

On the Structure of the Nuclei Between Helium and Oxygen

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In order to test the present assumptions on nuclear forces, the theory is applied to the nuclei in which the first p shell of protons and neutrons is being built up, i.e., to the nuclei with masses between 5 and 16. The Hartree-Fock approximation is used for the numerical calculations, but the more qualitative results are independent to a large degree of the approximations used. The angular momenta of the ground states appear to be given correctly by the theory. Although the wave functions used do not correspond to preformed alpha-particles, the first-order energies exhibit a marked four-shell structure. The experimental

energy difference between the nuclear pairs $(N, N+1)$, $(N+1, N)$ may possibly be somewhat larger than the difference in the electrostatic energies. Support for the use of spin exchange (Heisenberg) forces to account for the singlet-triplet separation in the deuteron is found in the singlet-triplet separation inferred from the $\text{Li}^6\text{-He}^6$ normal state energy difference. However, the $\text{B}^{10}\text{-Be}^{10}$ and $\text{N}^{14}\text{-C}^{14}$ normal state differences do not fit the simple theory which is adequate for the singlet-triplet spacing in the two and six particle problems.

I. INTRODUCTION

IT is generally accepted now that the explanation of the binding energies and scattering properties of the nuclei n , H^1 , H^2 , H^3 , He^3 , He^4 requires several kinds of forces.¹ The forces which are generally assumed at present between a proton and a neutron ("between unlike particles") are: (1) a "Majorana force" involving an exchange P of the Cartesian coordinates of the two particles and (2) a "Heisenberg force" involving the product of a Cartesian coordinate exchange P and a spin coordinate exchange Q . The Heisenberg force has about $\frac{1}{4}$ the depth of the Majorana force and the same range of action. For all present calculations the exact dependence of the potential on distance seems to be relatively unimportant and we shall use, for the sake of convenience, the usual

$$(1-g)A_{\nu\pi}e^{-r^2/r_0^2}P = (1-g)A_{\nu\pi}e^{-\alpha r^2}P \quad (1a)$$

for the Majorana force and

$$gA_{\nu\pi}e^{-\alpha r^2}PQ \quad (1b)$$

for the Heisenberg force between unlike particles. Here $A_{\nu\pi} = 72 mc^2$, $r_0 = 2.25 \cdot 10^{-13}$ cm, $g = 0.22$ and $\alpha = 16$ if r is measured in units of $\hbar/c(Mm)^{\frac{1}{2}} = 8.97 \cdot 10^{-13}$ cm.

The forces between like particles are less well

known. Between pairs of protons and pairs of neutrons one assumes a potential with a depth $A_{\nu\nu} = 41 mc^2$ and the same width as between unlike particles.¹ This force is either assumed to involve an exchange P of the Cartesian coordinates

$$A_{\nu\nu}e^{-\alpha r^2}P \quad (2a)$$

or else the scalar product of their spin operators:

$$-\frac{1}{3}A_{\nu\nu}e^{-\alpha r^2}(\sigma_1 \cdot \sigma_2). \quad (2b)$$

The latter possibility can be considered, because of Dirac's identity for antisymmetric wave functions²

$$-\frac{1}{3}(\sigma_1 \cdot \sigma_2) = \frac{1}{3} + \frac{2}{3}P_{12}, \quad (3)$$

as the sum of an ordinary and an exchange force. It is undecided, at present, which of the two forms of interaction deserves preference. In fact it has been proposed³ to assume interactions which would give equal attractions between pairs of like and unlike particles in the singlet state. We obtain a problem having this property by assuming for the interaction between like particles the same forces as between unlike particles:⁴

$$(1-g)A_{\nu\pi}e^{-\alpha r^2}P + gA_{\nu\pi}e^{-\alpha r^2}PQ. \quad (2c)$$

² Cf. P. A. M. Dirac, *Quantum Mechanics* (Oxford, 1935), §19, §61; J. H. Van Vleck, *Phys. Rev.* **48**, 367 (1935).

³ G. Breit and E. U. Condon, private communications.

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¹ E. Feenberg and J. K. Knipp, *Phys. Rev.* **48**, 906 (1935). E. Feenberg and S. S. Share, *Phys. Rev.* **50**, 253 (1936).

⁴ The assumption that the forces between all kinds of particles are the same was first put forward by L. A. Young, *Phys. Rev.* **47**, 972 (1935).

For like particles the second part of this expression can be written, because of the antisymmetry of the wave function in the coordinates of the protons (or neutrons), as an ordinary repulsive force $-gA_{\nu\pi}e^{-\alpha r^2}$. We shall consider this "all forces equal model" because it is at least a useful approximation, the properties of which in some cases, can be discussed with greater ease⁵ than the properties of a model with forces (2a) or (2b). In addition to the forces (2), there are between protons the electrostatic forces and also the ordinary spin-orbit and spin-spin interactions are generally assumed to exist. These latter interactions seem to play only a minor role in nuclear problems.

Of course, it is most probable that the interaction of the constituents of nuclei cannot be described at all rigorously by a Schrödinger equation, the variables of which are the Cartesian and spin coordinates of the protons and neutrons only. A similar description is impossible, strictly speaking, for the extranuclear electrons also. The cause of the interaction is probably some field in both cases—the electromagnetic field of light quanta in the second case and, perhaps, the electron-neutrino field in the first case. The elimination of the field variables is possible always in a certain approximation only and breaks down in a higher order approximation. This is manifested by the phenomena of spontaneous emission and line width for the extranuclear electrons and by the β disintegration, for example, in the case of nuclear constituents. Nevertheless, the usual Schrödinger equation gives a practically perfect description of atomic states and it may be hoped that a similar equation exists for the nuclear constituents also.

We shall attempt here to find experimental criteria which can be used to answer the following three questions:

- (1) Whether or not the difference between proton-proton and neutron-neutron interaction is only the Coulomb force.
- (2) Whether (2a), (2b) or (2c) is the more correct form for this interaction.
- (3) Whether or not the assumption is correct that the neutron-proton interaction operator is a linear combination of Majorana and Heisenberg terms.

Before examining these questions we apply the Hartree-Fock approximation method to the

⁵ It is equivalent to "approximation (2)" of the paper of one of the present writers in this issue.

light nuclei between and including Li⁶ and O¹⁶ and perform the same calculation for the position of the terms arising from the lowest configuration which Slater⁶ has made for atomic spectra.

II. HARTREE-FOCK CALCULATIONS

It is well known that the Hartree-Fock method gives only very roughly correct solutions of the nuclear wave equation. This has been pointed out by Weizsäcker, Flügge and Heisenberg⁷ for the older models (no interaction between like particles). It is true, however, also for the forces (1) and (2), though to a somewhat lesser degree. Bethe and Bacher⁸ performed similar calculations for both light and heavy nuclei using the newer model with the result that the method gives practically no binding energy for the observed nuclear densities. Since the Schrödinger equation without doubt has solutions with much lower characteristic values at the same densities (conglomerates of slightly compressed alpha-particles), the result obtained by Bethe and Bacher must be interpreted as revealing the inaccuracy of the Hartree approximation in nuclear problems. Considering the great similarity between the nuclear and metallic wave equations and the importance of the correlation energy in the latter,⁹ this is not surprising.

