetry did not exclude them, then on our altering the nuclear forces in such a way as to leave no tendency to grouping into alpha-particles, we should find these two levels merging into the first excited level, $n_{s2}=1$, of the lowest mode, $n=2$, of surface tension oscillation of the liquid model. Other levels of the alpha-particle model show similar correlation properties. Whether observation gives a scheme of vibrational levels corresponding more to the one model or the other will give important information on the extent to which the group structure of the nucleus is affected by diffusion.

D. Relation to Hartree-Fock type of calculation

Wigner[21] has shown that the symmetry character of the total nuclear wave function comes first in determining the energy of the nucleus; next come the finer details of dependence on coordinates and spin. The symmetry character is described by a certain partition of A, the number of particles in the nucleus; the nucleus has its lowest energy for the partition containing the largest possible number of 4's: $A=4+4+\cdots$.

The partition of lowest energy is automatically given by the alpha-particle treatment, which is not adapted as it stands, however, to the description of the highly excited nuclear states belonging to other partitions.

The Hartree-Fock description of the nuclear motion, on the other hand, (1) starts out with an equivalent central field to represent the interaction of a particle with its neighbors; (2) builds up out of single particle states in this field a wave function of the desired symmetry character for the whole system; and (3) by a perturbation calculation attempts to allow for the difference between the equivalent field and the actual field. In principle, such a treatment is no more (or less) complicated for one partition than for another. Carried through to a finish, it would yield a wave-function which recognized the collective types of motion common to the liquid and the alpha-particle models of the nucleus. In point of fact the perturbation calculation just mentioned is either carried out only to a low order of approximation,

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[21] E. Wigner, Proc. Nat. Acad. **22**, 662 (1936).

where it gives a false picture of the nuclear state of motion; or with more labor, is made to yield a better wave function, which is, however, too complicated in form to have its meaning read or its accuracy certified.

The relationship of the Hartree-Fock type of wave function to the alpha-particle description of the nucleus is most easily seen in the case of the ground state of oxygen. We compare the resonating group wave function of Eq. (7) with the zero-order wave function of the central field treatment:

$$\Psi(1,\cdots 16)=\begin{vmatrix} (sa|1) & (sb|1) & (sc|1) & \cdots & (p_1d|1) \\ (sa|2) & & & & \\ \cdots & & & & \\ \cdots & & & & \\ (sa|16) & \cdots & \cdots & \cdots & (p_1d|16) \end{vmatrix}. \tag{22}$$

In the determinant, **2** stands for the spatial, ordinary spin, and isotopic spin coordinates of particle 2 (which may be either a neutron or a proton).[9] The function $(p_1a|1)$ represents a single particle behaving as a proton with spin $+\frac{1}{2}$ and moving in a $2p$ state with $m_l=1$; b, c, d, refer, respectively, to a proton state of spin $-\frac{1}{2}$ and neutron states with spin $+\frac{1}{2}$ and $-\frac{1}{2}$; s indicates the $1s$ state. We add and subtract rows of the determinant without changing its value and obtain

$$\Psi=\begin{vmatrix} (\mathrm{I}a|1) & \cdots & (\mathrm{IV}d|1) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ (\mathrm{I}a|16) & \cdots & (\mathrm{IV}d|16) \end{vmatrix}, \tag{23}$$

where the new states I, II, III, IV exhibit the same type of tetrahedral symmetry (cf. Slater, Pauling, etc.) which characterizes the bond eigenfunctions of a carbon atom:

$$\begin{aligned} (\mathrm{I}|\ )&=2^{-1}\{(s|\ )+(x|\ )+(y|\ )+(z|\ )\}, \\ (\mathrm{II}|\ )&=2^{-1}\{(s|\ )-(x|\ )-(y|\ )+(z|\ )\}, \\ (\mathrm{III}|\ )&=2^{-1}\{(s|\ )+(x|\ )-(y|\ )-(z|\ )\}, \\ (\mathrm{IV}|\ )&=2^{-1}\{(s|\ )-(x|\ )+(y|\ )-(z|\ )\}. \end{aligned} \tag{24}$$

Here the states x, y, z are defined by

$$\begin{aligned} (x|\ )&=2^{-1}\{(p_{-1}|\ )-(p_{+1}|\ ), \\ (y|\ )&=i2^{-1}\{p_{-1}|\ )+(p_{+1}|\ ), \\ (z|\ )&=(p_0|\ ). \end{aligned} \tag{25}$$

In the form (23), the Hartree-Fock zero-order wave function is adapted to recognize the same spatial arrangement which the alpha-particle model attributed to the oxygen nucleus: the form of Ψ indicates that in the region of space near each vertex of a tetrahedron there exists a large probability of finding the building material of an alpha-particle—a pair of protons of opposite spins and a similar pair of neutrons. The wave function does not, however, seize this opportunity actually to construct alpha-particles; the exceedingly strong interaction between the four particles in the same region of space is not recognized by any dependence of the wave function upon the mutual separations of the neutrons and protons and the result is much the same as we should obtain on attempting to describe a helium nucleus in terms of particles moving in a central field. The initial advantage being lost by this neglect, a practicable series of perturbation calculations can regain it only in the case of light nuclei. For example, Inglis[22] has shown that successive approximations to the binding energy of He^4 converge well ($E=-27.6$ mMU in first order, -28.8 mMU in second order), but that for Li^6 higher order contributions fall off less rapidly ($E=-14.9$ mMU in first order, -27.7 mMU in second order). Already in the case of O^{16} a first-order calculation[20] misses the observed binding energy by 100 mMU. That the weakness of the central field procedure really lies in neglecting the alpha-particle type of correlation follows most readily from the fact that internal binding of four free alpha-particles is already 122 mMU, only 16 mMU less than the binding energy of the same neutrons and protons in oxygen.

Description of excited levels of the nucleus by the Hartree-Fock procedure in general requires

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[22] D. R. Inglis, Phys. Rev. **51**, 531 (1937).

the introduction in the wave function of terms representing the excitation of individual neutrons and protons to higher states. The dotted levels in Fig. 2 (carbon), for example, were obtained by considering[20] only 1*s* and 2*p* states of the individual particles in a central field. The next excited single particle states, 2*s* and 3*d*, lie about 10 mMU higher. At the least the ten states just mentioned must be taken into account if the central field calculation is to account for the levels of whose existence and approximate position the alpha-particle model informs us. There does not seem to be any physical argument to justify one in believing that this degree of approximation in the Hartree-Fock treatment will give one generally correct energy level differences, since the difference is so great between the starting point of the calculation and the actual modes of collective motion which characterize the nucleus. For example, the three low energy levels predicted by the central field approximation for Be^8 and C^{12} have been shown (Feenberg and Phillips, reference 20, Eq. (3a)) to satisfy the relation $E=E_0+L(L+1)K'$, which has the same form as the rotational energy given by the liquid model of the nucleus. Nevertheless, the constant K' represents an integral of potential energy rather than the influence of altered kinetic energy; moreover, K' has the same value for Be^8 and C^{12}, although the moment of inertia with which we should expect it to be correlated differs for the two nuclei by a factor between 1.6 and 2.

The lack of any valid picture of the nucleus with which to visualize the results of the Hartree-Fock treatment means unfortunately that one must depend on convergence arguments rather than physical insight to recognize in what cases one can rely on the results of the central-field calculation for binding energies and term differences. This emphasizes the desirability of developing a mathematical procedure more closely related to nuclear models which recognize the collective nature of the nuclear motion.

