case the numbers of neutrons and protons differ by more than 1, only a fraction of the configurations will occur in both Ψ_i and Φ_f with large coefficients.

We can illustrate this behavior by the following simple consideration. Let us assume that for the function Ψ_i a set of f configurations and for the function Φ_f another set of g configurations is important so that only part of them, in number h, occur in both sets. In the average over all possible distributions of the coefficients, that means all directions of the unit vectors Ψ and Φ in their respectively f and g dimensional spaces, we obtain

$$(M^2)_{\text{Av}}=(a\cdot b)^2=h(a_x{}^2)_{\text{Av}}(b_x{}^2)_{\text{Av}}=h/fg, \quad (16)$$

where x designates an arbitrary component of the corresponding vector.

It seems not possible at the present stage to obtain estimates of the number of configurations which are necessary to give a good approximation for higher nuclei. But we think that the empirical behavior of the matrix elements as found in §2 gives a definite indication about the nature of the wave functions in heavy nuclei. From the crude estimate (16) one would conclude that the number of configurations required for a fair approximation will increase more than linearly with the mass number. A factor of the order 1/100 for the heavy elements would be given by some such values as $f\sim g\sim 20$ of which $h\sim 4$ would coincide, which means that a rather large number of configurations would be necessary as would seem to be required by our other knowledge of the constitution of heavy nuclei.

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On the Structure of Nuclei Beyond Oxygen

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An attempt is made to correlate the kinks in the mass defect curve with the energy differences between isobars, both as obtained from direct measurements and also from the shift of the isotopic number to higher values with increasing number of particles. Since the single-particle picture is known to be an insufficient approximation, the symmetry property of the wave function, resulting from the use of a symmetric Hamiltonian is utilized. The average interaction between symmetrically and antisymmetrically coupled particles ($L+L'$ and $L-L'$) is determined mainly from the kinks in the mass defect curve and enables one to calculate the energy differences between isobars. The energy change at the end of the shell is obtained from experimental data. It should enable one to get some idea of the probabilities with which the particles are in excited configurations. For heavier elements, the formula obtained here should naturally be identical with Weizsäcker's semi-empirical formula and the connection between both is discussed.

(1)

THE extension of the calculations of E. Feenberg and the present author[1] on the spectroscopic characteristics of the normal state and the low excited states to higher elements encounters great computational difficulties. With the one-particle picture, after the $2p$ shell is completed at O^{16}, the $3d$ and $2s$ shells begin to be built up probably simultaneously. Even the normal states of these elements will contain wave functions from several configurations ($3d^n$, $3d^{n-1}2s$, $3d^{n-2}2s^2$, $\cdots$) with about equal coefficients. But even the d^n configuration alone gives rise to a large number of terms with the lowest partition and the explicit calculation of all the matrix elements between these states becomes increasingly difficult. Table I shows[2] the "low terms" of the d^n configurations up to d^4, together

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[1] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); also H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936), and F. Hund, Zeits. f. Physik, to appear soon.

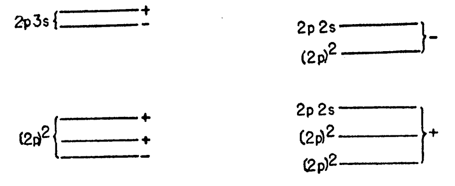
[2] For the preparation of Table I, cf. E. Wigner, Phys. Rev. **51**, 106 (1937), F. Hund, ref. 1. The terms for Table I have been first determined by E. Feenberg (private communication).

with the corresponding partition. In addition to the terms of the d^n configuration, one would have to take into account the terms from the $d^{n-1}s$, $d^{n-2}s^2$ etc. configurations, which are the same as those of the d^{n-1}, d^{n-2}, etc. configurations. Even for Ne^{20} one would have three S terms already. On the basis of the calculations of reference 1, one would expect that the term with the lowest azimuthal quantum number would be the lowest —that is, in all cases an S term. This, however, is certainly not correct for Na^{23}, which has a spin of $\frac{3}{2}$, although it is probably correct for the majority of nuclei.

It is well known also that the single-particle picture is not as good an approximation for nuclei as it is for atomic spectra.[3] Although it is probably at least a useful guiding principle for the determination of the lowest term[4] it may be that even for a light element like B^{10} or N^{14} it leads to erroneous results.[5] The greatest difference between nuclear and atomic spectra in this respect is doubtless the fact that while the terms of a configuration in atomic spectra are rather close together and more or less well separated from other configurations, the situation is exactly opposite in nuclear spectra. Even the lowest configuration spreads out its terms from low energy values to large, positive values. An example of this is given in reference 1, Table V, in the case of Li^6 where all terms (not only the "low terms") of the configuration p^2 are listed. The potential energy of the terms extends from $-12.7\ mc^2$ to $+5.5\ mc^2$, giving an actual repulsion in the latter case. This great difference between atomic and nuclear spectra is due to the fact that the interaction between nuclear constituents is not only a function of the distance but is for the same distance negative or positive according to the symmetric or antisymmetric character of the wave function for an interchange of the two particles. In consequence hereof, those configurations which give the strongest binding, give the strongest repulsion between the particles also. While in atomic spectra the terms of the same configuration are close together, in nuclear spectra this is much rather true of terms of the same symmetry character with respect to permutations, though, of course, for *extremely high* configurations (kinetic energy) the terms of every symmetry class become extremely high. The situation is schematically illustrated in Fig. 1. It is evident from this picture also that the configurations will be least mixed up for the ground state and the shell structure will be most noticeable for this state.

TABLE I. *Low terms of the d^n configurations.*

	d	d^2	d^3	d^4
Terms	D	SDG	$SDFGI$	$SDDGGHIK$
Partition	$4+\cdots+4+1$	$4+\cdots+4+2$	$4+\cdots+4+3$	$4+\cdots+4+4$



Spectrum of O^{++} Spectrum of Li^6 nucleus

FIG. 1. In atomic spectra, the states with the same configuration are usually close together while the effect of the symmetric (+) or antisymmetric (−) character of the wave function is comparatively unimportant. In nuclear spectra, the symmetry character of the wave function is much more important than the configuration. Observed values for O^{++}, crude estimates for the Li nucleus.

(2)

Since a calculation of the potential energies on the basis of the one particle picture is hardly practicable, a calculation on the more general lines of reference 1, Eqs. (4)–(8), suggests itself. Neglecting again spin dependent forces and differences between proton-proton, proton-neutron, and neutron-neutron forces (approximation 1 of reference 2) the wave functions belong to irreducible representations $D(R)$ of the whole permutation group. Consequently, if P_R is the permutation R of the coordinates[6]

$$P_R\psi_\kappa(x_1, \cdots, x_n) = \sum_{\lambda=1}^{f} D(R)_{\lambda\kappa}\psi_\lambda(x_1, \cdots, x_n), \quad (1)$$

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[3] This point has been especially emphasized lately by N. Bohr, Nature **137**, 344 (1936). Cf. also S. Flügge, C. F. v. Weizsäcker, W. Heisenberg, Zeits. f. Physik **96**, 459, 431, 473 (1935) also M. Ostrofsky, G. Breit, D. P. Johnson, Phys. Rev. **49**, 22 (1936).

[4] W. Elsasser, Phys. Rev. **51**, 144 (1937).

[5] According to E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937) the ground states of B^{10} and N^{14} are possibly D states, instead of S states as one would expect on the basis of the single particle picture (reference 1).

[6] Cf. reference 1, Eq. (6). The + sign in this equation is a misprint and should be replaced by an = sign. There is an error also in Table VIII, p. 104 of this paper, insofar as the experimental energy differences are too small by 1 mc^2 throughout the table, excepting the pair H^3-He^3.

where x_i always stands for the three positional coordinates of the ith particle. The ψ are supposed to be orthogonal and normalized and κ, λ take on the values $1, 2, \cdots, f$.