TABLE I. *Binding energies of elements with masses between 4 and 16.*

n_1	$n_2=0$	1	2	3	4	5	6
0 He	55	—	56				
1 Li		62	76	89			
2 Be			109	113	126		
3 B			110	125	147	153	
4 C				141	176	186	202
5 N					180	201	222
6 O						215	246

⁶ Cf. E. U. Condon and G. Shortley, *The Theory of Atomic Spectra* (Cambridge, 1936). In the Hartree approximation the nuclei of the group He⁶—O¹⁶ are obtained by successive additions to the p shell which is completed at O¹⁶. This has been first pointed out by J. H. Bartlett, *Nature* **130**, 165 (1932). Cf. also his letter in the *Phys. Rev.* **41**, 370 (1932) and G. Gamow, *Zeits. f. Physik* **89**, 592 (1934) and especially W. M. Elsasser, *J. de phys.* **4**, 549 (1933); **5**, 389 and 635 (1934).

⁷ C. F. v. Weizsäcker, *Zeits. f. Physik* **96**, 431 (1935); S. Flügge, p. 459; W. Heisenberg, p. 473.

⁸ H. A. Bethe and R. F. Bacher, *Rev. Mod. Phys.* **8**, 82 (1936).

⁹ It is more than half of the binding energy in Na.

Nevertheless, it can be expected that in the case of light nuclei the order of the terms will be given correctly by such a calculation. This is true for the corresponding atomic spectra⁶ although the calculated ratios of the term differences show marked deviations from the experimental values.

Table I gives the elements with which we shall be concerned; n_1+2 is the number of protons, n_2+2 the number of neutrons. The figures in the table are the binding energies¹⁰ in units of mc^2 . As long as we neglect the Coulomb forces, the constitutions of elements symmetric with respect to the main diagonal of the table are identical. The theory of holes¹¹ gives us furthermore the constitution of a nucleus from its mirror image with respect to the other diagonal.

In this section we shall neglect the Heisenberg forces. This makes the Hamiltonian operate on the Cartesian coordinates only and both protons and neutrons will have a "multiplicity"; each resultant spin angular momentum will be a good quantum number. In addition to these, we have the total azimuthal quantum number. The possible terms for Be⁹, for example, can be determined by combining every term of the configuration s^2p^2 of the protons with every term arising from the configuration s^2p^3 of the neutrons.¹² The former terms are 1S , 1D , 3P , the latter ones 2P , 2D , 4S . Taken together these two configurations yield for the whole nucleus the terms ^{12}P , ^{12}D , ^{14}S , ^{12}P , ^{12}D , ^{12}F , ^{12}S , ^{12}P , ^{12}D , ^{12}F , ^{12}G , ^{14}D , ^{32}S , ^{32}P , ^{32}D , ^{32}P , ^{32}D , ^{32}F , ^{34}P . The first index represents the multiplicity of the protons, the second the multiplicity of the neutrons. (The Heisenberg forces will introduce an interaction between proton spin and neutron spin and split many of these terms into several new ones.)

Since the number of terms is quite high, we first want a quick orientation as to which of the terms will be the lowest ones. It is clear that most of the terms just enumerated lie high in the continuous spectrum. We shall be interested in the low terms only.

For this first orientation, we shall assume the interaction between like particles to be the same

¹⁰ Computed from the mass values given by Bethe and Livingston at the Cornell Nuclear Conference, July, 1936. The C^{14} value is taken from a paper by T. W. Bonner (same conference).

¹¹ Cf. W. Heisenberg, Ann. d. Physik 10, 888 (1931); G. Shortley, Phys. Rev. 43, 451 (1933).

¹² Cf. reference 8, Table XIV for a complete list of terms.

as that between unlike particles. This amounts to assuming (1a) to be valid for all pairs of particles, since the interaction (1b) involving the spin is omitted in this section. We shall call this model the "equal orbital forces model." It does not constitute a good approximation to either (1a) and (2a) or to (1a) and (2b), but it is useful in obtaining a first orientation.

In the equal orbital forces model, the Hamiltonian is symmetric in all particles and acts on the Cartesian coordinates only. Four particles can be in the same orbit, namely two protons and two neutrons. Hence, not only those representations of the symmetric group will occur¹³ which occur in atomic spectra, but we shall have representations with 1, 2 and also 3, 4 as addends. The representations play a very great role in the calculation of the potential energy, because the interaction operator contains a permutation of the particles.

The kinetic energy is the same for all wave functions arising from the same configuration.

If $\psi_1, \psi_2, \dots, \psi_s$ belong to a certain representation D of the symmetric (permutation) group of $n = n_1 + n_2$ particles, we can calculate the matrix elements of the potential energy

$$V_{\kappa\kappa} = (\psi_{\kappa}(x_1 \dots x_n), \sum_{\alpha\beta} J(x_{\alpha} - x_{\beta}) P_{\alpha\beta} \psi_{\kappa}(x_1 \dots x_n)), \quad (4)$$

which is equal to

$$V_{\kappa\kappa} = \sum_{\alpha\beta} \sum_{\lambda} D(\alpha\beta)_{\lambda\kappa} (\psi_{\kappa}, J(x_{\alpha} - x_{\beta}) \psi_{\lambda}) \quad (5)$$

because of the relations

$$P_{\alpha\beta} \psi_{\kappa}(x_1 \dots x_n) + \sum_{\lambda} D(\alpha\beta)_{\lambda\kappa} \psi_{\lambda}(x_1 \dots x_n). \quad (6)$$

Only those terms will be great in the sum (5) for which $\kappa = \lambda$; in the other terms the positive and negative regions will about cancel each other. This canceling would be exact if the range of the forces were very long: in this case J could be taken out of the integral as a constant $J(0)$ and the remaining integral would vanish for $\kappa \neq \lambda$ and be 1 for $\kappa = \lambda$. The whole $V_{\kappa\kappa}$ is for very

¹³ Cf. E. Wigner, *Gruppentheorie* etc. (Braunschweig, 1931), Chap. 13. The "all orbital forces equal" model is equivalent to approximation (1) of the paper mentioned in reference 5.

TABLE II. The terms of a p^n configuration which correspond to a definite partition.

p^2				p^3			
2 $S D$ 1/1	1+1 P -1/1	3 $F P$ 1/1	2+1 $D P$ 0/2	1+1+1 S -1/1			
p^4				p^5			
4 $G D S$ 1/1	3+1 $F D P$ 1/3	2+2 $D S$ 0/2	2+1+1 P -1/3	4+1 $P D F G$ 2/4	3+2 $F D P$ 1/5	3+1+1 $D S$ 0/6	2+2+1 P -1/5
p^6							
4+2 $G F D D S$ 3/9	3+3 $F P$ 1/5	4+1+1 $F P$ 2/10	3+2+1 $D P$ 0/16	2+2+2 S -1/5			

long range forces

$$V_{\kappa\kappa} = J(0) \sum_{\alpha\beta} D(\alpha\beta)_{\kappa\kappa}. \quad (7)$$

The last sum has to be extended over all $n(n-1)/2$ transpositions and is equal to $n(n-1)/2$ times the character $\chi(T)$ corresponding to a transposition, divided by the dimension $\chi(E) = s$ of the representation:

$$V_{\kappa\kappa} = J(0) (n(n-1)/2) \chi(T) / \chi(E). \quad (8)$$

This expression is independent of the detailed shape of the wave function and depends only on the representation D to which the wave function belongs. The formulas for the characters of representations¹⁴ show that it is greatest for that

¹⁴ I. Schur, Berl. Ber. (1908), p. 664. We found the following formula most suitable for the calculation of characters:

$$\{\lambda_1, \lambda_2, \dots, \lambda_p\} = \{\lambda_1 - 1, \lambda_2, \dots, \lambda_p\} + \{\lambda_1, \lambda_2 - 1, \dots, \lambda_p\} + \dots + \{\lambda_1, \lambda_2, \dots, \lambda_p - 1\}.$$

Here the symbol $\{\lambda_1, \lambda_2, \dots, \lambda_p\}$ is the character of a permutation in the representation corresponding to the partition $\lambda_1 + \lambda_2 + \dots + \lambda_p = n$. The left side gives the character of the representation of the permutation group of n elements, the right side contains characters of representations of the permutation group of $n-1$ elements. The formula holds for every permutation and can be used as a recursive formula. On the right side, all symbols must be omitted in which one number in the bracket is greater than the preceding. If the last figure in a bracket is a zero, it can be dropped. Thus, e.g., for a transposition,

$$\{4+1+1\} = \{3+1+1\} + \{4+0+1\} + \{4+1+0\} \\ = \{3+1+1\} + \{4+1\}.$$

The values of these symbols can be obtained by a further application of the recursive formula, since, for a transposition, all symbols can be reduced finally to $\{2\}$ and $\{1+1\}$ which are $+1$ and -1 , respectively. They are, for a transposition, 0 and 2, so that the character corresponding to a transposition in $\{4+1+1\}$ is 2.