E. Refinement of the alpha-particle model

The alpha-particle model used in *A* and *B* above must be distinguished from a thoroughgoing application of the method of resonating group structure. The latter treatment takes explicitly into account the interchange of neutrons and protons between the various groupings, and allows us to derive the energies of the various nuclear levels directly from our information about the forces between elementary particles. The former description, of course, tacitly recognizes that such interchanges do go on and that they are in principle responsible for the forces between alpha-particles. Nevertheless, it depends for its validity on the assumption that the diffusion does not go on at such a rate that the alpha-particles altogether lose their identity during one characteristic period of the nuclear motion. We have seen that this assumption is reasonable for low nuclear excitation but grows progressively worse at high energies. It then becomes necessary to adopt the true method of resonating group structure to obtain a satisfactory treatment. It will be a good approximation to limit the groupings we consider to alpha-particles alone, if we are interested in nuclear levels of the symmetry type $4+4+\cdots$; levels of this kind in fact dominate the nuclear spectrum below 10 Mev. We wish now to indicate how our mathematical procedure can be simplified by taking over the results of the alpha-particle model as regards modes of vibration and allowed levels.

a. Analytic expression for alpha-particle wave function.—For definiteness, let us treat that level of C^{12} which in the simple alpha-particle model is described by the quantum numbers $n_1=2$, $n_2=0$, $J=3$, $K=3$, $m_J=3$. The system has six degrees of freedom exclusive of translation; they are associated with the three Eulerian angles φ, θ, χ of a symmetric top, the normal coordinate, ξ, of the dilatation vibration, and the coordinates η and ζ of the doubly degenerate tipping vibration. In terms of our molecular picture, the nuclear level in question would be described by the wave function

$$(\sin^6(\theta/2)e^{-3i\varphi}-\cos^6(\theta/2)e^{3i\varphi})e^{3i\chi}H_2(\xi') \\ \times H_0(\eta')H_0(\zeta')e^{-\frac{1}{2}(\xi'^2+\eta'^2+\zeta'^2)}. \tag{26}$$

Here the H's are the usual Hermitian polynomials encountered in the oscillator problem; ξ' is an abbreviation for $(m_\xi\omega_1/\hbar)^{\frac{1}{2}}\xi$; η', for $(m_\eta\omega_2/\hbar)^{\frac{1}{2}}\eta$ etc., where m_ξ is defined by $m_\xi\dot{\xi}^2$ $=$classical kinetic energy of the first mode of vibration.

In the method of resonating group structure, we write the nuclear wave function as an expression completely antisymmetric in the twelve neutrons and protons:

$$\Psi(1,\cdots,12)=\sum_{\text{perm}} F\left(\frac{1+2+3+4}{4},\cdots,\cdots\right)$$
$$\times\Phi(1\ 2\ 3\ 4)\Phi(5\ 6\ 7\ 8)\Phi(9\ 10\ 11\ 12). \quad (27)$$

The wave functions Φ describing the internal motion of the alpha-particles are taken to be fixed, while $F(\mathbf{X}_{\text{I}}, \mathbf{X}_{\text{II}}, \mathbf{X}_{\text{III}})$ varies from state to state of the system, and corresponds in our problem to the wave function (26). Our procedure is (1) to write down a suitable analytical expression for F, having approximately the same symmetry and transformation properties as (26), but containing certain adjustable parameters; (2) to adjust these parameters to obtain the best possible wave function Ψ in the sense of the variation principle; that is, Ψ shall be orthogonal to all lower states of the same J and m_J value, and at the same time shall minimize the energy of the system:

$$\int\Psi^*H\Psi d\tau\Big/\int\Psi^*\Psi d\tau=E=\text{a minimum}.$$

The function F exactly satisfying the condition of minimum energy (i.e., containing an infinite number of adjustable parameters) is found by solution of a wave equation, which is derived in the following paper and applied to the treatment of collision and disintegration problems. Here, however, F is taken to depend on a relatively small number of parameters, and to have a mathematical form which is at the same time simple enough to make practicable the analytic calculation of the energy of the system, and yet sufficiently flexible to give a good description of the nuclear motion. What is the most convenient form for F depends on the manner in which Φ and the nuclear interactions depend upon distance. We shall assume that the potential between elementary particles varies with distance as $\exp(-b^2\mathbf{r}_{ij}^2)$, and shall use for the alpha-particle wave function, Φ, an exponential expression which allows in the simplest manner for the close interaction between the four constituent particles:

$$\Phi(1\ 2\ 3\ 4)=24^{-\frac{1}{2}}\begin{vmatrix} a(1) & \cdots & a(4) \\ \vdots & & \vdots \\ d(1) & \cdots & d(4)\end{vmatrix} 8(2\alpha/\pi)^{9/4}$$
$$\times\exp -\alpha(\mathbf{r}_{12}^2+\mathbf{r}_{13}^2+\mathbf{r}_{14}^2+\mathbf{r}_{23}^2+\mathbf{r}_{24}^2+\mathbf{r}_{34}^2) \quad (28)$$

where a, b, c, and d are the four spin states defined in D. Then simplest of all choices for F is an e power with the exponent a second degree function of the Cartesian coordinates of the three alpha-particles:

$$\mathbf{F}\sim\exp\left\{-\sum_{i,j=\text{I}}^{\text{III}}\sum_{x}^{z}(ijx)X_iX_j-\sum_{i=\text{I}}^{\text{III}}\sum_{x}^{z}(ix)X_i\right\}. \quad (29)$$

Here the (ix) and (ijx) are numerical parameters; also $(ijx)=(jix)$, etc. The typical term in such an integral as

$$\int\mathbf{\Psi}_m{}^*H\mathbf{\Psi}_n d\tau \quad\text{or}\quad \int\mathbf{\Psi}_m{}^*\mathbf{\Psi}_n d\tau,$$

when expressed in terms of the coordinates of the twelve neutrons and protons, is thus representable as the product of three exponential integrals, associated respectively with the x, y, and z axes. The exponent of the first integral is a function of the Cartesian coordinates $x_1,\cdots,x_{12}$ and depends in addition on the constants α and b^2, the nine parameters $(ijx)_m$ and $(ix)_m$ of $\mathbf{\Psi}_m$, and nine similar parameters belonging to $\mathbf{\Psi}_n$:

$$I_x=\int\exp(-P_x)d\tau,$$

where
$$P_x=\sum_{k,l}^{12}{}_{1}[klx]x_kx_l+\sum_{k=1}^{12}[kx]x_k \quad (30)$$

and the brackets are combinations of the abovementioned parameters. The integration gives at once

$$I_x=\pi^{12/2}\Delta^{-\frac{1}{2}}\exp\Delta^{-1}\sum_{k,l}^{12}{}_{1}[kx][lx]\frac{\partial\Delta}{\partial[klx]}, \quad (31)$$

where
$$\Delta=\begin{vmatrix}[1,1x] & \cdots & [1,12x] \\ \vdots & & \vdots \\ [12,1x] & \cdots & [12,12x]\end{vmatrix}$$

and in the differentiation, $[klx]$ and $[lkx]$ are regarded as independent, being afterwards set equal to each other.

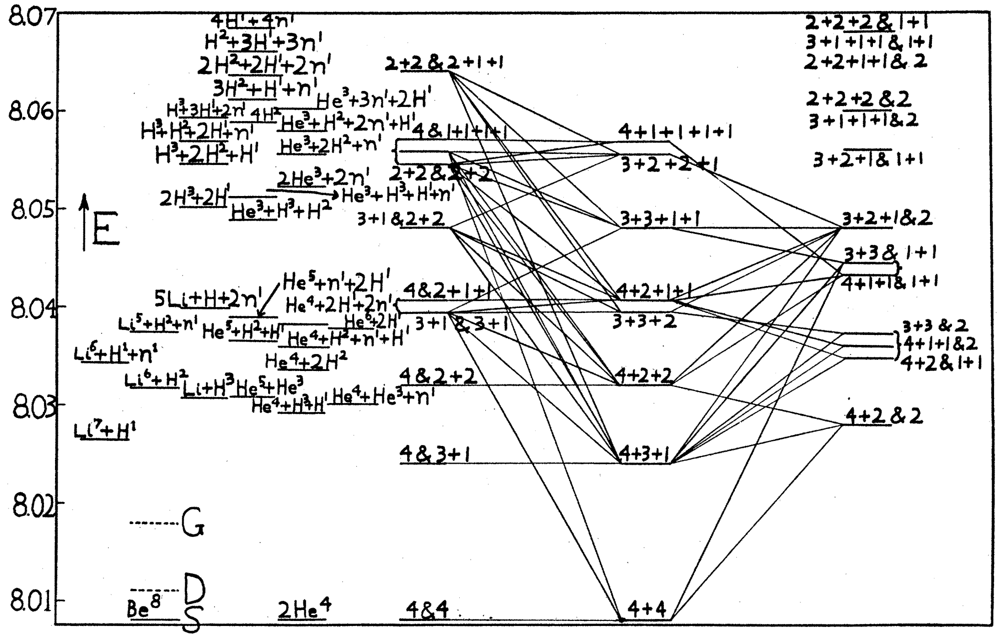

FIG. 4. At right: correlation (on long range force model) between levels of Be^8 (b) and those of groups into which the nucleus can be decomposed. Energy of levels is completely determined in this approximation by partition. The two groups whose energies are combined at (a) each contain four particles; those at (c), six and two particles. In the method of resonating group structure, states of Be^8 belonging to a particular partition are built up from those states of the various groups with which the given state is connected by correlation lines. The lowest states of Be^8 (partition 4+4) may be described approximately in terms of the normal states of two alpha-particles alone, since all other group states possess so much higher energy. At left: The experimental values for the normal energies of the constituent nuclei give a useful but less detailed guide to the choice of groups to represent a given state of the compound nucleus. (Energies in mass units.) Dotted levels predicted by alpha-particle model and by Hartree-Fock treatment.