The potential energy is symmetric in the n particles in approximation 1. Hence, it will be the same[7] for all f wave functions $\psi_1, \cdots, \psi_f$ and we can use instead of the potential energy of any of them, the mean value of all of them. Assuming only Majorana interaction, the potential energy V will be

$$\begin{aligned} V &= \sum_{\alpha<\beta}^{n} (\psi_\kappa, J(x_\alpha - x_\beta) P_{\alpha\beta} \psi_\kappa) \\ &= (1/f) \sum_{\kappa=1}^{f} \sum_{\alpha<\beta} (\psi_\kappa, J(x_\alpha - x_\beta) P_{\alpha\beta} \psi_\kappa), \end{aligned} \tag{2}$$

where $\alpha\beta$ is the transposition of α and β so that:

$$\begin{aligned} P_{\alpha\beta}\psi(x_1, \cdots, x_\alpha, \cdots, x_\beta, \cdots) \\ = \psi(x_1, \cdots, x_\beta, \cdots, x_\alpha, \cdots). \end{aligned}$$

Because of (1) the $(\psi_\kappa, J(x_\alpha - x_\beta) P_{\alpha\beta} \psi_\kappa)$ can be all expressed by the $(\psi_\lambda, J(x_1 - x_2) P_{12} \psi_\mu)$

$$\begin{aligned} (\psi_\kappa, J(x_\alpha - x_\beta) P_{\alpha\beta} \psi_\kappa) &= (P_{(\beta 2)(\alpha 1)}\psi_\kappa, P_{(\beta 2)(\alpha 1)} J(x_\alpha - x_\beta) P_{\alpha\beta} \psi_\kappa) \\ &= (P_{(\beta 2)(\alpha 1)}\psi_\kappa, J(x_1 - x_2) P_{12} P_{(\beta 2)(\alpha 1)} \psi_\kappa) \\ &= \sum_{\lambda,\, \mu=1}^{f} D((\beta 2)(\alpha 1))_{\lambda\kappa}{}^* D((\beta 2)(\alpha 1))_{\mu\kappa} \\ &\qquad \times (\psi_\lambda, J(x_1 - x_2) P_{12} \psi_\mu), \end{aligned} \tag{3}$$

where $(\beta 2)(\alpha 1)$ denotes two simultaneous transpositions of β with 2 and of α with 1. This inserted back into (2) gives, because of the unitary character of the representation

$$\begin{aligned} V &= (1/f) \sum_{\alpha<\beta} \sum_{\lambda\mu} \delta_{\lambda\mu} (\psi_\lambda, J(x_1 - x_2) P_{12} \psi_\mu) \\ &= [n(n-1)/(2f)] \sum_{\lambda} (\psi_\lambda, J(x_1 - x_2) P_{12} \psi_\lambda). \end{aligned} \tag{4}$$

This transformation can be performed with an arbitrary interaction of the form $\sum H_{\alpha\beta}$.

Using (1) again, to expand $P_{12}\psi_\lambda$ one obtains

$$V = [n(n-1)/(2f)] \sum_{\lambda\kappa} D(12)_{\kappa\lambda} (\psi_\lambda, J(x_1 - x_2) \psi_\kappa). \tag{5}$$

In reference 1, in order to make qualitative estimates of the integrals occurring on the right side of (4), $J(x_1 - x_2)$ was assumed to extend over a region which is large compared with nuclear dimensions. Then $J(0)$ could be taken out of the integral and this vanished for $\kappa \neq \lambda$. However, this can be done only for purposes of qualitative estimates, since the range of forces is certainly not very large compared with nuclear diameter.

There is a certain arbitrariness in the ψ_κ since all the matrices of the representation can be subjected to a common similarity transformation $D(R) \rightarrow SD(R)S^{-1}$. This will amount to a new choice of the set of wave functions $\psi_1, \cdots, \psi_f$. It is in particular possible in this way to make all ψ either symmetric, or antisymmetric in the first two particles, while still maintaining their orthogonality. Supposing that the first s wave functions $\psi_1, \psi_2, \cdots, \psi_s$ are symmetric in x_1 and x_2 and the rest are antisymmetric, then $D(12)_{\kappa\lambda}$ will be $\delta_{\kappa\lambda}$ if $\kappa \leq s$ and $-\delta_{\kappa\lambda}$ if $\kappa > s$. Thus (5) will become

$$\begin{aligned} V = [n(n-1)/(2f)] \Big[\sum_{\kappa=1}^{s} \int |\psi_\kappa|^2 J(x_1 - x_2) \\ - \sum_{\kappa=s+1}^{f} \int |\psi_\kappa|^2 J(x_1 - x_2) \Big]. \end{aligned} \tag{6}$$

The integrals in this expression can be interpreted in the following way:

$$g_s(x_1 - x_2) = (1/s) \sum_{\kappa=1}^{s} \int |\psi_\kappa|^2 dx_3 \cdots dx_n \tag{7a}$$

is the probability of the distance $|x_1 - x_2|$ for two symmetrically coupled particles and

$$g_a(x_1 - x_2) = (1/(f-s)) \sum_{\kappa=s+1}^{f} \int |\psi_\kappa|^2 dx_3 \cdots dx_n \tag{7b}$$

is the probability of the distance $|x_1 - x_2|$ for two antisymmetrically coupled particles. The expression

$$-\int g_s(x) J(x) dx = L + L' \tag{8a}$$

is the mean interaction for the former, and

$$-\int g_a(x) J(x) dx = L - L' \tag{8b}$$

is the mean interaction for the latter. Necessarily,

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[7] Cf. e.g., E. Wigner, *Gruppentheorie*, etc. (Braunschweig 1931), p. 124–125.

both $L+L'$ and $L-L'$ are positive. L and L' will be positive themselves for smoothly varying functions, J having the same sign throughout. In order to obtain s, we can remember that the matrix $D(12)_{\kappa\lambda}$ contained $+1$ in the main diagonal s times, -1 it contained $f-s$ times. The trace of any matrix corresponding to a transposition is therefore

$$\chi(T)=s-(f-s)=2s-f. \tag{9}$$

The whole potential energy thus reduces to

$$\begin{aligned} V&=-\Xi L-(n(n-1)/2)L', \\ \Xi&=n(n-1)\chi(T)/2f. \end{aligned} \tag{10}$$

For ordinary forces, the potential energy will have the same form (10), only the role of L and L' will be interchanged. Henceforth, L and L' will mean that combination of the left sides of (8a) and (8b) which enter into (10) allowing for ordinary forces. For Wheeler's velocity dependent forces,[8] (10) remains formally valid also, however $L=L'$ in this case. The approximation consisting in using linear combinations of Slater determinants with wave functions belonging to one configuration is a special case of (10) if one drops all small integrals (the K of reference 1) and keeps only the large ones (corresponding to L of reference 1).

Of course, (10) is only a formal expression for the potential energy in nuclei, since L will vary with atomic mass and will be different for different terms of the same element. However, the variation of L and L' can be reasonably assumed to be a smooth function of the number of particles and also of the excitation energy so that the rapid irregular variations may be attributed to Ξ. We shall be confined, thus, in our discussions to the comparison of the binding energies and stability of isobars and the kinks in the mass defect curve. It may be expected that the smoothness of the variation of L will be the better the more the configurations are mixed with each other. It will not be attempted in this paper to use (10) for the total binding energy.