TABLE III. Interaction energies between like particles.

INTER-ACTION	p^2 or π^2			p^3 or π^3		
	1S	1D	3P	2P	2D	4S
(2a) (2b) ORDINARY FORCE	$L+2K$ $L+2K$ $L+2K$	$L-K$ $L-K$ $L-K$	$-L+3K$ $(-L+3K)/3$ $L-3K$	$5K$ $L+2K$ $3L-4K$	$3K$ L $3L-6K$	$-3L+9K$ $-L+3K$ $3L-9K$
INTER-ACTION	p^4 or π^4		p^5 or π^5	p^6 or π^6		
(2a) (2b) ORDINARY FORCE	ADD $-L+8K$ to p^2 ADD $L+2K$ to p^2 ADD $5L-10K$ to p^2		$-2L+16K$ $2L+4K$ $10L-20K$	$-3L+24K$ $3L+6K$ $15L-30K$		

representation which contains as many 4's as possible and in addition to these only one other addend. The terms corresponding to this representation will be called the "low terms."

It is easy to determine the terms of a p^n configuration which correspond to a definite partition¹⁵ and the result is given in Table II. The last row contains $\chi(T)/\chi(E)$. The four s particles form a closed shell and contribute the same amount of energy to every term of a configuration.

We proceed now to the secular equations, using (2a) and (2b) for the interaction between like particles. The kinetic energy has been omitted, because it is the same for all terms of the same configuration. In the calculation of the potential energy, the closed s shell has been omitted for the same reason and the wave function for the p particles only used. To illustrate, for Li^7 the configuration is $p\pi^2$, the Greek letter π corresponding to the neutron, p to the proton. First the wave functions for the different π^2 terms 1S , 1D , 3P are written down. Then, for example, to calculate the ^{12}P term, wave functions with the azimuthal quantum number 1 are constructed from the proton wave function and the 1S neutron function, and from the proton function and the 1D neutron function. The spin functions can always be omitted when calculating in this way.

With this choice of wave functions, the interaction between like particles has only diagonal elements, these being the sum of the energies of the proton term and the neutron term used. The matrix elements for the proton-neutron

¹⁵ The simplest method for this is described in Section 4 of reference 3. (Cf. Table I there.)

interaction must be calculated separately for each case.

It is well known that all potential energy integrals for the p^n configuration can be expressed in terms of the following two,

$$L = \int \cdots \int (x/r)^2 R_p(r)^2 (x'/r')^2 R_p(r')^2 \times J(|r-r'|) d\tau d\tau',$$

$$K = \int \cdots \int (xy/r^2) R_p(r)^2 (x'y'/r'^2) \times R_p(r')^2 J(|r-r'|) d\tau d\tau' \quad (9)$$

by means of the identity

$$L - 2K = \int \cdots \int (x/r)^2 R_p(r)^2 (y'/r')^2 \times R_p(r')^2 J(|r-r'|) d\tau d\tau', \quad (10)$$

where $(x/r)R_p(r)$ denotes the p wave function. If there is any ambiguity we shall use L and K with the index $\nu\pi$ if J signifies the interaction between unlike particles, with an index $\nu\nu$ if it is the interaction between like particles, and with an index c if J is the electrostatic potential.

The matrix elements for the interaction energies between like particles are tabulated in Table III.