Evaluation of the expressions

$$\mathbf{H}_{mn}=\int \Psi_m{}^* H \Psi'_n d\tau \quad \text{and} \quad \mathbf{I}_{mn}=\int \Psi'_m{}^* \Psi'_n d\tau$$

actually requires calculation of a number of integrals of the above type, since Ψ contains $(12!/(4!\,4!\,4!\,3!))=5775$ terms $F\Phi\Phi\Phi$ corresponding to the different ways twelve particles can be grouped into alpha-particles. Symmetry arguments easily reduce the 5775 integrals entering I_{mn} to only nine, and in the same way enormously simplify the work of evaluating H_{mn}. The nine integrals all contain as first member of the integrand the expression $F_m((1+2+3+4)/4,\cdots,\cdots)\Phi(1\,2\,3\,4)\Phi(5\,6\,7\,8)\Phi(9\,10\,11\,12)$; in the last member, which contains F_n, the 12 particles have been interchanged. The 5775 integrals divide up as follows, according to the type of rearrangement:

1	belonging to the original arrangement
48	belonging to grouping (5234)(1678)(9 10 11 12)
128	(9234)(1678)(5 10 11 12)
54	(5634)(1278)(9 10 11 12)
576	(5934)(1678)(2 10 11 12)
216	(9 10 34)(1278)(5 6 11 12)
1728	(9234)(1 10 78)(5 6 11 12)
1728	(9634)(1 10 78)(52 11 12)
1296	(569 10)(127 11)(3 4 8 12).

The result of our choice of mathematical expression for F,

$$\mathbf{F}_m \sim \exp\{-\sum(ijx)_m X_i X_j - \sum(ix)_m X_i\}, \quad (32)$$

is to give simple analytical formulae for H_{mn}, I_{mn}, and the energy $E_m=\mathbf{H}_{mm}/\mathbf{I}_{mm}$, in terms of the parameters $(ijx)_m$, $(ix)_m$, $(ijx)_n$, and $(ix)_n$. We now observe that these three formulae, once obtained with the aid of the "generating functions" $\mathbf{F}_m$ and $\mathbf{F}_n$, enable us readily to find the corresponding quantities H_{mn}, I_{mn}, and E_m for any other pair of functions F_m and F_n. In fact, we have only to differentiate $\mathbf{F}_m$ a suitable number of times with respect to the parameters $(ijx)_m$ and $(ix)_m$, and then carry out linear combinations to build up a function F_m consisting of a general polynomial in the alpha particle coordinates, multiplied by the same exponential as occurs in $\mathbf{F}_m$ (cf. for example Eq. (35) below). But since the parameters behave as constants in those integrations over coordinates which gave H_{mn} and I_{mn}, the values of H_{mn} and I_{mn} are obtained by carrying out the same processes of differentiation and linear combination which produced F_m from $\mathbf{F}_m$ and F_n from $\mathbf{F}_n$. This point explains the motive for giving the exponent of the generating function a dependence on alpha-particle coordinates which is more general than allowed by symmetry and the requirements of proper transformation under rotation and translation. In fact, after the differentiations just referred to, the (ix) are set equal to zero, the diagonal coefficients (iix) are all equated to one constant, and the off-diagonal coefficients (ijx) are all identified with a second constant. The parameters which really make the wave function flexible are those which enter in the formation of the linear combinations referred to above.

c. Application of alpha-particle model to construction of wave functions.—We have obtained in (*b*) a relatively simple method for the determination of the matrix elements of the energy and of unity with respect to any two wave functions. We now have the problem of actually building up from the generating function $\mathbf{F}$ the proper type of wave function to represent a given state of the system.

Let us first take as example the ground state of C^{12}. There is no rotation of the nucleus, and only zero-point vibration. It is clear that the wave function $F(\mathbf{X}_{\mathrm{I}}, \mathbf{X}_{\mathrm{II}}, \mathbf{X}_{\mathrm{III}})$ has its maximum value when the three alpha-particles are equidistant from each other and at some equilibrium distance, a_0, from the center of gravity of the nucleus. Moreover, for a uniform dilatation of the triangle of alpha-particles, the wave function will fall off approximately as $e^{-\xi'^2/2}$ or as $1-(m_\xi\omega_1/2\hbar)\xi^2$, where ξ is the normal coordinate for the given mode of motion; a similar falling off will occur for the deformations of the triangle described by the normal coordinates η and ζ (cf. *a*, above). A suitable form for the ground state wave function, having the above properties, and expressible as a linear combination of generating functions, is the following:

$$F_0=e^{-\beta(1-\mu)(R_1^2+R_2^2+R_3^2)}$$
$$\times\{e^{-(\beta\mu/3)R_{12}^2}-e^{-(\beta/3)(\mu+\epsilon)R_{12}^2}\}$$
$$\times\{e^{-(\beta\mu/3)R_{13}^2}-e^{-(\beta/3)(\mu+\epsilon)R_{13}^2}\}$$
$$\times\{e^{-(\beta\mu/3)R_{23}^2}-e^{-(\beta/3)(\mu+\epsilon)R_{23}^2}\}. \quad (33)$$

The first factor tends to keep the alpha-particles near the center of the nucleus, whilst each successive factor represents a given pair of alpha-particles seeking to preserve a certain equilibrium separation. F_0 reaches its maximum value when $R_1=R_2=R_3=a_0$, where $\beta a_0^2=\epsilon^{-1}\ln(1+\epsilon)$; and in the neighborhood of the maximum we have

$$F_0=\epsilon^3(1+\epsilon)^{-3-3/\epsilon}\{1-6(1+\epsilon)\beta a_0^2\beta\xi^2$$
$$-3(1+\epsilon)\beta a_0^2\beta(\eta^2+\zeta^2)\}.$$

Minimization of the energy of the ground state with respect to β, ϵ, and μ fixes these parameters, from which we estimate directly $a_0=[\beta^{-1}\epsilon^{-1}\ln(1+\epsilon)]^{\frac{1}{2}}$ and $\hbar\omega_1\sim(\hbar^2/M_pa_0^2)\epsilon^{-2}(1+\epsilon)\ln^2(1+\epsilon)$.

Let us now consider as a more general example the state to which we referred in (a): $n_1=2$, $n_2=0$, $J=3$, $K=3$, $m_J=3$. To represent in terms of Cartesian coordinates the angular part of the wave function given by the alpha-particle model, we note that

$$(1/4)(\sin^6\theta/2e^{-3i\varphi}-\cos^6\theta/2e^{3i\varphi})e^{3i\chi}$$
$$\sim(X_1+iY_1)(X_2+iY_2)(X_3+iY_3)/a^3, \quad (34)$$

where $a\sim a_0+\xi$. To introduce the factors $X+iY$ into the wave function (33) used in the preceding example, we have only to operate on

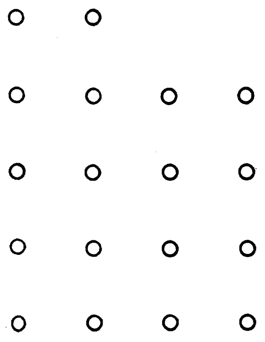

FIG. 5.

the generating function (29) with the differential expression

$$\left\{\frac{\partial}{\partial(1x)}+i\frac{\partial}{\partial(1y)}\right\}\left\{\frac{\partial}{\partial(2x)}+i\frac{\partial}{\partial(2y)}\right\}\times\left\{\frac{\partial}{\partial(3x)}+i\frac{\partial}{\partial(3y)}\right\}. \quad (35)$$

The proper dependence of the wave function on the normal coordinate ξ' which enters in the factor $H_2(\xi')=4\xi'^2-2$ is introduced by analogous differential operators.