The kinetic energy is usually estimated on the basis of the statistical model. This estimate is certainly too low since the wave function of the statistical model gives the smallest possible kinetic energy for a given radius r of the nucleus. This energy is

$$K=(9/20)(3^{\frac{1}{3}}\pi^{\frac{2}{3}}/2^{\frac{1}{3}})(\hbar^2/(Mr^2))(n_p{}^{5/3}+n_n{}^{5/3}), \tag{11}$$

where n_p is the number of protons, n_n the number of neutrons, r is the nuclear radius. Setting $n_n=\frac{1}{2}n+T_\zeta$ and $n_p=\frac{1}{2}n-T_\zeta$ and developing into a power series of T_ζ, this gives

$$K=[7.05n^{5/3}+15.7T_\zeta{}^2/n^{\frac{1}{3}}+\cdots]mc^2(r/r_e)^{-2}. \tag{12}$$

where $r_e=e^2/mc^2$ is the electronic radius.

If we confine ourselves again to the discussion of isobars and kinks in the mass defect curve, only the second term will be of importance. This gives us the increase in kinetic energy if the number of neutrons is greater than that of the protons. Although there is not much reason to expect that this term is correct in (12), it is not possible to tell whether it is too great or too small and it will therefore be used below.

The second part of (12) will be used as it stands for even n_n and n_p. If n_n or n_p is odd, it seems more appropriate to use the mean value of the corresponding expression for the adjoining even elements. This whole term will be only a small correction for most of the present paper.

The third important quantity for the nuclear binding energy is the electrostatic interaction of the protons. It will be dealt with in detail in Section 4.

(3)

The wave functions discussed so far contain only the positional coordinates of the particles. In order to form the complete wave function they must be completed[2] by functions of the ordinary and isotopic spin coordinates s and τ.

$$\Psi(x_1, s_1, \tau_1, \cdots, x_n, s_n, \tau_n) = \sum_{\kappa=1}^{f}\psi_\kappa(x_1, \cdots, x_n)F_\kappa(s_1, \tau_1, \cdots, s_n, \tau_n) \tag{13}$$

and Ψ will be antisymmetric if F_κ belongs to the κ row of the representation $\epsilon_R D(R)$, adjoint to $D(R)$; ($\epsilon_R=1$ for even and -1 for odd permutations R). In general there are several such sets

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[8] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).

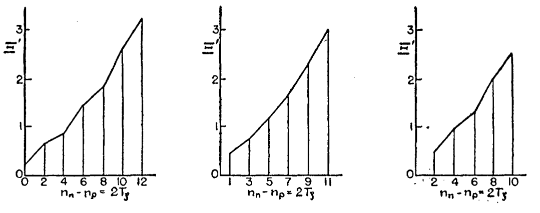

FIG. 2. The coefficient Ξ' of L as function of the difference between the number of protons and neutrons for the three types of nuclei with masses $4k$, $4k\pm 1$, $4k+2$. The plots for $4k$ and $4k+2$ show breaks, the plot for odd nuclei not.

of F_κ, which can be distinguished by three indices S_z, T_ζ, Y_ζ.

$$\begin{aligned}
\tfrac{1}{2}(s_{z1}+\cdots+s_{zn})F_{\kappa M}&=S_zF_{\kappa M},\\
\tfrac{1}{2}(\tau_{\zeta 1}+\cdots+\tau_{\zeta n})F_{\kappa M}&=T_\zeta F_{\kappa M},\\
\tfrac{1}{2}(s_{z1}\tau_{\zeta 1}+\cdots+s_{zn}\tau_{\zeta n})F_{\kappa M}&=Y_\zeta F_{\kappa M}.
\end{aligned} \tag{14}$$

M stands for $S_zT_\zeta Y_\zeta$ and even for given S_z, T_ζ, Y_ζ there may be several sets[9] of F_κ. As long as the ψ_κ are unchanged, all the wave functions (13) with the different S_z, T_ζ, etc., indices for F_κ have the same energy in our approximation 1, which neglects all spin forces, and form a multiplet if one introduces them. The character of the multiplet is determined by the representation $D(R)$ of the original wave functions $\psi_1, \cdots, \psi_f$. From the partition characterizing $D(R)$, the partition $\Lambda_4+\Lambda_3+\Lambda_2+\Lambda_1$ characterizing the adjoint representation $\epsilon_R D(R)$ can be obtained (Eq. (17), reference 2) and the latter partition can be replaced by the (STY) symbol (Eq. (11), reference 2). The S in this latter is the greatest value of S_z for which functions of the s and τ can be found belonging to $\epsilon_R D(R)$; T is the greatest value of T_ζ compatible with the greatest value S of S_z for such functions; and Y is the greatest value of Y_ζ compatible with both. Since the possible sets of $S_zT_\zeta Y_\zeta$ are the same as the possible sets of $T_\zeta S_z Y_\zeta$ or of any permutation of these numbers, S is at the same time also the greatest value of T_ζ (or also of Y_ζ) for which functions F_κ can be found such that (13) becomes antisymmetric.

The (STY) symbol has thus a more immediate physical significance than the partition by which the representation $D(R)$ usually is characterized and will be used instead of the latter henceforth.

The quantity $\Xi=n(n-1)\chi(T)/2f$ can be obtained either using Frobenius' character formula or by an argument similar to that of Dirac for atomic spectra which is given in the appendix.[10] It is

$$\begin{aligned}
\Xi&=-n^2/8+2n+5/2-\Xi',\\
\Xi'&=\tfrac{1}{2}\{(S+2)^2+(T+1)^2+Y^2\}.
\end{aligned} \tag{15}$$

Again, the first part of the expression for Ξ can be omitted for most applications because it is the same for all isobars. The restrictions on the S, T, Y are the following: $S\geq T\geq Y$, $T+Y\geq 0$. All three are integers for even n, half-integers for odd n; the sum $S+T+Y$ is even if n is a multiple of 4, it is odd if n has the form $4k+2$, it has in general the form $2k'-\frac{1}{2}n$ (k and k' are integers).

Since the quantity L in (10) is necessarily positive, the binding energy of a term will be the greater, the smaller Ξ' is. The stablest nucleus with given isotopic spin $\frac{1}{2}(n_n-n_p)=T_\zeta$ is obtained, therefore, by giving S its minimum value $S=T_\zeta$ and besides making T and Y as small as possible; for by doing so one gives to Ξ' its minimum value. Fig. 2 gives the smallest value of Ξ' for given n_n-n_p. The curve is smooth for odd masses, because T and Y can be $\frac{1}{2}$ and $\pm\frac{1}{2}$, respectively, which gives 5/4 for the sum of the last two terms of Ξ' throughout. The curves for

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[9] The number of such sets of F_κ for given S_z, T_ζ, Y_ζ is given by the numbers in the circles of Fig. 1, reference 2. The middle number in the figure for $(STY)=(110)$ (partition $2+1+1+0$, page 113) should be a 3 instead of a 2. The last sentence in the legend was meant to correct this mistake. There are two errors in Fig. 4 also. One is in the figure for the masses $4n+2$ for $(STY)=(210)$ where the point on the top should be a line, extending from $T_\zeta=-1$ to $T_\zeta=+1$, and the point below should have a 1 to the left, rather than a 0, denoting that the total ordinary spin momentum is 1 for this term. The other mistake is in the figure for the masses $4n$: for $(STY)=(200)$ a point is missing which should have a 0 to the left, denoting that the term is a singlet.