Table IV contains the matrix elements

TABLE IV. Matrix elements of the interaction between unlike particles.¹⁶

Li ⁶				Be ⁹					
^(2P²P) 2 ² S	$L+2K,$	^(2P²P) 2 ² D	$L-K,$	^(2P²P) 2 ² P	$-L+3K$	^(1S²P) 1 ² P	^(1D²P) 1 ² P	^(1D²D) 1 ² P	
Li ⁷				Be ¹⁰					
^(2P¹D) 2 ¹ F	$2L-2K,$	^(2P²P) 2 ² P	$L+2K,$	^(2P²P) 2 ² D	$L,$	^(1S²P) 1 ² P	^(1D²P) 1 ² P	^(1D²D) 1 ² P	
^(2P¹D) 2 ¹ D	$-L+4K,$	^(2P²P) 2 ² S	$-2L+6K,$			^(1D²D) 1 ² D	^(1D²D) 1 ² D	^(1D²P) 1 ² D	
Be ⁸				Be ¹⁰					
		^(2P¹S) 2 ¹ P			^(2P¹D) 2 ¹ P				
^(2P¹S) 2 ¹ P	^(2P¹D) 2 ¹ P	$2(L+2K)/3$	$(20)\frac{1}{2}(L-K)/3$	$2(L+4K)/3$	$(20)\frac{1}{2}(L-K)/3$				
Be ⁸				Be ¹⁰					
		^(1S¹S) 1 ¹ S ₊			^(1D¹D) 1 ¹ S ₊				
^(1S¹S) 1 ¹ S ₊	^(1D¹D) 1 ¹ S ₊	$4(L+2K)/3$	$(80)\frac{1}{2}(L-K)/3$	$(80)\frac{1}{2}(L-K)/3$	$2(L+14K)/3$				
Li ⁸				Be ¹⁰					
		^(1S¹D) 1 ¹ D ₊			^(1D¹D) 1 ¹ D ₊				
^(1S¹D) 1 ¹ D ₊	^(1D¹D) 1 ¹ D ₊	$4(2L+K)/3$	$(56)\frac{1}{2}(L-K)/3$	$(56)\frac{1}{2}(L-K)/3$	$(-2L+23K)/3$				
Li ⁸				Be ¹⁰					
		^(1D¹D) 1 ¹ G ₊	$4L-4K.$						
Li ⁸				Be ¹⁰					
		^(2P²P) 2 ² P			^(2P²D) 2 ² P				
^(2P²P) 2 ² P	^(2P²D) 2 ² P	$(L+7K)/2$	$(15)\frac{1}{2}(L-K)/2$	$(15)\frac{1}{2}(L-K)/2$	$(-L+13K)/2$				
Be ¹⁰				Be ¹⁰					
		^(2P²D) 2 ² D			^(2P²P) 2 ² D				
^(2P²D) 2 ² D	^(2P²P) 2 ² D	$(L+7K)/2$	$3\frac{1}{2}(L-K)/2$	$3\frac{1}{2}(L-K)/2$	$(3L+K)/2$				
Be ¹⁰				Be ¹⁰					
		^(2P²D) 2 ² F	$2L-K.$						
Be ⁹				Be ¹⁰					
		^(1S²P) 1 ² P	$2L+4K$	0	$(20/3)\frac{1}{2}(L-K)$	0	$(L+17K)/2$	$-(49/12)\frac{1}{2}(L-K)$	$-(20/3)\frac{1}{2}(L-K)$
^(1D²P) 1 ² P	^(1D²D) 1 ² P	$(20/3)\frac{1}{2}(L-K)$	$-(49/12)\frac{1}{2}(L-K)$	$-(49/12)\frac{1}{2}(L-K)$	$(-L+23K)/2$				
Be ¹⁰				Be ¹⁰					
		^(1S²D) 1 ² D	$2L+4K$	0	$2(L-K)$	0	$(L+17K)/2$	$(7/4)\frac{1}{2}(L-K)$	$2(L-K)$
^(1D²D) 1 ² D	^(1D²P) 1 ² D	$2(L-K)$	$(7/4)\frac{1}{2}(L-K)$	$(7/4)\frac{1}{2}(L-K)$	$(3L+11K)/2$				
Be ¹⁰				Be ¹⁰					
		^(1D²P) 1 ² F			^(1D¹D) 1 ² F				
^(1D²P) 1 ² F	^(1D²D) 1 ² F	$3L+K$	$-2\frac{1}{2}(L-K)$	$-2\frac{1}{2}(L-K)$	$2L+4K$				
Be ¹⁰				Be ¹⁰					
		^(1D²D) 1 ² G	$4L-2K.$						
Be ¹⁰				Be ¹⁰					
		^(1S¹S) 1 ¹ S			^(1D¹D) 1 ¹ S				
^(1S¹S) 1 ¹ S	^(1D¹D) 1 ¹ S	$8(L+2K)/3$	$-(80)\frac{1}{2}(L-K)/3$	$-(80)\frac{1}{2}(L-K)/3$	$(-8L+50K)/3$				
Be ¹⁰				Be ¹⁰					
		^(1S¹D-1D¹S) 1 ¹ D	$4L+4K$	0	$(4L+20K)/3$	0	$(56)\frac{1}{2}(L-K)/3$	0	$(56)\frac{1}{2}(L-K)/3$
^(1S¹D-1D¹S) 1 ¹ D	^(1S¹D+1D¹S) 1 ¹ D	$(4L+20K)/3$	$(56)\frac{1}{2}(L-K)/3$	$(56)\frac{1}{2}(L-K)/3$	$(5L+28K)/3$				
Be ¹⁰				Be ¹⁰					
		^(1D¹D) 1 ¹ F	$4L+4K,$	$(1D1D)11G$	$4L.$				
Be ¹⁰				Be ¹⁰					
		^(2P²P) 2 ² S ₊			^(2D²D) 2 ² S ₊				
^(2P²P) 2 ² S ₊	^(2D²D) 2 ² S ₊	$2L+9K$	$(15)\frac{1}{2}(L-K)$	$15K$	$15K$				
Be ¹⁰				Be ¹⁰					
		^(2P²P) 2 ² D ₊	$(7L+9K)/2$	0	$(21)\frac{1}{2}(L-K)/2$	0	$(21)\frac{1}{2}(L-K)/2$	0	$(21)\frac{1}{2}(L-K)/2$
^(2P²P) 2 ² D ₊	^(2P²D+2D²P) 2 ² D ₊	$(21)\frac{1}{2}(L-K)/2$	$5L+5K$	0	$(3L+21K)/2$				
Be ¹⁰				Be ¹⁰					
		^(2D²P+2P²D) 2 ² F ₊	$5L+2K,$	$(2D2D)22G+$	$5L.$				

The + sign signifies that the wave function remains unchanged if the proton coordinates are interchanged with the neutron coordinates. This quantum number exists only for elements with equal numbers of protons and neutrons and also in this case only if the multiplicity for both is the same. If the multiplicities for protons and neutrons are different, the + term coincides with the - term and the degeneracy is doubled.

¹⁶ The matrix elements for Li⁶ are given by Bethe and Bacher, reference 8, §36.

for the proton-neutron interaction. The symbol $({}^1D^3P)13F$ denotes the wave function with the azimuthal quantum number 3 which can be constructed out of the 1D protonic and the 3P neutronic wave functions.

Table IV contains all matrix elements for Li^6 and Li^7 , but for the other nuclei only those which are necessary for the calculation of the "low terms" (as determined by the equal orbital forces approximation). The matrix elements $\bar{V}_{\kappa\lambda}$ for the nucleus with $6-n_1$ protons and $6-n_2$ neutrons are obtained from the corresponding matrix elements $V_{\kappa\lambda}$ of the nucleus with n_1 protons and n_2 neutrons by means of the relation

$$\bar{V}_{\kappa\lambda} = V_{\kappa\lambda} + (6-n_1-n_2)(2L+4K)\delta_{\kappa\lambda}, \quad (11)$$

which can be derived from the theory of holes. Thus we have all the matrix elements for the Majorana forces.

The next task is the calculation of the integrals L and K . Because of

$$\int \dots \int (xy' - x'y)^2 / r^2 r'^2 \cdot R_p(r)^2 R_p(r')^2 \times J(|r-r'|) d\tau d\tau' = 2L - 6K > 0 \quad (12)$$

we know that at any rate $3K < L$. The actual calculation of the ratio and absolute values of L and K is given in the appendix. It is clear that these quantities depend principally on the radius r_p of the p shell and on the range r_0 of the nuclear forces. If r_0/r_p is very large, K will vanish. This corresponds to the approximation made in Eqs. (7) and (8). The appendix shows that K , though not zero, is 7 to 11 times smaller than L .

For the calculation of energies of terms which do not appear as solutions of secular equations, the interaction energy of like particles has to be added directly to the matrix element of Table IV. In the other cases, the like particle interaction matrix elements must be added to the diagonal elements of the secular determinant of Table IV. It must be remembered that the L and K in Table III are not identical with the corresponding quantities in Table IV. If, however, we suppose that the two different L 's are equal and set $K=0$, the result must be that given by Eq. (8). This shows that in the equal orbital forces model, the differences between the "low terms" are due to the finite value of K . With the

other models the numerical results for the spacing of the "low terms" in a multiplet depend essentially on the value of K and hardly at all on the L/K ratio.

The actual solutions of the secular equations are given in Table V in units of mc^2 for the terms which are "low" in the equal orbital forces approximation. Although the term differences are independent of the choice of like particle interaction, we do not believe them to be correct (because of the use of the Hartree method). However, we think that the order of the terms will be right. The situation is the same in the corresponding atomic spectra.⁶

The symbols (i) and (ii) designate different methods of making the numerical calculations which are fully described in the appendix. Models (2a) and (2b) (and consequently also model (2c)) give the same order and spacing of the "low terms" within a multiplet. The spacing is also practically independent of the method of calculation, because K has nearly the same value in both methods. Fig. 1 gives the calculated total energy (method (ii)) of the lowest term and the experimental binding energy. Only the most stable nucleus with a given total mass is shown on the diagram. To obtain the theoretical curve a linear function $c(n_1+n_2)+d$ is added to the computed total energy and the constants c and d determined to fit the experimental points at He^4 and O^{16} . The observed "4 shell" structure is clearly evident in the computed curve, although the wave functions used do not correspond to preformed alpha-particles. In general the agreement is somewhat worse than for the similar calculation in atomic spectra.¹⁷

For the comparison of the calculated excited levels with experiment there is available only a limited amount of experimental material. P. Savel¹⁸ and W. Bothe¹⁹ have investigated the gamma-radiation associated with some nuclear transmutations. Unfortunately, Savel's results do not allow any conclusions to be drawn as to the character of the excited states. Bothe seems to have found three states in the C^{12} nucleus, the positions of which agree fairly well with our "low" terms. However an interpretation of Bothe's terms on these lines hardly seems

¹⁷ R. Peierls, Zeits. f. Physik **59**, 738 (1929).