In summary, then, we use the method of resonating group structure as follows to calculate the position of the levels predicted by the alpha-particle model: (1) we build up the type of wave function suggested by the simple model, by operating on a generating function with the proper combination of differentiation and linear superposition; (2), we calculate the matrix element of unity between this state (s) and any lower state (t) of the same J and m_J value by carrying out on the generator $\mathbf{I}_{mn}$ the same set of operations which give F_s and F_t from the generating function $\mathbf{F}$; (3), we put on the parameters of F_s the condition that this matrix element shall vanish (requirement of orthogonality to lower states); (4), we obtain the diagonal matrix elements H_{ss} and I_{ss} by a similar operational procedure; (5), we minimize the energy $E_s=H_{ss}/I_{ss}$ by varying the parameters in F_s, subject to condition (3); (6) we then have the energy and wave function of the state s in a form which (a) bears a close relation to the results of the alpha-particle model, but (b) maintains its validity even when the rate of exchange of neutrons and protons between alpha-"particles" is very rapid, and in fact (c) brings into evidence the fact that the forces between alpha-particles are intimately bound up with this interchange phenomenon.

Relative Importance of Various Groupings

The usefulness of the alpha-particle model treated above, and the importance of alpha-particle groupings in the description of the nucleus by the method of resonating group structure, both depend on those properties of nuclear forces (cf. p. 1085) which make the alpha-particle grouping particularly stable. Other types of group structure must be taken into account, however, if we desire a more accurate treatment of a nucleus of the type $4N$, or if we wish to apply our method to nuclei which cannot be built up from alpha-particles alone. In general, the various groups in which we may be interested will be characterized by their state of excitation as well as by the number of neutrons and protons they contain. In the following we develop with the aid of group theory and energy considerations some general qualitative arguments as to which types of grouping will be most important for the description of a certain state of a given nucleus.

We can discuss the general relationship between the energy levels of a compound nucleus (C) and the nuclei (A and B) from which it is synthesized with the aid of group theory alone, if we adopt a model of the nucleus (Wigner's[23] model (1)) in which the forces depend only on space coordinates and are the same between all pairs of particles. Then the eigenfunctions have the form

$$\psi^{(R)}(\mathbf{x}_1\sigma_1\tau_1,\cdots,\mathbf{x}_n\sigma_n\tau_n)$$
$$=\sum_{k=1}^{\chi(E)} f_k{}^{(R)}(\sigma_1\tau_1,\cdots,\sigma_n\tau_n)u_k{}^{(R')}(\mathbf{x}_1,\cdots\mathbf{x}_n). \quad (36)$$

Here x_i denotes the position and σ_i and τ_i, the ordinary and isotopic spin variables, of the ith particle. $\chi(E)$ is the character of the identity operation. The $f_k{}^{(R)}$ constitute the normalized orthogonal basis of an irreducible representation, R, through unitary matrices, of the group H_n of permutations on n symbols. Similarly, the $u_k{}^{(R')}$ compose the substratum of that matrix representation, R', the "associate" of R, obtained by changing the sign of $(-1)^{\frac{1}{2}}$ and multiplying by ϵ_p(-1 for odd, $+1$ for even permutations P).

With each irreducible representation of H_n is associated, according to Young, a certain "symmetry pattern" constructed by arranging n dots in rows and columns, all rows starting at a fixed vertical axis and extending to the right, all columns running upwards from a fixed base line. Successive columns do not increase in length, nor is a given row ever longer than the rows beneath it. (Fig. 5 represents that symmetry pattern of H_{18} which is said to correspond to the partition

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[23] E. Wigner, Phys. Rev. **51**, 107 (1937).

$4+4+4+4+2$ or $4^4 2$.) Start with any function of n variables, put the n labels on the blocks in any order, symmetrize the function with respect to the variables associated with the rows, antisymmetrize with respect to the columns. The result (if not identically zero) gives a function belonging to the corresponding representation of the symmetry group. The pattern of the associated representation, R', is obtained by rotating the pattern R by 180° about the diagonal which begins at the lower left-hand corner. Since the pair of numbers σ, τ take on only four sets of values, it follows that the spin functions $f_k^{(R)}(\sigma_1 \cdots \tau_n)$ belong to a symmetry pattern whose columns do not exceed four dots in length. The rows of physically allowed partitions R' therefore never contain more than four dots.

The interaction of n_1 particles in nucleus A with n_2 particles in nucleus B will be described to the first order in terms of the zero-order wave functions of the individual systems. In their dependence on spin, these functions are linear combinations of products of the type

$$f_i^A(\sigma_1, \cdots, \tau_{n_1}) f_j^B(\sigma_{n_1+1}, \cdots, \tau_n), \qquad (37)$$

obtained by distributing the $n=n_1+n_2$ pairs of variables $\sigma\tau$ between A and B in the $n!/n_1!n_2!$ possible different ways. The $\chi^A(E)\chi^B(E)n!/n_1!n_2!$ products give a representation of H_n. Broken into its different irreducible parts, this representation provides the proper linear combinations of wave functions for treating the energy levels of C. To find the symmetry patterns belonging to these allowed levels of C from the patterns of A and B, we make use of the following procedure given by Littlewood and Richardson[24] and drop those compound patterns not physically possible:

"Take the tableau (pattern) A intact, and add to it the symbols (dots) of the first (bottom) row of B. These may be added to one row of A, or the symbols may be divided (proceeding from the left, and) not disturbing their order, into any number of sets, the first set being added to one row of A, the second set to a subsequent (higher) fow, the third to a row subsequent to this, and so on. After the addition no row must contain more symbols than a preceding (lower) row, and no two of the added symbols may be in the same column.

"Next add the second row of symbols from B, according to the same rules, with this added restriction. Each symbol from the second row of B must appear in a later (higher) row of the compound tableau (pattern) than the symbol (if any) from the first row (of B which now lies) in the same column (of the compound pattern).

"Similarly add each subsequent row of symbols from B, each symbol being placed in a later row of the compound tableau than the symbol in the same column (which came originally) from the preceding row of B, until all the symbols of B have been used."

The application of this simple rule is based upon Wigner's observation that the gross structure of the nuclear energy level system appears to be determined by the symmetry of the wave function, the splitting into a multiplicity of levels arising from the finer details of the dependence of the wave function on coordinates and spins. Assuming an exactly saturated mixture of ordinary and Majorana long range forces (so that the splitting is negligible) he has in fact shown that the energy levels of an n particle system are given by

$$E = -J[(1/4) - \chi^{(R)}(2)/\chi^{(R)}(E)]n(n-1)/2 + \text{kinetic energy.} \qquad (38)$$

Here J is the constant of the two-body interaction $V = -J[(\frac{1}{4}) + P_M]$. R indicates the representation to which the spin part of the wave function belongs and $\chi(2)$ is the character of a transposition in this representation[25] (see Table V in Appendix). In this simplified model it is very easy to trace the relation between the energy levels of the compound nucleus and those of the groups into which it can be decomposed. Fig. 4 gives the correlation diagram for the lower "gross structure" levels of a system of eight particles.

Decrease of the range of the forces and introduction of electrostatic and spin interactions splits up the levels in the correlation diagram and complicates the tracing of correspondences. Also,

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[24] D. E. Littlewood and A. R. Richardson, Proc. Lond. Math. Soc. **A233** (1934).

[25] Reference is made to F. D. Murnaghan, J. Am. Math. Soc. **59**, 437 (1937), for a treatment of the properties of the symmetric group in a form convenient for application to nuclear theory. I am indebted to Professor Murnaghan for informing me of his results before publication.

the symmetry character of the wave function is no longer an exact integral of the equations of motion. Nevertheless, it is approximately an integral, and dominates the discussion of the composition of the compound wave function from the wave functions of the constituent groups. Particularly, in applying the method of resonating group structure in its simplest form to approximate the description of a certain level of C, we can neglect all but the lowest of the "symmetry levels" of the groups A and B into which C may be decomposed. The usefulness of this approximation is based on the relatively enormous dependence of the energy of the system on the symmetry character, in comparison to other effects.

With the above considerations as a guide, we are able to make an approximate estimate as to which types of groupings are most important in describing a given state of a given nucleus. This is particularly important for the development of the following paper, in which the mathematical treatment starts out with the assumption that the important groupings are already known.

Two-Body and Many-Body Forces

So far we have applied the concepts of resonating group structure to the nucleus in a way which has been independent of all but the simplest assumptions about the properties of the interactions between elementary particles. Nevertheless, the question of the detailed nature of these interactions must have an essential, if indirect, connection with the types of collective motion occurring in the nucleus.