[10] Formulas similar to (10) and (15) can be derived on the basis of the Hartree model also. This was pointed out in reference 1, p. 100, bottom of first column, and also Proc. Nat. Acad. **22**, 662 (1936), last paragraph. The calculation on this basis has been performed recently by D. R. Inglis and L. A. Young, Phys. Rev. **51**, 525 (1937). In this paper, for sake of further simplification, the orbital and spin degeneracy is neglected also, although they will occur, of course. The argument in references 1, 2 shows that not only the orbital and spin degeneracy does not matter but also that the results are valid also outside the limits of the Hartree scheme. The condition $g<g_m$, derived by Inglis and Young for the instability of odd-odd nuclei should read $g+2g_\sigma<g_m$.

The kinetic energy is, of course, the same for all terms of the same (STY) multiplet. Hence, the quantity which enters into (12) should not be T_ζ but rather the highest T_ζ of the multiplet, i.e. the S of the (STY).

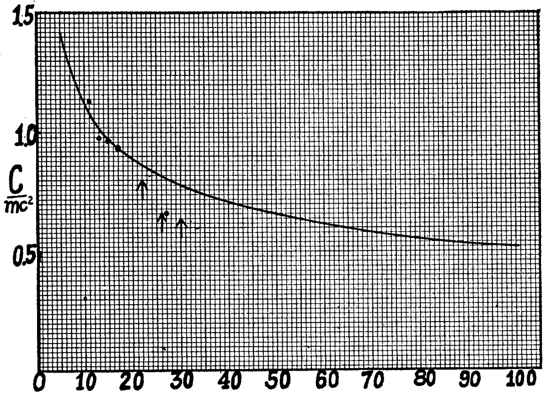

FIG. 3. The coefficient C in the expression (16) for the Coulomb energy as function of the total number n of the particles. The points result from the comparison of pairs of isobars in which the number of protons and neutrons is interchanged.

even masses show breaks, because for masses of the form $4k+2$, for even $S=T_\zeta$ the $T+Y$ must be odd and $T=1$, $Y=0$ is the best choice, while for odd $S=T_\zeta$, the $T+Y$ must be even and both T and Y can be 0. Thus the last two terms of Ξ' give 2 in the former, $\frac{1}{2}$ in the latter case. For masses of the form $4k$ it is the other way around. The breaks in the two latter curves will be made responsible for the instability of nuclei with odd number of protons and neutrons, i.e., of elements with masses $4k+2$ and even T_ζ and elements with masses $4k$ and odd T_ζ.

(4)

The last important term in the Hamiltonian is the Coulomb energy. This is supposed, at present, to represent the only difference between protons and neutrons in the Hamiltonian.[11] The Coulomb energy will be assumed to have the form

$$E_c=\tfrac{1}{2}Cn_p(n_p-1)=\tfrac{1}{2}C(\tfrac{1}{2}n-T_\zeta)(\tfrac{1}{2}n-T_\zeta-1)$$
$$=C(n^2-2n)/8-\tfrac{1}{2}C(nT_\zeta-T_\zeta(T_\zeta+1)). \qquad (16)$$

If we approximate the charge distribution in a nucleus by assuming it to be uniform within a sphere of radius r and zero outside, the significance of C is $C=1{\cdot}2e^2/r$. Actually the exchange energy should be substracted from this. We shall assume C to be proportional to $n^{-\frac{1}{3}}$ which corresponds to neglecting the exchange energy and assuming a constant density for the nuclei. The latter approximation probably introduces a greater error than the former, and there is a considerable uncertainty in the values of C assumed here. This will affect the values of L as well. Since L/C is less sensitive under a change of C than L itself, the latter quantity will be used in most cases.

C can be determined experimentally almost directly by comparing the binding energies of two nuclei[1] with $T_\zeta=\pm\frac{1}{2}$. The difference in binding energy is according to Heisenberg's hypothesis, only the Coulomb energy, which can be expressed by (16) and the resulting C plotted against n. The plot is given in Fig. 3, the solid curve is $2{\cdot}4n^{-\frac{1}{3}}$, the energies are given in mc^2.

Unfortunately, most of the points are at low n. Only one is as high as $n=27$, corresponding to the transition from $_{14}\mathrm{Si}^{27}$ to $_{13}\mathrm{Al}^{27}$ and this one falls off the curve. This would indicate that the radius increases more quickly than with the $\frac{1}{3}$ power of n, or that the energy of the transition is greater than the value measured by I. Curie and Joliot[12] for the limit of the β-ray spectrum. The latter alternative is rendered probable by the energy differences[13] $_{11}\mathrm{Na}^{22}-{}_{10}\mathrm{Ne}^{22}$ and $_{15}\mathrm{P}^{30}-{}_{14}\mathrm{Si}^{30}$. Assuming a symmetric Hamiltonian there will be a term of Na^{22} equal to every term of Ne^{22}, even using "approximation 3" of reference 2, i.e., taking ordinary spin forces into account. The term of Na^{22} which has in approximation 3 equal

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[11] The hypothesis that the proton-proton interaction differs only by the electrostatic energy from the neutron-neutron interaction was first introduced by W. Heisenberg, Zeits. f. Physik **77**, 1 (1932) and has not been questioned seriously so far. The most direct experimental verification for the $\mathrm{H}^3-\mathrm{He}^3$ pair leads to a moderate agreement only: S. Share's calculation (Phys. Rev. **50**, 488 (1936)), yields 1.37 mc^2 or the electrostatic energy in He^3, the experimental difference in the binding energies is 1.58 according to the most recent masses of H. A. Bethe and M. S. Livingston (cf. reference 18). The experimental energy difference would be even much greater, 2.0 mc^2, on the basis of the masses of S. Flügge and A. Krebs (Physik. Zeits. **38**, 13 (1937)). These are thus definitely in contradiction to Heisenberg's hypothesis.

Another piece of evidence pointing in the direction that the difference between neutron-neutron and proton-proton interaction is greater than the electrostatic energy, is given in Table VIII, reference 1, if the experimental values are increased by 1 mc^2 (cf. reference 6). In order to eliminate the discrepancy here, it was necessary to decrease the radii to $\frac{3}{4}$ of the values assumed in reference 1. For the present considerations this is not very important if one considers (16) as the total difference between proton-proton and neutron-neutron interaction.

[12] I. Curie and F. Joliot, J. de phys. et rad. **5**, 153 (1934).

[13] L. Meitner, Naturwiss. **22**, 420 (1934); C. D. Ellis and W. J. Henderson, Proc. Roy. Soc. **A152**, 714 (1935).

binding energy to that of Ne^{22} can be the ground state or an excited state of Na^{22}. The latter possibility is the more probable one, on account of the Heisenberg forces which have a similar effect in the nuclei Li^6, B^{10}, N^{14}. At any rate, the binding energy of Na^{22} is at least equal to that of Ne^{22} in approximation 3. If the departure from the symmetric Hamiltonian is caused by the electrostatic forces alone, these must be at least equal to the experimental binding energy difference of $8{\cdot}1\ mc^2$ between Ne^{22} and Na^{22}. This gives $10C > 8{\cdot}1\ mc^2$ or $C > .81\ mc^2$ for $n = 22$. The $P^{30} - Si^{30}$ pair gives in the same way $14C > 9\ mc^2$. The corresponding points are inserted as small arrows in Fig. 3 and render the value obtained directly for $n = 27$ improbable. Further experiments would be needed, however, to definitely settle this point.

(5)

In the following we shall assume $C = 2{\cdot}4\ mc^2/n^{\frac{1}{3}}$ which leads to $r = n^{\frac{1}{3}} r_e/2$, a radius for the nucleus which is much smaller than that obtained by Bethe.[14] It must be remembered, however, that our radius is the distance over which the wave function extends while Bethe's radius is the distance in which the electrostatic repulsion on an α-particle becomes as strong as the attraction due to specifically nuclear forces. It is natural that Bethe's radii are greater than ours.