¹⁸ P. Savel, Ann. d. Physik **4**, 88 (1935).

¹⁹ W. Bothe, Zeits. f. Physik **100**, 273 (1936).

TABLE V. Calculated p shell potential energies in units mc^2 .

TERM	(2a)		(2b)		TERM	(2a)		(2b)	
	(i)	(ii)	(i)	(ii)		(i)	(ii)	(i)	(ii)
Li ⁶					Be ¹⁰ or C ¹⁰				
²² S	-18.6	-12.7	-18.6	-12.7	¹¹ S	-90.0	-62.6	-104.7	-69.6
²² D	-14.3	-8.4	-14.3	-8.4	¹¹ D	-87.6	-60.2	-102.3	-67.2
²² P	11.5	5.5	11.5	5.5	¹¹ D	-86.8	-59.4	-101.5	-66.4
For N ¹⁴ add	-137.1	-105.7	-195.9	-133.6	¹¹ F	-84.1	-56.7	-98.8	-63.7
Li ⁷					B ¹⁰				
¹² P	-44.0	-28.4	-44.0	-28.4	²² S	-96.9	-67.6	-111.6	-74.6
¹² F	-37.8	-22.1	-37.8	-22.1	²² D	-92.9	-63.5	-107.6	-70.5
For C ¹³ or N ¹³ add	-102.8	-79.3	-146.9	-100.2	²² D	-92.5	-63.0	-107.2	-70.0
Li ⁸					C ¹⁴ or O ¹⁴				
²² P	-39.2	-27.4	-46.5	-30.9	¹¹ S	-148.9	-113.7	-207.7	-141.6
²² D	-36.2	-23.9	-43.5	-27.4	¹¹ D	-146.2	-111.0	-205.0	-138.9
²² F	-32.8	-22.4	-40.1	-25.9	N ¹⁵ or O ¹⁵				
For B ¹² add	-68.5	-52.9	-97.9	-66.8	¹² P	-171.4	-132.2	-244.9	-167.1
Be ⁸					O ¹⁶				
¹¹ S	-88.1	-56.8	-88.1	-56.8	¹¹ S	-205.6	-158.6	-293.8	-200.5
¹¹ D	-84.3	-53.0	-84.3	-53.0	Be ⁹ or B ⁹				
¹¹ G	-75.6	-44.1	-75.6	-44.1	¹² P	-83.1	-55.8	-90.5	-59.3
For C ¹² add	-68.5	-52.9	-97.9	-66.8	¹² D	-80.7	-53.3	-88.1	-56.8
Be ⁹ or B ⁹					¹² F	-77.0	-49.5	-84.4	-53.0
¹² P	-83.1	-55.8	-90.5	-59.3	¹² G	-72.0	-44.4	-79.4	-47.9
¹² D	-80.7	-53.3	-88.1	-56.8	For B ¹¹ or C ¹¹ add	-34.3	-26.4	-49.0	-33.4
¹² F	-77.0	-49.5	-84.4	-53.0	Be ⁹ or B ⁹				
¹² G	-72.0	-44.4	-79.4	-47.9	¹² P	-83.1	-55.8	-90.5	-59.3
For B ¹¹ or C ¹¹ add	-34.3	-26.4	-49.0	-33.4	¹² D	-80.7	-53.3	-88.1	-56.8

feasible, because the transition from ¹¹G to ¹¹S should be so much less probable than the transition to the ¹¹D level that the gamma-ray corresponding to the former transition should be unobservable. Hence, if our results are correct, at least one of his lines must have another interpretation. A possible interpretation is that two of the lines observed by Bothe come from the transitions ¹¹G-¹¹D, ¹¹D-¹¹S and the third from another excited level which happens to be near ¹¹G. Such an excited level might arise from an excited configuration. This interpretation gains in plausibility from the fact that N¹⁵ has an excited state^{20, 21} with an excitation energy of 10.5 mc^2 . Since the normal state configuration of

²⁰ E. O. Lawrence, E. McMillan and M. C. Henderson, Phys. Rev. **47**, 273 (1935).

²¹ J. D. Cockcroft and W. B. Lewis, Proc. Roy. Soc. **A154**, 261 (1935).

N¹⁵ has only one term, all the excited states must be ascribed to configurations in which one or more particles are excited. The occurrence of short and long range groups of particles together in transmutations in which C¹² is produced confirms the existence of an excited level about 10 mc^2 above the normal state.^{20, 21, 22} Cockcroft and Lewis²¹ find two excited states in B¹¹, one of which falls between the levels ¹²G, ¹²F and the other between the levels ¹²F, ¹²D of the theoretical calculation. Bonner and Brubaker²² observe three excited states in B¹⁰ with the energy 8.4 mc^2 available to produce excitation. The theoretical calculation shows three triplet levels and one singlet level in this range.

If we assume in analogy with H² that there

²² T. W. Bonner and W. M. Brubaker, Phys. Rev. **50**, 308 (1936).

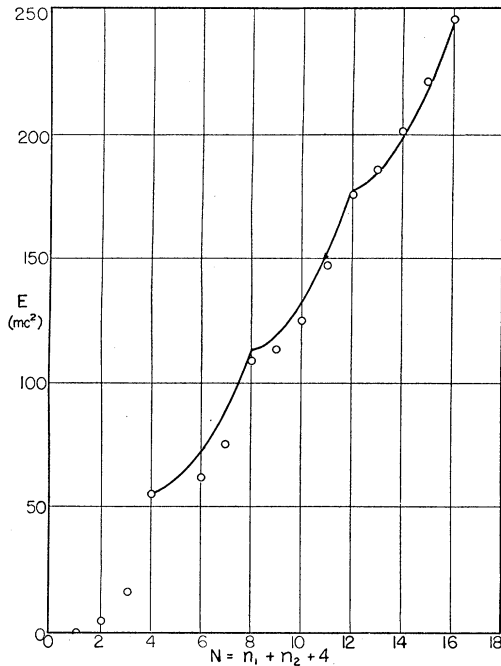


FIG. 1. Binding energy against nuclear mass. The calculated values lie on the solid curve; the circles mark experimental points.

are forces which tend to direct the proton spin parallel to the neutron spin (the Heisenberg forces will be shown to have such an effect), we obtain Table VI for the total angular momenta of the ground states. The ambiguity, in the cases for which several possible values of the angular momentum are given, is caused by the lack of knowledge concerning forces coupling the spin to the orbit. Experimentally the following momenta are known: Li^6-1 , $\text{Li}^7-3/2$, $\text{C}^{12}-0$, $\text{N}^{14}-1$, $\text{O}^{16}-0$. These agree with the theory.

III. THE HEISENBERG FORCE BETWEEN UNLIKE PARTICLES

The comparatively large singlet-triplet separation in H^2 ($\sim 4 \text{ mc}^2$) points to an interaction between protons and neutrons which involves the spin in some way. The simplest assumption for such an interaction—though not the only possible one—is the Heisenberg force (1b) which involves the exchange of both Cartesian and spin coordinates of the interacting particles.

It is characteristic of this interaction that it is invariant with respect to the rotation of all

spin coordinates—though it is not invariant with respect to the rotation of the proton spin alone any more, as were the Majorana forces. Consequently, it will leave the total spin angular momentum an exact quantum number, not, however, the proton spin and the neutron spin separately.