Can nuclear properties be accounted for by forces acting between pairs of particles, or is a field theory necessary from the beginning in any reasonably accurate account of nuclear properties? Although present theory supplies no answer, we can at least expect *a priori* an intimate connection between electron theory and forces within the nucleus. The disturbance of the electron-positron field near the heavy nuclear particles owing to the rapidly alternating electromagnetic fields within the nucleus necessarily counter induces a reaction of the electron positron distribution upon the heavy particles. In the terminology of the quantum theory, we may say that virtual pairs induce forces between the heavy particles in the same sense in which we speak of virtual quanta being responsible for the Coulomb interaction.

It is instructive in this connection to recall the method by which we may abstract from the existence of the electromagnetic field to obtain the ordinary expression for the interactions in a system of charged particles: from the fields $\mathbf{E}_i$ and $\mathbf{H}_i$ of the individual particles we obtain the total fields $\mathbf{E}=\Sigma\mathbf{E}_i$ and $\mathbf{H}=\Sigma\mathbf{H}_i$ and thence that part of the field energy which does not contain self-energy terms:

$$(8\pi)^{-1}\int(\mathbf{E}^2+\mathbf{H}^2-\sum\mathbf{E}_i^2-\sum\mathbf{H}_i^2)d\tau. \quad (39)$$

In the static case the result is

$$\sum_{i<j} e_i e_j / r_{ij},$$

and also in general it is the sum of terms, each of which represents the interaction between only two particles, a consequence of the quadratic expression for the field energy.

A similar calculation of how much the electron-positron field contributes to the interaction of a system of particles might be made in the same way if it were not that a general expression for the energy density, U', of the field is lacking. However, even in the expansion for U' given by Euler and Kockel[26]

$$\begin{aligned} U' = {} & (180\pi^2)^{-1}mc^2(\hbar/mc)^{-3} \\ & \times\{(\tfrac{1}{2})(\mathbf{f}^2-\mathbf{g}^2)(3\mathbf{f}^2+\mathbf{g}^2)+(7/2)(\mathbf{f}\cdot\mathbf{g})^2 \\ & +(2/7)(\mathbf{f}^2-\mathbf{g}^2)^2(5\mathbf{f}^2+\mathbf{g}^2) \\ & +(13/7)(\mathbf{f}\cdot\mathbf{g})^2(3\mathbf{f}^2-\mathbf{g}^2)+\cdots\} \quad (40) \end{aligned}$$

$$[(\hbar/mc)e\mathbf{E}=mc^2\mathbf{f};\quad (\hbar/mc)e\mathbf{H}=mc^2\mathbf{g}],$$

valid only for not too strong or too rapidly varying fields,

$$(\hbar/mc)|\text{grad }\mathbf{F}|\ll|\mathbf{F}|,\quad (\hbar/mc^2)|\dot{\mathbf{F}}|\ll|\mathbf{F}|, \quad (41)$$

it is clear that that part of the interaction energy of a group of particles at nuclear distances $(|\mathbf{f}|, |\mathbf{g}|\gg 1)$ which is due to the electron positron field involves many-body forces and two-body forces of the same order of magnitude.

The observed dependence of nuclear binding on numbers of neutrons and protons gives little un-

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[26] H. Euler and B. Kockel, Naturwiss. **23**, 246 (1935); also W. Heisenberg and H. Euler, Zeits. f. Physik **38**, 714 (1936).

ambiguous information on the question of the existence of many-body forces. Nevertheless, if we adopt the long range force model by which Wigner[21] freed the group theory discussion of nuclear binding from complications due only to the dependence of forces on distance and spin, we can draw one important conclusion; namely, that no compromise is available between two-body forces and many-body forces. More precisely, to account for observed binding energies, either (a), the many-body forces must be much weaker than the two-body forces; or (b), with increasing number of particles in the nucleus, each successive group of many-body forces coming into action must make a contribution comparable with the total binding energy. Excluded is situation (c): 2, 3, $\cdots$, k-body forces strong ($238 \gg k > 2$), t-body forces negligible ($t > k$). This result can be seen in a general way from a comparison of the number of adjustable force constants with the requirements of observation: saturation character of nuclear binding, observed energy differences between isotopes, relative stability of nuclei of the types $4N$, $4N+1$, $4N+2$, $4N+3$, How it comes about in detail is shown in the appendix for the cases $k=3$ and $k=4$.

It must be emphasized[27] that the above conclusion depends on the (incorrect) assumption that nuclear forces are of long range; for example, it does not follow that the conclusion is true for short range many-body forces; and it is probably false if the many-body forces depend on velocity. Our arguments, in fact, are not quite general enough to allow us the satisfaction of dividing possible theories of nuclear forces into two sharply differentiated groups: (1), the usual, and plausible, description in terms of two-body forces; and (2), a description whose mathematical formulation requires from the beginning essentially a complete field theory of nuclear forces.

That many particle forces are consistent with our present information about the nucleus emphasizes the almost insuperable mathematical difficulties in the way of attempts to make nuclear spectroscopy alone yield any complete and unambiguous account of nuclear *forces*. Concerning the symmetry properties and certain other general features of the nuclear *wave function*, however, it appears possible to draw definite conclusions which depend on only a few very general assumptions about the exchange nature of nuclear forces—not, for example, on the number of particles involved in the interactions, nor particularly on the way the forces may vary with velocity. Although the nuclear liquid model of Bohr and Kalckar does not attempt to take account of symmetry properties, the qualitative correspondence in other respects between the modes of nuclear motion predicted by this and by the alpha particle treatment emphasizes to what a small extent the details of the interactions really enter into the determination of the general structure of the nuclear energy level spectrum. From this point of view, the most permanent value of detailed calculations based on special types of two-body forces is the correlation achieved between the properties of nuclear wave functions or states of motion and energy level systems.

Appendix I. Treatment of the Nuclear Three-Body Problem by the Method of Resonating Group Structure

Particles **1** and **2** are protons (neutrons); **3** is a neutron (proton). Consider the system as composed of proton (neutron) plus deuteron. The spin of the deuteron has three possible orientations:

$$\begin{aligned}
\Phi_1(2,3) &= \alpha(2)\alpha(3)\varphi(3-2),\\
\Phi_0(2,3) &= 2^{-\frac{1}{2}}[\alpha(2)\beta(3)+\alpha(3)\beta(2)]\varphi(3-2),\\
\Phi_{-1}(2,3) &= \beta(2)\beta(3)\varphi(3-2).
\end{aligned} \tag{42}$$

In these deuteron wave functions, $\varphi(3-2)$ is supposed to be a known function, depending only on the separation $r=|3-2|$ of neutron and proton, and to be normalized so that $\int \varphi^2(r)4\pi r^2 dr = 1$.

According as $S=\frac{3}{2}$ or $S=\frac{1}{2}$ for the combined system $He^3(H^3)$, we have two types of antisymmetric total wave function, typified, respectively, by

$$\begin{aligned}
\Psi(1,2,3) = 2^{-\frac{1}{2}}\alpha(1)\alpha(2)\alpha(3)[&F((2+3)/2-1)\varphi(3-2)\\
&-F((1+3)/2-2)\varphi(3-1)]
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\Psi(1,2,3) = 12^{-\frac{1}{2}}&[\alpha(1)\alpha(2)\beta(3)+\alpha(1)\beta(2)\alpha(3)\\
&-2\beta(1)\alpha(2)\alpha(3)]F((2+3)/2-1)\varphi(3-2)\\
&-12^{-\frac{1}{2}}[\alpha(2)\alpha(1)\beta(3)+\alpha(2)\beta(1)\alpha(3)\\
&-2\beta(2)\alpha(1)\alpha(3)]F((1+3)/2-2)\varphi(3-1).
\end{aligned} \tag{44}$$

With the help of these expressions, we calculate in terms of F the average energy of the system:

$$E = \int \Psi(T+V)\Psi d\tau / \int \Psi^2 d\tau.$$

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[27] I am indebted to Professor Wigner for making this observation.