The value of r can be inserted into (12), the second part of which becomes $(63T_\zeta^2/n)\ mc^2$. This value for the change in kinetic energy with increasing isotopic number is too high within a shell and too low at the end of the shell. In Hartree's model, the kinetic energy of all isobars is the same, as long as all the particles can be in the same shell; it is very different, however, if one particle is forced out of the shell. This will not be true actually, of course, but it is to be expected that our expression is too high in the middle of the shell. At the end of the shells, there will be an additional decrease in binding energy for all particles which are outside the shell. This decrease, on the other hand, will not be as great as one would expect on the basis of Hartree's model.[15]

The energy difference between a particle in the $2p$ and $3d$ shells should be, according to the central field model,[15] about 41 mc^2, while it actually is only[16] about 10 mc^2. The energy change at the end of the $(3d)(2s)$ shell should be according to Hartree's model[15] about 17 mc^2 while the experiments indicate about 3 mc^2. It seems that roughly $\frac{1}{5}$ of the increase in kinetic energy is concentrated at the end of the shells, the rest is probably rather uniformly distributed all over the shell. We shall decrease, hence, the expression for increase in kinetic energy with isotopic number to $(50T_\zeta^2/n)\ mc^2$.

The final expression for the binding energy $-E$ will be, hence,[16a]

$$\begin{aligned} -E = -E_0(n) - \Xi' L + \tfrac{1}{2} C(n - T_\zeta - 1) T_\zeta& \\ -(50 T_\zeta^2/n) mc^2& \\ C = 2{\cdot}4 mc^2/n^{\frac{1}{3}},& \end{aligned} \tag{17}$$

where Ξ' is defined in (15), plotted for the lowest state in Fig. 2. $E_0(n)$ is a smooth function of n which contains all terms omitted and is responsible for the greatest part of the binding. L will be supposed also to depend mainly on n, and a smooth function of this.

According to (17), there are (apart from the small mass difference between neutron and H^1) two causes which tend to keep T_ζ small: the potential energy and the kinetic energy. For small T_ζ the former one will be more effective, for large T_ζ their values will be in a constant ratio. The Coulomb energy is the only term which tends to increase the isotopic spin.

The following sections will be devoted to a comparison of (17) with experiment. The only unknown quantity in (17) is L. This L will be derived from the kinks in the mass defect curve,

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[14] H. A. Bethe, Phys. Rev. **50**, 977 (1936).

[15] W. Elsasser, J. de phys. et rad. **4**, 549 (1933); **5**, 389, 635 (1934); cf. also K. Guggenheimer, J. de phys. et rad. **5**, 253, 475 (1934). What is called central field model here is really only a somewhat crude potential well picture, without direct interactions between the particles. Also the term Hartree's approximation, as used in papers dealing with nuclei, often refers only to the kind of conclusions which one can draw, according to Slater, (Phys. Rev. **34**, 1293 (1929)), for the relative positions of terms, without a detailed knowledge of the wave functions. The direct interaction between the particles is taken into account here at least in first approximation.

[16] According to E. O. Lawrence, E. McMillan and M. C. Henderson, Phys. Rev. **47**, 273 (1935) and to J. D. Cockcroft and W. B. Lewis, Proc. Roy. Soc. **A154**, 261 (1935) there is an excited state of N^{16} in this height above the normal state.

[16a] In the last term of (17), it would be more logical to write S instead of T_ζ (cf. footnote 10). However, the S of the lowest term of a nucleus is T_ζ.

from the energy difference between isobars, and from stability conditions for isobars; the results then compared.

(6) KINKS IN THE MASS DEFECT CURVE

Through the work of Brasefield and Pollard and of Aston[17] the masses of most stable elements up to 40 are known with sufficient accuracy to make a comparison with (17) possible.

Formula (17) written for the atomic mass M instead of the binding energy reads (electronic mass units)

$$M(n)=M_0(n)+\Xi' L-\tfrac{1}{2}C(n-T_\zeta-1)T_\zeta +50T_\zeta^2/n+1.4T_\zeta \quad (18)$$

(1.4 is the difference between the mass of H^1 and the neutron). Fig. 4 contains the difference between the actually observed masses[17] and the next integer number, multiplied by 1825. The points for the three kinds of masses, $4k$, $4k\pm 1$, $4k+2$ are connected by broken lines. Around $n=30$, the line for the $4k+2$ type of masses runs just in the middle between the lines for the two other types. Using this as an empirical fact, one obtains

$$15L/8-C(n-\tfrac{3}{2})/4+50/(2n)+0.7 =2(5L/2-C(n-2)/2+50/n+1.4)$$

or $L/C=4.7$; $L=3.5\ mc^2$.

Some corrections, applied by Bethe[18] on the masses[17] make the lines for $4k$ and $4k+2$ just cross at $n=28$ and we obtain $5L/2-C(n-2)/2+50/n+1.4=0$, or the somewhat smaller values $L/C=4.0$; $L=3.15$. This is as accurate a value as can be expected with the present masses.

At lower masses, one can make a calculation for individual points also. It gives for $n=13$, 14, 15 the values $L/C=4.6$, 4.5, 4.4, respectively, with an average error of 0.3 each. For $n=21$, 22, 23, one obtains $L/C=4.2$, 4.8, 4.0, respectively; the average error is here 0.5.

From the value of L at 28 or 30, one can calculate the actual distance of the broken lines for the elements $4k$ and $4k\pm 1$. The result is 2 mc^2 (with $L=3.15\ mc^2$) or 3 mc^2 (with the original values, $L=3.5\ mc^2$). This is in fair agreement with the "observed" 3 mc^2 (corrected masses) and 2 mc^2 (original values).

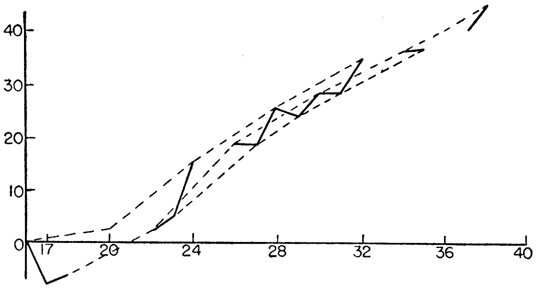

FIG. 4. The difference between n times one-sixteenth of the mass of oxygen and the mass of the stable element with n particles, given in units of the electronic mass. Experiments of Aston and Pollard and Brasefield.[17]

The increase of isotopic number with nuclear mass will be discussed in more detail later. However, the change from $\frac{1}{2}$ at $n=35$ to $\frac{3}{2}$ at $n=37$ may be treated here. The condition for the stability of ${}_{17}Cl^{35}$ against electron emission is

$$35L/8-C(n-\tfrac{3}{2})/4+50/(2n)+0.7 <59L/8-C(n-5/2)/(\tfrac{3}{4})+(50/n)/(5/2)+2.1,$$

which gives $L/C>3.4$ for $n=35$. The condition that Cl^{37} be stable against the capture of an extranuclear electron is the opposite of this inequality. This gives $L/C<3.8$.

A knowledge of the masses of S^{33}, A^{36}, K^{39} would be very valuable for a further comparison. The values of L obtained in this section are plotted in Fig. 5.

(7) DIFFERENCES BETWEEN MASSES OF ISOBARS

The previous section yielded values for the quantity L which should enable us to calculate the differences between isobaric masses from about $n=12$ to $n=40$. The value of L obtained in the previous section decreases a little less strongly than with the inverse first power of n. This is indeed what would be expected as soon as the nuclear dimension exceeds the range of forces.