By Dirac's identity the exchange of the spin coordinates can be written

$$Q_{12} = \frac{1}{2} + \frac{1}{2}(\sigma_1 \cdot \sigma_2). \quad (13)$$

The first part gives matrix elements which are, apart from the factor $g/(1-g)$, just half as great as the matrix elements of the Majorana force, thus effectively increasing this by the factor $(1-\frac{1}{2}g)/(1-g)$. The matrix elements of the second part vanish by the selection rules if both wave functions have a proton spin 0 or a neutron spin 0 or if the proton spins or the neutron spins differ by 2 or more. In these cases the addition of the Heisenberg force amounts only to an increase of the Majorana force by the factor $(1-\frac{1}{2}g)/(1-g)$. For most purposes it is more practical to unite the first part of the Heisenberg force with the Majorana force and consider the rest as an interaction involving P , the interchange of the Cartesian coordinates, and the scalar product of the spin operators. The effect of the first part of the Heisenberg force is numerically much greater than the second in most cases. It has been included in Tables IV and V and in Fig. 1 by calculating with a modified form of (1a) in which g is replaced by $\frac{1}{2}g$.

The splitting of a term with proton multiplicity s and neutron multiplicity σ is caused by the second part only. There are new components with the total spin angular momenta $S = |s-\sigma|, |s-\sigma|+1, \dots, s+\sigma$ the displacement of the term with the total spin S being proportional to

$$s(s+1) + \sigma(\sigma+1) - S(S+1). \quad (14)$$

The matrix elements of the second part of (1b) for the ground state when they do not vanish

TABLE VI. Angular momenta for normal states.

Li^6 1	Li^7 $\frac{1}{2}, \frac{3}{2}$	Li^8 0, 1, 2	Be^9 0	Be^9 $\frac{1}{2}, \frac{3}{2}$	B^9 $\frac{1}{2}, \frac{3}{2}$	Be^{10} 0	B^{10} 1	C^{10} 0
B^{11} $\frac{1}{2}, \frac{3}{2}$	C^{11} $\frac{1}{2}, \frac{3}{2}$	B^{12} 0, 1, 2	C^{12} 0	C^{13} $\frac{1}{2}, \frac{3}{2}$	N^{13} $\frac{1}{2}, \frac{3}{2}$	N^{14} 1	N^{15} $\frac{1}{2}, \frac{3}{2}$	O^{16} 0

(in all these cases $s = \sigma = \frac{1}{2}$) are given in Table VII for the triplet state only, since the elements for the singlet state can be obtained by multiplication with -3 . Here L and K are identical with the same symbols in Table IV. These matrix elements must be multiplied by the factor $g/2(1 - \frac{1}{2}g)$ and added to the corresponding effective Majorana matrix elements in Table IV.

The numerical calculation (method (i)) yields a splitting of 9 mc^2 for both Li^6 and B^{10} (and thus also for N^{14}), the triplet term being always the lowest. Method (ii) gives a splitting of 6.3 mc^2 for Li^6 and N^{14} and of 6.6 mc^2 for B^{10} . For Li^8 both methods give very little splitting so that it is impossible, in this case, to tell what should be the normal state.

Since the azimuthal quantum number and the resultant spin angular momentum both remain good quantum numbers, the Heisenberg force does not introduce a splitting of the 3P or similar terms into the fine structure components 3P_0 , 3P_1 , 3P_2 . It has been suggested by D. R. Inglis (at the summer symposium at Ann Arbor, as yet unpublished) that the spin orbit coupling should have the same sign (giving the larger j the lower energy) for both neutrons and protons in nuclei. A relativistic (Thomas) term results from the acceleration of the particles by the nuclear forces. According to Inglis this is the preponderating spin orbit coupling term in nuclei.

IV. THE COULOMB ENERGY

A further question of interest is that concerning the difference between proton-proton and neutron-neutron interaction. The simplest assumption is that the only difference is the electrostatic interaction between protons.

According to this assumption the binding energy and the whole spectrum of two nuclei

TABLE VII. Matrix elements of the second part of the Heisenberg force.

Li^6	$({}^2P^2P) {}^2S$	$L+2K$
Li^8	$({}^2P^2P) {}^3P$	$({}^2P^2D) {}^3P$
$({}^2P^2P) {}^3P$ $({}^2P^2D) {}^3P$	$(-L+3K)/2$ $(15)^{\frac{1}{2}}(L-3K)/6$	$(15)^{\frac{1}{2}}(L-3K)/6$ $(-5L+11K)/6$
B^{10}	$({}^2P^2P) {}^3S$	$({}^2D^2D) {}^3S$
$({}^2P^2P) {}^3S$ $({}^2D^2D) {}^3S$	$(2L-K)$ $(15)^{\frac{1}{2}}(-L+3K)/3$	$(15)^{\frac{1}{2}}(-L+3K)/3$ $(8L-5K)/3$

should differ only in the Coulomb energy if the number of neutrons in the first element is equal to the number of protons in the second element and *vice versa*. Known pairs of this type are $\text{H}^2 - \text{He}^3$, $\text{Be}^9 - \text{B}^9$, $\text{B}^{11} - \text{C}^{11}$, $\text{C}^{13} - \text{N}^{13}$, $\text{N}^{15} - \text{O}^{15}$, $\text{O}^{17} - \text{F}^{17}$.

For He^3 there already exists a simple calculation of the Coulomb energy¹ and a more accurate calculation giving substantially the same result has been made by S. S. Share.²³ The computed Coulomb energy for He^3 appears to be about 15 percent smaller than the experimental $\text{H}^2 - \text{He}^3$ energy difference. For the other pairs we have performed the calculation, adding the electrostatic energy to the like particle interaction. The details of the computation are given in the appendix. For all cases in which the positron spectrum has been observed the experimental energy difference is obtained as a sum of three terms: (1) the upper limit of the positron spectrum (in electron mass units, taking the inspection value for the upper limit),²⁴ (2) the $n^1 - \text{H}^1$ mass difference, (3) the mass of two electrons. Table VIII exhibits the electrostatic and experimental differences. The agreement is quite good. Nevertheless the experimental values are always somewhat greater than the theoretical ones. Considering the rather large uncertainty of these calculations, one cannot claim with certainty at present that the neutron-neutron interaction is stronger than the proton-proton interaction.

V. COMPARISON OF ISOBARIC NUCLEI

In this section we wish to make a few remarks concerning the spectra of isobars and the possibility of deciding between the alternatives (2a), (2b) and (2c) for the interaction between like particles.

It is clear from Table V that for the same values of the depth and range of the potentials, (2a) gives much less binding than (2b). Hence using the statistical method one obtains a smaller discrepancy with (2b) than with (2a). This cannot be considered, however, to be decisive evidence in favor of (2b), since it is possible that

²³ S. S. Share, Phys. Rev. **50**, 488 (1936).

²⁴ W. A. Fowler, L. A. Delsasso and C. C. Lauritsen, Phys. Rev. **49**, 561 (1936); F. N. D. Kurie, J. R. Richardson and H. C. Paxton, Phys. Rev. **48**, 167 (1935); L. Meitner, Nature **22**, 420 (1934).