We represent the interactions between the constituent particles by combinations of Majorana and Heisenberg forces.[28] We find by integration

$$E=\{(3\hbar^2/4M)\int(\nabla F)^2d\mathbf{X} +E_0\int F^2(\mathbf{X})d\mathbf{X}+\int F^2(\mathbf{X})V(\mathbf{X})d\mathbf{X} +\int\int F(\mathbf{X})J(\mathbf{X},\boldsymbol{\xi})F(\boldsymbol{\xi})d\mathbf{X}d\boldsymbol{\xi}\}/ \{\int F^2(\mathbf{X})d\mathbf{X}+\int\int F(\mathbf{X})I(\mathbf{X},\boldsymbol{\xi})F(\boldsymbol{\xi})d\mathbf{X}d\boldsymbol{\xi}\}. \quad (45)$$

Here M represents the proton mass and E_0 the average energy of a free deuteron described by the wave function φ:

$$E_0=(\hbar^2/M)\int(\nabla\varphi)^2d\boldsymbol{\sigma}-B\int\varphi^2(\boldsymbol{\sigma})\exp(-b^2\boldsymbol{\sigma}^2)d\boldsymbol{\sigma}. \quad (46)$$

Brackets are used below to represent the dependence of the ordinary interaction potential V and the integral operators I and J upon spin; the upper component belongs to $S=\frac{3}{2}$, the lower to $S=\frac{1}{2}$. $\mathbf{X}$, $\boldsymbol{\xi}$, $\boldsymbol{\sigma}$ are vectors, as are also $\mathbf{u}=(4\boldsymbol{\xi}+2\mathbf{X})/3$ and $\mathbf{v}=(4\mathbf{X}+2\boldsymbol{\xi})/3$.

$$I(\mathbf{X},\boldsymbol{\xi})=(64/27)\binom{-1}{1/2}\varphi(\mathbf{u})\varphi(\mathbf{v}), \quad (47)$$

$$V(\mathbf{X})=(1-2g)B\binom{1/3}{-2/3}\int\varphi^2(\boldsymbol{\sigma})\exp[-b^2(\mathbf{X}-\boldsymbol{\sigma}/2)^2]d\boldsymbol{\sigma} +e^2\int\varphi^2|\mathbf{X}-\boldsymbol{\sigma}/2|^{-1}d\boldsymbol{\sigma}, \quad (48)$$

$$J(\mathbf{X},\boldsymbol{\xi})=(64/27)\varphi(\mathbf{u})\varphi(\mathbf{v})[-B\binom{4/3-2g/3}{11/6-19g/6} \exp[-b^2(\mathbf{u}-\mathbf{v})^2]+B\binom{1}{-1/2}[\exp(-b^2\mathbf{u}^2) +\exp(-b^2\mathbf{v}^2)]-\binom{1}{-1/2}e^2|\mathbf{u}-\mathbf{v}|^{-1} +(256\hbar^2/81M)\binom{1}{-1/2}[\varphi(\mathbf{u})\nabla^2\varphi(\mathbf{v}) +\varphi(\mathbf{v})\nabla^2\varphi(\mathbf{u})+\nabla\varphi(\mathbf{u})\cdot\nabla\varphi(\mathbf{v})]. \quad (49)$$

Before proceeding further, we investigate how accurate a treatment of the binding energy of H^3 is possible with the simplest analytical expressions for the deuteron wave function and the intergroup wave function:

$$\varphi(\boldsymbol{\sigma})=\exp(-\alpha\boldsymbol{\sigma}^2); \quad F(\mathbf{X})=\exp(-\beta\mathbf{X}^2). \quad (50)$$

No electrostatic terms enter this problem. The state of the system is ^2S. The energy becomes a simple function of α and β. Its minimum value, -6.4mMU, is reached when $\alpha=0.87(mc^2/e^2)^2$, $\beta=0.68(mc^2/e^2)^2$. In comparison, a wave function of the form

$$\Psi(1,2,3)=2^{-\frac{1}{2}}\alpha(3)[\alpha(2)\beta(1)-\alpha(1)\beta(2)]N \times\exp[-(\nu/2)(\mathbf{r}_{13}{}^2+\mathbf{r}_{23}{}^2)-(\mu/2)\mathbf{r}_{12}{}^2] \quad (51)$$

gives an expression[29] for the energy of H^3, whose minimum value, -5.6 mMU, is obtained when $\mu=0.8$ and $\nu=1.0(mc^2/e^2)^2$. (The same values of the force constants are used in both calculations. We estimate by the equivalent two-body method that an accurate calculation using these constants would give $E=-8.7$ mMU for the binding energy.) We compare the spatial parts of the wave function (51) and the resonating group wave function (44):

$$\exp\{-(\mu+\nu)\mathbf{X}^2/2-(\mu+5\nu)\boldsymbol{\sigma}^2/8+(\mu-\nu)\mathbf{X}\cdot\boldsymbol{\sigma}/2\} \leftrightarrow \exp\{-\beta\mathbf{X}^2-\alpha\boldsymbol{\sigma}^2\}. \quad (52)$$

The polarization term, $\mathbf{X}\cdot\boldsymbol{\sigma}$, in the first function is in the right direction to allow for polarization of the deuteron by the neutron, but is small owing to the near equality of μ and ν. That the second function gives a binding 0.8 mMU better than the first in this problem must be attributed mainly to its having a better form of spin dependence, built up from deuteron spin functions (cf. discussion below).

How can one find the best possible wave function of the form (44), supposing φ to be given? We use the variation principle, varying $F(\mathbf{X})$ to make the expression (45) for the energy a minimum. The necessary and sufficient condition for stationary E is found to be [30] that F satisfy the wave equation[31]

$$(3\hbar^2/4M)\nabla^2F(\mathbf{X})+(E-E_0)F(\mathbf{X})=V(\mathbf{X})F(\mathbf{X}) +\int[J(\mathbf{X},\boldsymbol{\xi})-EI(\mathbf{X},\boldsymbol{\xi})]F(\boldsymbol{\xi})d\boldsymbol{\xi}. \quad (53)$$

Thus the relative motion of deuteron and neutron is governed by an ordinary potential $V(\mathbf{X})$ and a velocity dependent potential, represented by an integral operator with the kernel $K(\mathbf{X},\boldsymbol{\xi})=J-EI$. The three dimensional equation of motion (53) is reduced[32] to a radial equation by expressing $\mathbf{X}$ and $\boldsymbol{\xi}$ in polar coordinates r, θ, φ and ρ, σ, τ and making the substitutions

$$F(\mathbf{X})=r^{-1}f_L(r)P_L(\cos\theta);$$
$$\cos(\mathbf{X},\boldsymbol{\xi})=\mu;$$
$$K(\mathbf{X},\boldsymbol{\xi})=\Sigma(2L+1)K_L(r,\rho)P_L(\mu)/4\pi r\rho. \quad (54)$$

For S states $(L=0)$ we find

$$(3\hbar^2/4M)d^2f/dr^2+(E-E_0)f(r)=V(r)f +\int[J_0(r,\rho)-EI_0(r,\rho)]f(\rho)d\rho, \quad (55)$$

which is thus the wave equation applying to the ground states of H^3 and He^3.

We determine the dependence of neutron-deuteron interaction on distance by substituting into (47–49) the expression $(2\alpha/\pi)^{\frac{3}{4}}\exp(-\alpha r^2)$ for the deuteron wavefunction, with $\alpha=0.87$ (see Fig. 6d). We find

$$V(r,\rho)=-(2B/3)(1-2g)(1+b^2/8\alpha)^{-\frac{3}{2}} \times\exp[-b^2r^2(1+b^2/8\alpha)^{-1}]; \quad (56)$$

$$I_0(r,\rho)=(8/3)(2\alpha/\pi)^{\frac{3}{2}}\sinh(32\alpha r\rho/9) \times\exp[-(20\alpha/9)(r^2+\rho^2)]; \quad (57)$$

$$J_0(r,\rho)=-(8/3)(2\alpha/\pi)^{\frac{3}{2}}(1+b^2/4\alpha)^{-1}B(11/3-19g/3) \cdot\sinh(32\alpha/9+8b^2/9)r\rho\exp[-(20\alpha/9+4b^2/9) \times(r^2+\rho^2)]-(8/3)(2\alpha/\pi)^{\frac{3}{2}}(1+b^2/2\alpha)^{-1}B\sinh (32\alpha/9+16b^2/9)r\rho\cdot\{\exp[-(20\alpha/9+4b^2/9)r^2 -(20\alpha/9+16b^2/9)\rho^2]+\exp[-(20\alpha/9+16b^2/9)r^2 -(20\alpha/9+4b^2/9)\rho^2]\}+(64\hbar^2\alpha/9M)(2\alpha/\pi)^{\frac{3}{2}} \times\exp[-(20\alpha/9)(r^2+\rho^2)]\{[(11/4 -(56\alpha/9)(r^2+\rho^2)]\sinh(32\alpha r\rho/9) +(104\alpha r\rho/9)\cosh(32\alpha r\rho/9)\}. \quad (58)$$

V, I_0 and J_0 are charted in Fig. 6.