For elements of the form $4k$, the (STY) symbol for the lowest state with isotopic spin 1 is (110). The difference in potential energy between this state and the lowest state of the element with isotopic spin zero $[(STY)=(000)]$ is $4L$. The

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[17] E. Pollard and C. J. Brasefield, Phys. Rev. **51**, 8 (1937); F. W. Aston, Nature **138**, 1094 (1936).

[18] I am much indebted to Professor H. A. Bethe for communicating to me his results before publication.

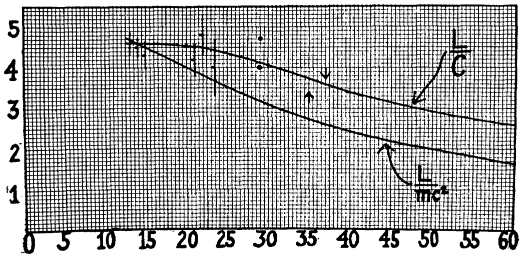

FIG. 5. L/C and L as function of the number of particles. The experimental points (with mean errors for the light elements) all refer to L/C. The higher point at $n=29$ refers to the original measurements,[17] the lower one to the corrected ones.[18]

difference between the mass defects is, therefore, in units of the electronic mass

$$4L-C(n-2)/2+100/n+1{\cdot}4.$$

The expression for the mass difference for odd elements with isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$ is

$$3L-C(n-3)/2+100/n+1{\cdot}4.$$

The theoretical and experimental values are collected in Table II. Except for F^{20}, the agreement is quite satisfactory. For F^{20} one may infer the existence of γ-radiation. The disintegrations in which a neutron outside the *3d2s* shell is transformed into a proton within this shell, are excluded from this comparison (Cl^{38}, A^{41}, K^{42}).

(8) STABLE ISOBARS

The first important change in the isotopic number occurs, as we saw, at Cl^{37} where it becomes $\frac{3}{2}$. We know, however, that K^{39} has $T_\zeta=\frac{1}{2}$ again. This can be attributed only to the closing of a shell, since the Coulomb energy is certainly even more effective in ${}_{19}K^{39}$ than in ${}_{17}Cl^{37}$. The instability of ${}_{18}A^{39}$ can be explained by assuming that one of its 21 neutrons is outside a closed shell and its transformation into a proton which can be inside the shell represents a large energy decrease. One is thus led to the assumption that a shell is closed with 20 neutrons, or 10 orbital states. This agrees with the calculations of Margenau[19] with the potential well picture.

It is possible to determine the magnitude of the increase X in kinetic energy of a particle lifted from the *3d2s* shell into the next one from the energies of the ${}_{17}Cl^{38}$, ${}_{18}A^{41}$ and ${}_{19}K^{42}$ nuclei which are known from β-disintegration, if one assumes that they are not accompanied by γ-rays. In both cases a neutron outside the *ds* shell is transformed into a proton inside this shell. The change in potential and Coulomb energy can be calculated for the disintegrations if one extrapolates the value of L in Fig. 5, assuming that one is already in the region where L is inversely proportional to n and L/C to $n^{\frac{2}{3}}$. For ${}_{17}Cl^{38}$ and ${}_{19}K^{42}$ the formula $5L-C(n-4)/2+4.50/n+1{\cdot}4$ yields $6{\cdot}9$ and $4\ mc^2$, respectively; for ${}_{18}A^{41}$ $4L-C(n-5/2)+4.50/n+1.4=3$. The experimental values[20] are $9{\cdot}4$, $7{\cdot}2$ and $5{\cdot}8\ mc^2$, respectively. This gives approximately $X=3\ mc^2$ for the energy difference between the *sd* shell and the next shell. (Cf. Section 5.)

Assuming this value, ${}_{19}K^{39}$ is by $1{\cdot}5\ mc^2$ more stable than ${}_{18}A^{39}$, which explains the curious backward movement of the isotopic spin.[19] It has been pointed out before[2] that simultaneously with the first change in isotopic spin for odd elements there should be a change for the $4k$ elements also. The following Table III gives for $n=36$, 40, 44 the calculated mass differences between the elements with isotopic number 1 and 2 on the one hand and the element with $T_\zeta=0$ on the other hand, in units of the electronic mass. One sees, first of all, that all the nuclei with isotopic spin 1 (proton and neutron number both odd) are unstable, for $n=36$ and 40 even against both $T_\zeta=0$ and $T_\zeta=2$. From the nuclei with equal

TABLE II. *Theoretical and experimental values of the mass difference for odd elements with isotopic spin 3/2 and 1/2 (electronic mass units).*

n	calc.	observed
12	24	21.5
20	14.4	9.8
24	10.7	9.8
27	4.6	4.0
28	7.7	6.4
31	3.2	3.45
32	5.1	4.3
36	2.7	?

Sources for the experimental results: For $n=12$ and 20: W. A. Fowler, L. A. Delsasso and C. C. Lauritsen, Phys. Rev. **49**, 561 (1936). For $n=24$ E. O. Lawrence, Phys. Rev. **47**, 17 (1935) and N. D. Kurie, J. R. Richardson and H. C. Paxton, Phys. Rev. **49**, 368 (1936). For $n=27$ M. C. Henderson, Phys. Rev. **48**, 855 (1935). For $n=28$ E. McMillan and E. O. Lawrence, Phys. Rev. **47**, 343 (1935). For $n=31$ and 32 Kurie, Richardson and Paxton, reference **20**, and E. M. Lyman, Phys. Rev. **51**, 1 (1937).

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[19] H. Margenau, Phys. Rev. **46**, 613 (1934); H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936), § 32.

[20] F. N. D. Kurie, J. R. Richardson and H. C. Paxton, Phys. Rev. **49**, 368 (1936); J. Ambrosen, Zeits. f. Physik **91**, 43 (1934).

number of protons and neutrons, masses 36 and 40 are stable, 44 unstable, in agreement with experiment. However, all three isobars 36, 40, 44 with $T_\zeta=2$ should be stable against the emission of an electron.[21] The first one, S^{36}, is not known (S^{35} is, of course, unstable).

The next change in the isotopic number of odd elements occurs at $n=49$, where it changes to 5/2. Using further the extrapolated value of L along the curve $L=96/n$, one obtains that the mass difference between the isobars with isotopic spin $\frac{3}{2}$ and 5/2

$$4L-C(n-5)/2+200/n+1{\cdot}4$$

becomes just 0 at $n=47$, while the latter should be by about 1.2 mc^2 more stable than the former for $n=49$. In fact, the element $T_\zeta=5/2$ is the stable one for $n=49$ while only $T_\zeta=\frac{3}{2}$ is stable for $n=47$.[22]

It has been pointed out before[2] that simultaneously with this jump in isotopic number, that of the elements $4k+2$ should increase by 2. Table IV gives in units of electronic mass the calculated mass differences between the elements $T_\zeta=2$, 3 on one hand and $T_\zeta=1$ on the other hand, according to the formulae

$$5L-C(n-4)/2+200/n+1{\cdot}4$$
$$8L-C(n-5)+400/n+2{\cdot}8.$$

The situation is very similar to that of Table III. Again, all odd-odd elements are unstable, ${}_{21}\mathrm{Sc}^{46}$ even against disintegration into both ${}_{20}\mathrm{Ca}^{46}$ and ${}_{22}\mathrm{Ti}^{46}$. The elements $n=42$, 46, 50 with $T_\zeta=1$ are stable, in agreement with experiment. From the elements with isotopic spin 3, the mass number 42 should be unstable, the rest stable against transformation into the element with isotopic spin 2. Again, one of the elements which should be stable, ${}_{20}\mathrm{Ca}^{46}$, is not known. In additon to this, the element with mass 54 and $T_\zeta=1$ is stable. Probably, L is somewhat larger than the extrapolated curve (Fig. 5) would indicate.