TABLE VIII. *Electrostatic and measured energy differences.*

	H ³ —He ³	Be ⁹ —B ⁹	B ¹¹ —C ¹¹	C ¹³ —N ¹³	N ¹⁵ —O ¹⁵	O ¹⁷ —F ¹⁷
Electrostat.	1.37	3.11	4.06	4.62	5.54	5.62
Experim.	1.57	3.20	4.90	5.10	6.00	6.80

the correlation energy can account for the greater difference between the observed energies and the results of the statistical calculation with (2a). The discrepancy for (2c) is even greater than for (2a), but (2c) also cannot be excluded on this basis.²⁵

In all cases only a small fraction of the binding energy is due to the interaction between like particles. This makes it very difficult to draw conclusions concerning these forces from experimental term values. If we consider, however, the Schrödinger equations of two isobars in the equal orbital forces approximation, their characteristic values will be absolutely equal. Nevertheless, the energy values will not necessarily be equal even in this approximation, because a characteristic value which is allowed for one isobar, may be forbidden for the other by Pauli's principle. Thus the low terms of Be⁸ arise from the partition 4+4 of the equal orbital force approximation, but this gives no allowed terms for Li⁸, the low terms of which come from the 4+3+1 partitions. These are allowed for Be⁸ also, but correspond to excited states. For Be¹⁰ and B¹⁰, however, the low states come in both cases from the 4+4+2 partition, in the first case giving singlet-singlet, in the second doublet-doublet terms. These coincide exactly in the equal orbital forces approximation and their similarity can be seen even in Table V. If one now introduces the difference between (2) and (1a) as a perturbation, the Be¹⁰, B¹⁰ energy separation will appear as a difference of two small quantities.

Experimentally the binding energy of Be¹⁰ is greater than that of B¹⁰ by 0.8 mc². From Table VIII we estimate that the difference between

²⁵ H. A. Bethe and R. F. Bacher, reference 8, §6, have shown that a symmetrical Hamiltonian gives stable nuclei consisting of protons or of neutrons alone if one does not take into account the Pauli principle. However the Pauli principle causes the effective interaction between like particles to be smaller than that between unlike particles, even though the fundamental Hamiltonian does not distinguish between like and unlike particles. A good example in this connection is Li⁸ which is much less stable than Be⁸.

the proton-proton and the neutron-neutron interactions (Coulomb energy) has decreased the binding energy by 3.2 mc². Thus the binding energy due to nuclear forces alone is greater for B¹⁰ than for Be¹⁰ by 2.4 mc². The corresponding quantity for the N¹⁴—C¹⁴ pair is 5.1—0.8 = 4.3 mc². *Such small differences are possible only if the constants in the interactions between like and unlike particles are essentially equal.*

With (2a) or (2b) the difference between the Majorana energies of the B¹⁰ and Be¹⁰ ground states is seen from Table V to be 5.0 mc² (method (ii)). This must be increased by the amount 1.6 mc² by which the normal state of B¹⁰ is lowered when account is taken of the second part of the Heisenberg force between unlike particles. This gives a total difference of 6.6 mc². Using (2c) it can be shown that the ground state of Be¹⁰ coincides with the singlet state arising from ²²S in B and hence the difference between the binding energies of Be¹⁰ and B¹⁰ is 6.6 mc² for all three models. The agreement between models (2a), (2b) and (2c) in this case is not due to numerical coincidence, but can be shown to be true for all values of *L* and *K* simply by comparing the like particle contributions to the secular equations of the two systems for each of the three different models.

We see that the experimental difference between the binding energies of Be¹⁰ and B¹⁰ is much smaller than the theoretical difference. For C¹⁴ and N¹⁴ the situation is similar. Both sets of low terms belong to the partition 4+4+2 and hence coincide in the equal orbital forces approximation. Here also the energy difference is independent of the model. From Table V (method (ii)) the C¹⁴—N¹⁴ energy difference is 4.7+1.6=6.3 mc², while experimentally the difference is only 4.3 mc².

In view of these discrepancies it is surprising to find excellent agreement between the computed and experimental Li⁶—He⁶ energy differences. Bjerger and Broström²⁶ report the value 7.5±1.0 mc² for the maximum kinetic energy of the electrons produced in the transformation of He⁶ into Li⁶. Since the additional Coulomb energy in Li⁶ is almost exactly balanced by the greater mass of the constituents of He⁶, we

²⁶ T. Bjerger, Nature **138**, 400 (1936), and T. Bjerger and K. J. Broström, Nature **138**, 400 (1936).

obtain 7.5 ± 1.0 mc² for the experimental singlet-triplet splitting in Li⁶. The computed value is 6.3 mc² (method (ii)) or 9.0 mc² (method (i)).

The secular equations for the states of Be⁸ corresponding to the 4+3+1 partitions are given in Table IX.

We obtain -1.0 mc² for the (Li⁸) ³P – (Be⁸) ¹³P energy difference using model (2a) and -3.4 mc² using model (2b). These numbers, added to the experimental normal state energy difference, yield the results 21 mc² (2a) or 24.4 mc² (2b) for the excitation energy of the ¹³P term in Be⁸. This term could be observed directly since it is stabilized by the spin conservation law.²⁷ A similar result can be deduced for C¹² also, by comparison with B¹², since the experimental binding energy difference is nearly the same. It should be noted that method (ii) gives a Li⁸–Be⁸ normal state spacing in fair agreement with experiment. Here, as well as in the problems of the “four shell” structure and the singlet-triplet separation, method (ii) gives better agreement with the experimental facts than method (i).

APPENDIX

We assume that each particle has a separate wave function (Hartree-Fock model) and, furthermore, that this has the form

$$R_s(r) = ce^{-r^2/r_s^2} = ce^{-\alpha\sigma r^2} \quad (15)$$

for the s particles and

$$\begin{aligned} (x/r)R_p(r), \quad (y/r)R_p(r), \quad (z/r)R_p(r), \\ R_p(r) = c'r e^{-r^2/r_p^2} = c'e^{-\tau\alpha r^2} \end{aligned} \quad (16)$$

for the p particles, where α is the reciprocal square of the range of the forces and is defined in (1).

It is well known that if a determinantal wave function for the whole system is built up out of these functions and the energy calculated, choosing the parameters σ and τ so as to make the energy a minimum, the absolute value of this minimum will be very small and the corre-

²⁷ The spin conservation law is practically rigorous for nuclei if the singlet-triplet splitting is produced by a force of the form (1b). We should mention at this occasion that it is not sufficient for the validity of such a conservation law that the forces acting on the spin be small. It is necessary, in addition to this, that the separation of states with different spin should not be small also. For atomic spectra, this follows from the fact that the singlet-triplet separation is of electrostatic nature, while the spin forces are magnetic. For nuclei, if (1b) is correct, the singlet-triplet separation is due to the Heisenberg force, while the spin forces still are, under this assumption, of magnetic nature. Thus if (1b) is correct, the situation is much better for the spin law in nuclei than in the external shells. It is invalidated, however, by any force, coupling the spin to orbit, as e.g. that assumed by Inglis (cf. Section III, end).

TABLE IX. Continuation of Table V. Be⁸.

	(¹ S ³ P) ¹³ P	(¹ D ³ P) ¹³ P
(¹ S ³ P) ¹³ P	$4(L+2K)/3$	$-(20)^{\frac{1}{2}}(L-K)/3$
(¹ D ³ P) ¹³ P	$-(20)^{\frac{1}{2}}(L-K)/3$	$-(4L-25K)/3$
(¹ D ³ P) ¹³ D	$2L+3K,$	(¹ D ³ P) ¹³ F $2L$

sponding radii greater than the radius of the nucleus. The reason for this discrepancy is that it is impossible to express the finer statistical correlations between the positions of the particles assuming a separate wave function for each particle. In order to take into account at least approximately the correlation energy, a semi-empirical method is used, modifying the Hamiltonian in such a way that it gives the experimental binding energy for O¹⁶.