The function $f(r)=r\exp(-\beta r^2)$, with $\beta=0.68$, represents an approximate solution of the radial equation. Curve A in Fig. 6 shows how this function is related to the range of the neutron-deuteron interaction.

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[28] Eqs. (22a, b, c), "The Interaction Between Two Normal Alpha-Particles," give the exact expressions used. (To be published in the *Physical Review*.)

[29] E. Feenberg, Phys. Rev. **47**, 854 (1935), Eq. (14).

[30] For the variational calculation, see following paper, Eq. (17).

[31] In a recent paper [Proc. Phys. Math. Soc. Japan 19, 542 (1937)], Yukawa and Sakata discuss the scattering of neutrons in deuterium and derive an integro-differential equation which is similar to (53) but contains no term analogous to I, since their treatment was not based on the variation principle.

[32] Wheeler, Phys. Rev. 50, 647 (1936), Eq. (10).

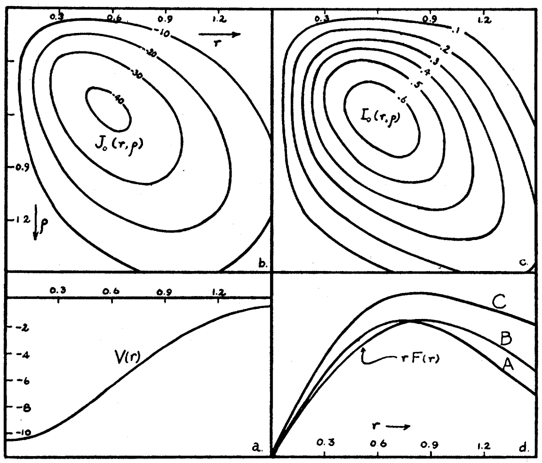

FIG. 6. Interaction of neutron and deuteron in ^2S states of H^3. (*a*) Ordinary potential $V(r)$ in mMU as a function of r (in units $e^2/mc^2=2.80\times10^{-13}$ cm). (*b*) Contour map of the kernel, $J_0(r,\rho)$, of the velocity dependent interaction, in units mMU $(e^2/mc^2)^{-1}$, as function of r and ρ. As one carries through the transition from the nuclear to the molecular three-body problem, the trough in J_0 becomes deeper and narrows down in the ratio $m/M=1/1840$, representing finally an interaction nearly independent of velocity. (*c*) $I_0(r,\rho)$, in units $(e^2/mc^2)^{-1}$. (*d*) B, approximate solution of the radial wave equation for relative motion of neutron and deuteron. A, internal wave function of deuteron used in calculating V, I_0, and $J_0(r\varphi(r)=Kr\exp(-\alpha r^2))$. C, exact wave function of free deuteron, for comparison. (A, B, and C not normalized.)

A calculation of V, I_0 and J_0 using for φ the exact wave function for a free deuteron, although complicated, would not give interactions very different from those shown in the figure, as one sees from a comparison of curves B and C. This choice of φ, however, makes one essential simplification: the wave function (44) for the three-body problem has exactly the right asymptotic behavior to describe dissociation and scattering. Solution of (53) then gives phase shifts and scattering cross sections.

The connection between the method of resonating group structure and the Heitler-London procedure is easily traced. The forces between 3 and 1, and 3 and 2, no longer depend appreciably on spin. In contrast to the deuteron, the atom has stable states of both the singlet and triplet type. Out of these we can build up two types of doublet wave functions, for the molecule, one of the form (44), the other being

$$\begin{aligned}&2^{-1}\alpha(1)[\alpha(2)\beta(3)-\alpha(3)\beta(2)]G(\mathbf{X}_{\mathrm{I}})\chi(3-2)\\&-2^{-1}\alpha(2)[\alpha(1)\beta(3)-\alpha(3)\beta(1)]G(\mathbf{X}_{\mathrm{II}})\chi(3-1).\end{aligned}\tag{59}$$

The states of the whole system are actually represented by linear combinations of the two types of doublet function. In the nuclear problem, the slight admixture of the second type was neglected, although our treatment would have been improved by taking into account the virtual ^1S level of the deuteron, describable by χ. In the molecular case, the atomic wave functions χ and φ are to be identified, as the dynamical effect of spin is negligible. The proper linear combinations are easily found: $F=(3^{\frac{1}{2}}/2)F_1$, $G=-(\frac{1}{2})F_1$ gives the molecular state built on a singlet state of the two protons; the doublet state built on a triplet is obtained when $F=(\frac{1}{2})F_3$, $G=(3^{\frac{1}{2}}/2)F_3$. It is to the former that the nuclear ground state is most closely related. The associated wave functions are

$$\begin{aligned}\Psi_1=2^{-1}[\alpha(1)\beta(2)-\alpha(2)\beta(1)]\alpha(3)[F_1(\mathbf{X}_{\mathrm{I}})\varphi(3-2)\\+F_1(\mathbf{X}_{\mathrm{II}})\varphi(3-1)],\\\Psi_3=12^{-\frac{1}{2}}[2\alpha(1)\alpha(2)\beta(3)-\alpha(1)\beta(2)\alpha(3)\\-\alpha(2)\beta(1)\alpha(3)]\cdot[F_3(\mathbf{X}_{\mathrm{I}})\varphi(3-2)-F_3(\mathbf{X}_{\mathrm{II}})\varphi(3-1)].\end{aligned}\tag{60}$$

In addition we also have four quartet states built on triplet states of the two protons. These are like (43) in spin dependence, and have the same coordinate dependence as Ψ_3.

The different spin dependence of atomic and nuclear forces governed the just mentioned changes in the spin part of the compound wave function, changes which had nothing to do, however, with the decrease of the mass of particle 3 from M to $m=M/1839$. At the end of the latter transition, in the limit $m/M\rightarrow0$, the separation of particle 1 from group (23), namely, $\mathbf{X}_{\mathrm{I}}=(M\mathbf{x}_2+m\mathbf{x}_3)/(M+m)-\mathbf{x}_1$, approaches the negative of the complementary separation $\mathbf{X}_{\mathrm{II}}$. The calculation of the energy of the system in this limit is very simple and gives the same type of expression as occurred in the nuclear problem, (Eq. (45)), with $\hbar^2/M$ as coefficient of the kinetic energy term and

$$\begin{aligned}V(\mathbf{X})&=J(\mathbf{X})+e^2/|\mathbf{X}|,\\J(\mathbf{X},\boldsymbol{\xi})&=\delta(\mathbf{X}+\boldsymbol{\xi})[2K(\mathbf{X})+D(\mathbf{X})e^2/|\mathbf{X}|\\&\qquad-\hbar^2/2m\nabla^2D(\mathbf{X})],\\I(\mathbf{X},\boldsymbol{\xi})&=\delta(\mathbf{X}+\boldsymbol{\xi})D(\mathbf{X})\end{aligned}\tag{61}$$

(for singlet states; I and J have opposite sign for triplet states). Here D is the orthogonality integral, and J and K the direct and exchange integrals, which are familiar from the Heitler-London approximation:

$$\begin{aligned}D(\mathbf{X})&=\int\varphi(-\mathbf{X}/2+\boldsymbol{\lambda})\varphi(\mathbf{X}/2+\boldsymbol{\lambda})d\boldsymbol{\lambda},\\J(\mathbf{X})&=\int\varphi^2(-\mathbf{X}/2+\boldsymbol{\lambda})\frac{e^2}{|\mathbf{X}/2+\boldsymbol{\lambda}|}d\boldsymbol{\lambda},\\K(\mathbf{X})&=\int\varphi(-\mathbf{X}/2+\boldsymbol{\lambda})\frac{e^2}{|\mathbf{X}/2+\boldsymbol{\lambda}|}\varphi(\mathbf{X}/2+\boldsymbol{\lambda})d\boldsymbol{\lambda}.\end{aligned}\tag{62}$$

The velocity dependence of the interaction between the two groups has degenerated in the molecular problem to a kernel which is approximately a δ function. Two cases are possible: (1), $F(\mathbf{X})$ an even function of $\mathbf{X}(L=0,2,\cdots)$. Then $\int\delta(\mathbf{X}+\boldsymbol{\xi})F(\boldsymbol{\xi})d\boldsymbol{\xi}=F(\mathbf{X})$; the effective molecular potential represents attraction for singlet states (ionized parahydrogen), instability for triplets (derived from orthohydrogen); (2), $L=1,3,\cdots$; $\int\delta(\mathbf{X}+\boldsymbol{\xi})F(\boldsymbol{\xi})d\boldsymbol{\xi}=-F(\mathbf{X})$; and the roles of the ortho- and parahydrogen molecular ion are interchanged.