TABLE III. *Calculated mass differences between elements with isotopic number 1 and 2 and the element with $T_\zeta=0$ (electronic mass units).*

n	$T_\zeta=1$	$T_\zeta=2$
36	2.3	0
40	$0+X$	$-4.1+2X$
44	-2.2	-8

(9)

It is hardly necessary to mention that the somewhat surprising agreements between the rather rough theory given in this paper and the experimental findings must be partly fortuitous.[22] Approximation 1 is certainly very crude around $n=50$ and it would be desirable to take into account the Heisenberg force at least in first approximation and, what is probably even more important, the second approximation for the Coulomb force. Also, the shells become probably even less important for higher masses and the factor 50 should be somewhat increased in (17). All these factors work for a decrease of the isotopic spin and, indeed, they seem to be noticeable already in the region of Table IV.

For very large nuclei, the probabilities $g_s(x)$ and $g_a(x)$ of a distance x between two particles will be inversely proportional to the volume of the nucleus as long as x is small compared with the nuclear diameter. This follows from the assumption that the $g(x)$ approach limiting functions with increasing number of particles.[23] In the region in which this is true, L and L' will be inversely proportional to the total number n of the particles, since $V(x)$ will extend over a region which is small compared with the nuclear diameter. For nuclei with about 30 or 40 particles, this limiting case will not yet be reached, because the surface effects are still appreciable. Their gradual disappearance will cause L to decrease more slowly than indicated in Fig. 5. This point

TABLE IV. *Mass differences between elements with $T_\zeta=2$, 3 and with $T_\zeta=1$ (electronic mass units).*

n	$T_\zeta=2$	$T_\zeta=3$
42	4.4	5
46	2	0.7
50	0.1	-3.6
54	-2	-6.9

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[21] M. Goeppert-Mayer, Phys. Rev. **48**, 512 (1935) has shown that this means an enormously long lifetime even if the nucleus is unstable against the emission of two electrons or positrons.

[22] The stability or instability of a nucleus depends often on mass differences of less than the electronic mass. Certainly none of the formulas of this paper is accurate to that amount.

[23] In metals, where the situation is very similar from the mathematical point of view, the existence of such a limiting function is quite obvious.

has been fully discussed by Weizsäcker[24] into whose well-known quasiempiric formula (18) should go over for high n.

Since for heavy nuclei Ξ' is proportional to T_ζ^2, and L to $1/n$, the term $\Xi' L$ is proportional to $50\ T_\zeta^2/n$. As far as the experimental values for heavy nuclei are concerned, the factor of T_ζ^2/n could be changed if the magnitude of L is changed also to compensate for it. For light nuclei, the other terms in Ξ' are important also and keep the possibility of a change of the numerical value of L within certain limits.

It is a pleasure to remember the interesting conversations I had on the subject of this paper with Drs. Breit, Feenberg and Wheeler.

APPENDIX

Formula (15) is of course, an immediate consequence of the character formulas of Frobenius.[25] There seems to be a general desire, however, for a derivation which does not make use of the whole theory of representations of the symmetric permutation group. The derivation to be given here is a generalization of that given by Dirac[26] for the theory without isotopic spin which is used in atomic and molecular problems.

Instead of considering the positional part of the wave function, one may consider the spin function alone. Since the corresponding representations are adjoint, the respective values of $n(n-1)\chi(T)/(2f)$ have opposite signs.

There are several sets of spin functions for every representation which may be distinguished, e.g., by the total S_z, T_ζ, and Y_ζ. The value of Ξ depends, however, only on the representation which is specified by S, T, Y. It is thus immaterial which S_z, T_ζ, Y_ζ one takes, and the highest set will be taken here: $S_z=S$, $T_\zeta=T$, $Y_\zeta=Y$.

Consider the operators is_x, is_y, is_z, $i\tau_\xi$, $i\tau_\eta$, $i\tau_\zeta$ and the nine products $s_x\tau_\xi$, $s_x\tau_\eta, \cdots$ etc., acting on the spin variables of one particle. These 15 matrices, taken with both $\pm$ signs, together with the unit matrix, taken again with both signs, form a group of the order $h=32$. It is the same irreducible group which is generated by Dirac's α matrices. These α have the important property which will be repeatedly used later that the trace of every element is zero, except that of ± 1. This follows from the fact that they can be transformed into their negative values.

The interchange R_{12} of the spin coordinates $s_1\tau_1$ and $s_2\tau_2$ is identical with the following sum over the whole group.

$$R_{12}=(2h)^{-1}\sum_i \alpha_{i1}\alpha_{i2}^{-1}, \tag{I}$$

where α_{i1} acts on the spin of the first, α_{i2} on the spin of the second particle. Eq. (I) will be proved by showing that $R_{12}^2=1$ and then that $R_{12}\alpha_{k1}R_{12}^{-1}=\alpha_{k2}$. This proves that R_{12} is to within a sign the interchange of the spin 1 with spin 2. The sign can be obtained by considering the trace of R_{12}, which, according to (I), is positive. If R_{12} were the negative of an interchange, its trace would be negative, since the number of symmetric linearly independent spin functions is greater than the number of antisymmetric spin functions. We have

$$R_{12}^2=(2h)^{-1}\sum_{ij}\alpha_{i1}\alpha_{i2}^{-1}\alpha_{j1}\alpha_{j2}^{-1}.$$

Setting $\alpha_{j2}\alpha_{i2}=\alpha_{l2}$, $\alpha_{j1}\alpha_{i1}=\alpha_{l1}$ we have $\alpha_{i2}^{-1}\alpha_{j2}^{-1}=\alpha_{l2}^{-1}$ and similarly $\alpha_{i1}=\alpha_{j1}^{-1}\alpha_{l1}$,

$$R_{12}^2=(2h)^{-1}\sum_{jl}\alpha_{j1}^{-1}\alpha_{l1}\alpha_{j1}\alpha_{l2}^{-1}.$$

Now $\sum_j \alpha_{j1}^{-1}\alpha_{l1}\alpha_{j1}$, commutes with all α_{k1} and is, hence, a multiple of the unit matrix. Since its trace is h times the trace of α_{l1} it is different from zero only for $\alpha_{l1}=\pm 1$. Hence

$$R_{12}^2=(2h)^{-1}[h\cdot 1+(-h)\cdot(-1)]=1. \tag{II}$$

Furthermore we have to show that

$$R_{12}\alpha_{k1}R_{12}^{-1}=R_{12}\alpha_{k1}R_{12}=\alpha_{k2}.$$

We have

$$R_{12}\alpha_{k1}R_{12}=(2h)^{-1}\sum_{ij}\alpha_{i1}\alpha_{i2}^{-1}\alpha_{k1}\alpha_{j1}\alpha_{j2}^{-1}.$$

Setting again $\alpha_{j2}\alpha_{i2}=\alpha_{l2}$, $\alpha_{j1}\alpha_{i1}=\alpha_{l1}$, we have

$$R_{12}\alpha_{k1}R_{12}=\sum_{jl}(2h)^{-1}\alpha_{j1}^{-1}\alpha_{l1}\alpha_{k1}\alpha_{j1}\alpha_{l2}^{-1}.$$

The sum over j is again different from zero only if $\alpha_{l1}\alpha_{k1}=\pm 1$ in which case it is $\pm h$. Hence

$$R_{12}\alpha_{k1}R_{12}=(2h)^{-1}\{h\alpha_{k2}+(-h)(-\alpha_{k2})\}=\alpha_{k2}.$$

This proves Eq. (I). Dirac's equation for ordinary spin $Q_{12}=\frac{1}{2}(1+s_{x1}s_{x2}+s_{y1}s_{y2}+s_{z1}s_{z2})$ is a special case hereof, for $h=8$.