The kinetic energy per particle is $(3/2)\alpha\sigma$ and $(5/2)\alpha\tau$ for the s and p particles, respectively. For a system with N particles the total kinetic energy is taken to be

$$(1-1/N)(6\sigma-10\tau+(5/2)N\tau)\alpha. \quad (17)$$

The factor $(1-1/N)$ serves to eliminate the spurious kinetic energy of the center of gravity which arises from the use of single particle wave functions with coordinates measured from a fixed point.²⁸ For the Coulomb energy we obtain

$$\begin{aligned} CE(s) &= (\alpha\sigma/8)^{\frac{1}{2}}, \quad (\text{within the } s \text{ shell}), \\ CE(sp) &= (1/3)(\sigma\tau)^{\frac{1}{2}}(\sigma+\tau)^{-\frac{3}{2}}(3\tau+2\sigma-2\tau^2\sigma(\sigma+\tau)^{-2})\alpha^{\frac{1}{2}}, \end{aligned} \quad (18)$$

(between the s shell and each proton in the p shell),

$$\begin{aligned} L_c &= (49/120)(\alpha\tau/2)^{\frac{1}{2}}, \\ K_c &= (1/40)(\alpha\tau/2)^{\frac{1}{2}}. \end{aligned} \quad (19)$$

If we use the relation $A_{\nu\nu} \sim (1-2g)A_{\nu\pi}$ the total potential energy in the s shell is

$$6(1-g)A_{\nu\pi}B(\sigma), \quad B(\sigma) = (\sigma/\sigma+1)^{\frac{3}{2}}, \quad (20)$$

for all three forms of the like particle interaction.

The total interaction energy between the s shell and a single particle in the p shell (including both like and unlike particle forces) has the form

$$\begin{aligned} 3(1-g)A_{\nu\pi}D + (1-2g)A_{\nu\pi}(D-C), \quad (1a+2a), \\ 3(1-g)A_{\nu\pi}D, \quad (1a+2b), \\ 3(1-g)A_{\nu\pi}D + (1+g)A_{\nu\pi}(D-C), \quad (1a+2c), \end{aligned} \quad (21)$$

$$\begin{aligned} \text{with } D(\sigma, \tau) &= 16\sigma^{\frac{3}{2}}(\tau/(\sigma+\tau))^{5/2}(\sigma+\tau+2)^{-5/2}, \\ C(\sigma, \tau) &= (2\sigma)^{\frac{3}{2}}\tau^{5/2}(1+2\sigma)(2\sigma\tau+\sigma+\tau)^{-5/2}. \end{aligned} \quad (22)$$

Within the p shell

$$\begin{aligned} L_{\nu\pi} &= A_{\nu\pi}(1-g/2)L^0, \quad L_{\nu\nu} = A_{\nu\pi}(1-2g)L^0, \\ & \quad (\text{for } 2a \text{ and } 2b), \\ K_{\nu\pi} &= A_{\nu\pi}(1-g/2)K^0, \quad K_{\nu\nu} = A_{\nu\pi}(1-2g)K^0, \\ L^0 &= (\tau/\tau+1)^{\frac{3}{2}}\{1-(4\tau+1)/4(\tau+1)^2\} \\ K^0 &= (\tau/\tau+1)^{\frac{3}{2}}/4(\tau+1)^2. \end{aligned} \quad (23)$$

In computing Eqs. (21) and (22) the Majorana interaction between unlike particles is given the depth $(1-g/2)A_{\nu\pi}$ in order to take into account part of the Heisenberg interaction (as discussed in Section III).

It appears to be sensible to try to compensate for the correlation energy by either multiplying the potential

²⁸ S. Flügge, reference 7.

energy by a suitable constant,^{7,8} or by allowing the particles to interact at a greater distance than they really do. With (2b) we found that multiplying the potentials by 1.36 yields a function for the sum of all energies which has a minimum value of -246 mc^2 at $\sigma=0.80$ (taking $\tau=\sigma$ since very little improvement is obtained from varying both τ and σ independently). The second method (method (i)) is believed to be somewhat more reliable. With (2b) the radius of action must be increased by the factor 1.3 to make the minimum value of $E(\text{O}^{16})$ agree with experiment. The minimum is assumed at $\sigma=1.01$ (again taking $\sigma=\tau$ for the same reason as before). To obtain the numerical results discussed in the text we use the values

$$\begin{aligned} L_{\nu\pi} &= 15.74, & K_{\nu\pi} &= 1.41, \\ L_{\nu\nu} &= 10.05, & K_{\nu\nu} &= 0.90, \\ L_c &= 0.892, & K_c &= 0.055, & \text{Method (i).} & (24) \\ CE(s) &= 1.09, & CE(ps) &= 1.63, \\ & & 3(1-g)D(\sigma, \sigma)A_{\nu\pi} &= 14.95, \\ & & 6(1-g)B(\sigma)A_{\nu\pi} &= 120.00. \end{aligned}$$

In computing the kinetic and Coulomb energies α is replaced by $\alpha/(1.3)^2$.

The straight-forward application of this procedure to the models (2a) and (2c) is not very satisfactory, because an increase in the strength of the potentials or in the radius of action which serves to fit the experimental O^{16} energy will give far too much energy to the lighter elements.

Finally it was found most satisfactory to make the calculations without modifying either the range or depth of the potentials. With $\sigma=1.01/(1.3)^2$ the kinetic and Coulomb energies are exactly as in method (i). The other matrix elements have the values

$$\begin{aligned} L_{\nu\pi} &= 9.83, & K_{\nu\pi} &= 1.44, \\ L_{\nu\nu} &= 6.19, & K_{\nu\nu} &= 0.90, & \text{Method (ii).} & (25) \\ & & 3(1-g)D(\sigma, \sigma)A_{\nu\pi} &= 12.13, \\ & & 6(1-g)B(\sigma)A_{\nu\pi} &= 77.37. \end{aligned}$$

The correlation energy is introduced by adding to the total computed energy a linear function of the number of particles. Since the general linear function contains two parameters it is possible in this way to fit exactly the measured binding energies of two different nuclei, i.e., He^4 and O^{16} . Then the energies of all the others are uniquely determined.

On the Consequences of the Symmetry of the Nuclear Hamiltonian on the Spectroscopy of Nuclei

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The structure of the multiplets of nuclear terms is investigated, using as first approximation a Hamiltonian which does not involve the ordinary spin and corresponds to equal forces between all nuclear constituents, protons and neutrons. The multiplets turn out to have a rather complicated structure, instead of the S of atomic spectroscopy, one has three quantum numbers S, T, Y . The second approximation can either introduce spin forces (method 2), or else can discriminate between protons and neutrons (method 3). The last approximation discriminates between protons and neutrons in method 2 and takes the spin forces into account in method 3. The method 2 is worked out schematically and is shown to explain qualitatively the table of stable nuclei to about Mo.

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RECENT investigations¹ appear to show that the forces between all pairs of constituents of the nucleus are approximately equal. This makes it desirable to treat the protons and neutrons on an equal footing. A scheme for this was devised in his original paper by W. Heisen-

berg² who considered protons and neutrons as different states of the same particle. Heisenberg introduced a variable τ which we shall call the isotopic spin, the value -1 of this variable can be assigned to the proton state of the particle, the value $+1$ to the neutron state. The assumption that the forces between all pairs of particles are equal is equivalent, then, to the assumption that they do not depend on τ or that the Hamiltonian does not involve the isotopic spin.

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