If φ is the exact wave function for the ground state, E_0, of the hydrogen atom, we have

$$-(\hbar^2/2m)\nabla^2D(\mathbf{X})=E_0D(\mathbf{X})-K(\mathbf{X})$$

and the wave equation (53) can be put in a form which exhibits the usual Heitler-London molecular potential:

$$(1\pm D)^{-1}(\hbar^2/M)\nabla^2F+(E-E_0)F=\left[\frac{e^2}{|\mathbf{X}|}+\frac{J\pm K}{1\pm D}\right]F.\tag{63}$$

TABLE V. *Table for determining the character ratios* $\chi(2)/\chi(E)$, $\chi(3)/\chi(E)$, $\chi(4)/\chi(E)$, $\chi(2^2)/\chi(E)$ *for certan simple partitions. Example: For the partition* $(t+1)^3t$ *of the number* $n=4t+3$, $\chi(2)/\chi(E)=(4t^2-10t-6)/n(n-1)$.

PARTITION (SPIN)	ADJOINT (COORDINATE)	ORDER n	$n!\chi(2)/(n-2)!\chi(E)$			$n!\chi(3)/(n-3)!\chi(E)$				$n!\chi(4)/(n-4)!\chi(E)$					$n!\chi(2^2)/(n-4)!\chi(E)$				
			t^2	t	1	t^3	t^2	t	1	t^4	t^3	t^2	t	1	t^4	t^3	t^2	t	1
t^4	4^t	$4t$	4	-16	0	4	-48	$+68$	0	4	-96	$+404$	-336	0	16	-144	$+416$	-264	0
$(t+1)^3t^3$	4^t1	$4t+1$	4	-14	0	4	-45	$+56$	0	4	-92	$+356$	-268	0	16	-128	$+344$	-232	0
$(t+1)^2t^2$	4^t2	$4t+2$	4	-12	-2	4	-42	$+38$	0	4	-88	$+296$	-172	0	16	-112	$+264$	-128	0
$(t+1)^3t$	4^t3	$4t+3$	4	-10	-6	4	-39	$+14$	$+6$	4	-84	$+224$	-24	0	16	-96	$+176$	$+24$	0
$(t+2)t^3$	4^t1^2	$4t+2$	4	-12	$+2$	4	-42	$+50$	0	4	-88	$+320$	-236	0	16	-112	$+296$	-272	0
$(t+2)(t+1)t^2$	4^t21	$4t+3$	4	-10	0	4	-39	$+32$	-3	4	-84	$+260$	-156	0	16	-96	$+224$	-168	0
$(t+2)^2t^2$	4^t2^2	$4t+4$	4	-8	0	4	-36	$+20$	-12	4	-80	$+212$	-136	0	16	-80	$+176$	-136	$+24$
$(t+2)(t+1)^2t$	4^t31	$4t+4$	4	-8	-4	4	-36	$+8$	0	4	-80	$+188$	-24	$+8$	16	-80	$+144$	-24	-8
$(t+2)^2(t+1)t$	4^t32	$4t+5$	4	-6	-4	4	-33	-4	-12	4	-76	$+140$	-20	$+24$	16	-64	$+104$	-8	$+24$
$(t+2)^3t$	4^t3^2	$4t+6$	4	-4	-6	4	-30	-22	-24	4	-72	$+80$	$+12$	$+72$	16	-48	$+56$	$+48$	$+72$

That the factor $(1\pm D)^{-1}$ occurs in (63), although absent in the customary molecular treatment, is due to the fact that we have consistently applied the variation principle. Indeed, without this factor, the wave functions Ψ derived from F's for two different energies would not have the proper orthogonality. The modification which the above factor introduces in the mass of the system gives an improvement, in principle, on the H–L approximation. Although the Born-Oppenheimer theory[3] requires use of the customary value for the reduced mass, this does not contradict our conclusions, for this theory assumes an accurate calculation of the molecular potential.

APPENDIX II. TWO-, THREE- AND FOUR-BODY LONG RANGE EXCHANGE FORCES

A number of important consequences were drawn by Wigner[21] by considering a Hamiltonian arising from long range two-body interactions, linear combinations of ordinary and Majorana forces, equal between all pairs of particles:

$$H=\sum_{i<j}[-J_0-J(ij)]. \tag{64}$$

The symbol (ij) represents the Majorana coordinate exchange operator. The J's are numerical constants. The expression

$$\begin{aligned}H=\sum_{\text{pairs}}[-J_0-J(ij)]+\sum_{\text{triples}}[-K_0-K(ijk)]\\ +\sum_{\text{quadruples}}[-L_0-L(ijkl)]\\ +\sum_{\text{double pairs}}[-M_0-M(ij)(kl)]\end{aligned} \tag{65}$$

represents the corresponding Hamiltonian operator if forces exist between larger groups of particles. The permutations are defined by equations like

$$(ijk)u(\mathbf{x}_1\cdots\mathbf{x}_i\cdots\mathbf{x}_j\cdots\mathbf{x}_l\cdots)=u(\mathbf{x}_1\cdots\mathbf{x}_l\cdots\mathbf{x}_i\cdots\mathbf{x}_j). \tag{66}$$

In terms of the number, n, of particles and the characters, χ, of the representation of the symmetric group, H_n to which the coordinate part of the wave function belongs, the energy, W, of the system is given exactly by

$$\begin{aligned}W=&[-J_0-J\chi(2)/\chi(E)]n(n-1)/2\\ &+[-K_0-K\chi(3)/\chi(E)]n(n-1)(n-2)/6\\ &+[-L_0-L\chi(4)/\chi(E)]n(n-1)(n-2)(n-3)/24\\ &+[-M_0-M\chi(2^2)/\chi(E)]\\ &\times n(n-1)(n-2)(n-3)/8.\end{aligned} \tag{67}$$

From Table V one finds that exact saturation requires that

$$M_0=-M/16,\ L_0=L/64,\ M=\mathrm{L}/4,\ K_0=-K/16,$$
$$J_0=J/4+45K/48+575L/512,$$

J, K, and L being arbitrary. Under these conditions, the normal energy level of the system is given by

$$\begin{aligned}-W&=n(15J/8+75K/32+915L/768) && (n=4N)\\ &=n(15J/8+90K/32+945L/768)\\ &\quad-(15J/8+90K/32+6750L/768) && (n=4N+1)\\ &=n(15J/8+95K/32+1195L/768)\\ &\quad-(20J/8+160K/32+2000L/768) && (n=4N+2)\\ &=n(15J/8+90K/32+1305L/768)\\ &\quad-(15J/8+150K/32+2745L/768)\\ &&& (n=4N+3)\end{aligned} \tag{68}$$

and when $n=4N$, the first excited level is determined by

$$\begin{aligned}-W&=n(15J/8+107K/32+1363L/768)\\ &\quad-(32J/8+256K/32+3584L/768),\end{aligned} \tag{69}$$

the partition being $4+4+\cdots+4+3+1$. The excitation energy

$$W=4J+(8-n)(K+7L/12) \tag{70}$$

corresponds in the case of actual nuclei to the difference of binding energies in pairs such as ${}_3\mathrm{Li}^8$, ${}_4\mathrm{Be}^8$. Allowance being made in the experimental values for electrostatic energy, this difference does not change by as much as 50 percent in the light nuclei up to $n=16$, whence

$$|K+7L/12|<J/8. \tag{71}$$

From the almost periodic manner in which the binding energy of light nuclei is observed to vary with atomic number, we conclude by the same type of argument that K and L individually cannot exceed approximately 10 percent of J. In conclusion: if a mixture of two-, three-, and four-body exchange and ordinary forces is to account for the characteristic features of nuclear structure, the two-body forces must be responsible for at least 90 percent of the binding energy of the light nuclei up to oxygen, provided first, that the forces are of long range, and second, that they do not depend on velocity.