We shall consider now the operators for the "total spin."

$$A_i=\sum_{\kappa=1}^{n}\alpha_{i\kappa}. \tag{III}$$

Since the α are unitary,

$$A_i^\dagger=\sum_{\kappa=1}^{n}\alpha_{i\kappa}^{-1}. \tag{IIIa}$$

We shall calculate now the sum of all $n(n-1)/2$ interchange operators $R_{\kappa\lambda}$. We shall denote this sum by Ξ because it will turn out that, if applied to a function F belonging to a definite multiplet system, it merely multiplies this F with $n(n-1)\chi(T)/(2f)$ where $\chi(T)$ is the trace of the matrix $D(R)$ in (1) corresponding to a transposition.

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[24] C. F. v. Weizsäcker, Zeits. f. Physik **96**, 431 (1935).

[25] G. Frobenius, Berl. Ber. (1900) p. 516. F. Hund (ref. 1) has derived the very simple formula for

$$\Xi=\tfrac{1}{2}[\lambda_1(\lambda_1-1)+\lambda_2(\lambda_2-3)+\lambda_3(\lambda_3-5)+\cdots].$$
$$(\lambda_1\geqq\lambda_2\geqq\lambda_3\geqq\cdots)$$

[26] P. A. M. Dirac, *The Principles of Quantum Mechanics*, second edition (Oxford 1935), § 61.

We have

$$\begin{aligned}\Xi=\sum_{\kappa<\lambda} R_{\kappa\lambda}&=\tfrac{1}{2}\sum_{\kappa\lambda}R_{\kappa\lambda}-\tfrac{1}{2}\sum_{\kappa}R_{\kappa\kappa}\\ &=\tfrac{1}{2}(2h)^{-\frac{1}{2}}\sum_{\kappa\lambda}\sum_{i}\alpha_{i\kappa}\alpha_{i\lambda}{}^{-1}-\tfrac{1}{2}(2h)^{-\frac{1}{2}}\sum_{\kappa}\sum_{i}1\\ &=\tfrac{1}{2}(2h)^{-\frac{1}{2}}\sum_{i}A_iA_i{}^{\dagger}-\tfrac{1}{2}nh(2h)^{-\frac{1}{2}}.\end{aligned}\qquad\text{(IV)}$$

So far, the first index of α referred in an arbitrary way to the elements of the group generated by Dirac's α matrices. In the following, however, we shall use single indices for Dirac's original four matrices α_1, α_2, α_3, α_4 only, and write $\alpha_{12}=\alpha_1\alpha_2$, $\alpha_{123}=\alpha_1\alpha_2\alpha_3$, etc. The notation for the A will be changed in a similar way. The A with one or four indices will be Hermitean, those with two or three indices skew-Hermitean. Hence (IV) will read (in the summation in (IV), A_i occurred with both signs; $h=32$)

$$\begin{aligned}\Xi=-2n+(1/8)\{&n^2+A_1{}^2+A_2{}^2+A_3{}^2+A_4{}^2\\ &-A_{12}{}^2-A_{13}{}^2-A_{14}{}^2-A_{23}{}^2-A_{24}{}^2-A_{34}{}^2\\ &-A_{123}{}^2-A_{124}{}^2-A_{134}{}^2-A_{234}{}^2+A_{1234}{}^2\}.\end{aligned}\qquad\text{(V)}$$

The S_z, T_ζ are, apart from a trivial factor, two commuting A's. Perhaps the most symmetric choice is

$$S_z=\tfrac{1}{2}iA_{12};\qquad T_\zeta=\tfrac{1}{2}iA_{34};\qquad \text{then}\qquad Y_\zeta=-\tfrac{1}{2}A_{1234}.$$

Let us consider now the functions F which belong to a certain multiplet system (STY) and let us assume that they have definite "magnetic quantum numbers" S_z, T_ζ, Y_ζ. This is always possible since iA_{12}, iA_{34}, $-A_{1234}$ commute. Since Ξ is a sum over all elements of a class, it follows from the general theory that it is a constant within the multiplet. One can convince oneself also directly with the help of (V) and the commutation relations that it commutes with all A and thus, if applied to the functions F of an (STY) multiplet, it merely multiplies them with the same number. This number is, according to (IV), $n(n-1)/2$ times the mean value of the diagonal elements of the interchange operator $R_{\kappa\lambda}$. Since the latter quantity is $\chi(T)/f$ it was justified to denote the sum in (IV) by Ξ.

In order to find another expression for the value of Ξ, we apply it to the F with the highest set of magnetic quantum numbers $S_z=S$, $T_\zeta=T$, $Y_\zeta=Y$. For this F_{STY} we have

$$\begin{aligned}iA_{12}F_{STY}=2SF_{STY},\quad iA_{34}F_{STY}=2TF_{STY},\\ -A_{1234}F_{STY}=2YF_{STY}.\end{aligned}\qquad\text{(VI)}$$

Repeated use will now be made of the fact that for every equation giving the commutator of two α, there is a similar equation for the A. Thus

$$[\alpha_1,\alpha_2]=2\alpha_{12};\quad [A_1,A_2]=2\sum\alpha_{1\kappa}\alpha_{2\kappa}=2A_{12}.$$

From $[iA_{12}, A_1-iA_2]=2(A_1-iA_2)$ it follows that

$$iA_{12}(A_1-iA_2)F_{STY}=2(S+1)(A_1-iA_2)F_{STY},$$

so that the S_z of $(A_1-iA_2)F_{STY}$ is $S+1$. Since no F exists with a higher S_z than S, this must be zero. In this way we obtain the equations

$$\begin{aligned}(A_1-iA_2)F_{STY}=0,\qquad (A_{13}-iA_{23})F_{STY}=0,\\ (A_{14}-iA_{24})F_{STY}=0,\qquad (A_{134}-iA_{234})F_{STY}=0.\end{aligned}\qquad\text{(VII)}$$

A_3-iA_4 commutes with iA_{12} and it does not change thus the value of S_z. However, $[iA_{34}, A_3-iA_4]=2(A_3-iA_4)$ so that the T_ζ of $(A_3-iA_4)F_{STY}$ is $T+1$. Therefore $(A_3-iA_4)F_{STY}$ must be zero, since the highest T_ζ for all the F with $S_z=S$ is T. In this way one obtains

$$(A_3-iA_4)F_{STY}=0,\qquad (A_{123}-iA_{124})F_{STY}=0.\qquad\text{(VIII)}$$

We can rewrite now (V)

$$\begin{aligned}\Xi=-2n+(1/8)\{&n^2+(A_1+iA_2)(A_1-iA_2)+(A_3+iA_4)\\ &\times(A_3-iA_4)-A_{12}{}^2-(A_{13}+iA_{23})(A_{13}-iA_{23})\\ &-(A_{14}+iA_{24})(A_{14}-iA_{24})-A_3{}^2-(A_{123}+iA_{124})\\ &\times(A_{123}-iA_{124})-(A_{134}+iA_{234})(A_{134}-iA_{234})\\ &+A_{1234}{}^2\}+(1/8)\{2iA_{12}+2iA_{34}+2iA_{12}+2iA_{12}\\ &+2iA_{34}+2iA_{12}\}.\end{aligned}\qquad\text{(IX)}$$

This gives with (VI), (VII) and (VIII)

$$\begin{aligned}\Xi F_{STY}=\{-2n+(1/8)(&n^2+4S^2\\ &+4T^2+4Y^2+16S+8T)\}F_{STY}\end{aligned}\qquad\text{(X)}$$

for the Ξ of the representation of the spin functions. Taking into account the change of sign in passing to space functions this proves (15) of the text.