On the Matrix Element in Fermi's Theory of β-Decay

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It is investigated how the Fermi theory of β-decay has to be formulated for a complex nucleus containing many particles. By symmetrizing the problem in all particles (neutrons and protons) a general prescription is obtained which would allow one to calculate the transition probabilities when the wave functions for the nucleus in its initial and final state are known. The discussion of the heavy particle matrix element which is involved shows that a considerable decrease of the transition probability with the complexity of the nucleus is to be expected. A new reduction of the empirical data shows that the influence of the Coulomb field on the β-disintegration probability is larger than heretofore assumed. The difference obtained in reduced lifetimes of light and heavy nuclei (of the order of a factor 50 to 100) is of the same order of magnitude as estimated from the theoretical considerations.

1. INTRODUCTION

IN the Fermi theory of β-decay the probability for a definite disintegration process contains besides other factors a matrix element depending on the wave functions of the heavy particle which goes over from a neutron to a proton state or *vice versa*. Hitherto one has assumed that this matrix element, whose square enters in the total transition probability, should be of the order unity for the so-called allowed transitions.[1] This was suggested by the fact that the lifetimes of different elements with approximately the same maximum energy are of the same order of magnitude. (See Table I.) However, a more careful consideration of the theory shows that this apparent equality in the lifetimes for heavy and light nuclei actually means a considerable variation of the matrix element. This is due to the large influence of the Coulomb field on the electronic density near the nucleus.[2] We have made a new calculation of this effect (see §2) and we have found it considerably larger than heretofore assumed. In fact, the reduced lifetimes between light and heavy elements differ by a factor of the order of 100.

The only possible explanation for this effect seems to be that it is due to the above mentioned matrix element of the heavy particles. We have investigated, therefore, whether one can make any statements concerning this matrix element from theoretical considerations. The first question is then, how one can take into account the fact that we have many particles in the nucleus which virtually can make the neutron-proton transition. It seems to us that a unique formulation of the Fermi theory for this case can be obtained which would make it possible to calculate the matrix element if the correct wave functions were known. From the form of this matrix element one would expect rather large individual deviations and it seems to be a natural consequence that the matrix element becomes smaller with increasing complexity of the nucleus.

2. EFFECT OF THE COULOMB FIELD

In the Fermi theory of β-decay the probability that an electron is emitted by a nucleus with total energy between ϵ and $\epsilon+d\epsilon$ (in units mc^2) is

$$wd\epsilon=\frac{|M|^2}{\tau_0}F(Z,\epsilon)(\epsilon_0-\epsilon)^{2n}(\epsilon^2-1)^{\frac{1}{2}}\epsilon d\epsilon, \quad (1)$$

if the mass of the neutrino is taken to be zero. In (1) ϵ_0 is the upper limit of the energy. The exponent n depends on the basic assumption regarding the interaction, n being 1 for Fermi's original form and 2 for the one used by Konopinski and Uhlenbeck,[3] which gives a better agreement with the experimental distribution curves. τ_0 is a universal constant of the dimension

[1] We say that a transition is an allowed one when the corresponding element belongs to the group which shows the shortest reduced life times among all elements of comparable atomic weight.

[2] The influence of the Coulomb field has already been included in Fermi's original treatment (Zeits. f. Physik **88**, 161 (1934)) and H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936) give an estimate similar to ours, but they obtain much smaller differences between heavy and light nuclei.

[3] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **48**, 7 (1935).

TABLE I. *Lifetimes, maximum energies and comparative values for the heavy particle matrix elements for β radioactive nuclei.*

Nucleus	±	τ(sec.)	$(\epsilon_0-1)(mc^2)$	f_F	f_{KU}	$10^5\times 1/\tau f_F$		$10^6\times 1/\tau f_{KU}$	
$_6C^{11}$	+	1,200	2.5	11	29	8.0		29	
$_7N^{13}$	+	660	2.8	17	57	8.9		26	
$_8O^{15}$	+	150	3.9	63	360	11		19	
$_9F^{17}$	+	70	4.9	170	1,400	8.6		9.9	
$_{12}Mg^{27}$	−	620	(3.9)	110	620	1.5		2.6	
$_{14}Si^{27}$	+	150	3.9	54	300	12		22	
$_{46}Rh^{105}$	−	44	5.5	1600	18,000	1.5		1.3	
$_{47}Ag^{108}$	−	22	5.5	1700	19,000	2.7		2.4	
$_{49}In$	−	13	6.3	3200	46,000	2.4		1.7	
$_{49}In$	−	3,240	2.5	83	230	.37		1.3	
$_{81}AcC''^{207}$	−	410	2.7	750	2,700	.33	(.20)	.91	(.44)
$_{81}ThC''^{208}$	−	275	3.6	2200	12,000	.17	(.11)	.32	(.18)
$_{82}ThB^{212}$	−	55,000	0.7	8.2	2.4	.22	(.05)	7.7	(.79)
$_{82}RaB^{214}$	−	2,300	1.3	54	48	.81	(.33)	9.1	(2.3)
$_{91}UX_2^{234}$	−	94	4.5	7100	57,000	.15	(.11)	.19	(.12)

of a time which determines the absolute magnitude of the interaction.

$$\frac{1}{\tau_0}=\frac{G}{(2\pi)^3}\frac{mc^2}{\hbar};\quad G\sim 10^{-13}.$$

M is the matrix element of the wave functions of proton and neutron which can be factored out under the assumption that the electron and neutrino functions can be taken as constant over the nucleus (allowed transitions).

$$M=\int\Phi^*(x)O\Psi(x)dx, \tag{2}$$

where Ψ and Φ are the wave functions for the initial and final state and where x stands for spin and space variables. O might be an operator acting on the spin variable depending on the special form of the interaction Hamiltonian.

$F(Z,\epsilon)$, finally, is a factor expressing the dependence on the Coulomb field of the nucleus. Its physical meaning is the change in the electronic density $|\psi|^2$ at the nucleus, introduced by the Coulomb field and obtained by summing over all electronic s states in the interval $d\epsilon$. The relativistic s functions, of course, diverge at the center of the field. However, as the field in a nucleus with finite range will not increase infinitely, the electronic density inside of it will not vary rapidly and we can use for the β-decay the value at the border of the nucleus. The expression for $F(Z,\epsilon)$ already determined by Fermi[2] is then

$$F(Z,\epsilon)=\frac{4}{[\Gamma(1+2s)]^2}\left(\frac{2(\epsilon^2-1)^{\frac{1}{2}}}{\hbar}mcR\right)^{2s-2}\times e^{\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}}|\Gamma(s+i\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}})|^2 \tag{3}$$

with Z being the nuclear charge, $\alpha=1/137$, $\gamma=\alpha Z$, $s=(1-\gamma^2)^{\frac{1}{2}}$ and R the nuclear radius.[4] The quantity γ is positive for electron emitters and negative for positron emitters. For $Z\to 0$, i.e., $\gamma\to 0$, $s\to 1$, we obtain $F\to 1$ as it should be. Even for the largest Z, $x=s-1\sim-\gamma^2/2$ and $y=\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}$ will be <1 (excluding the smallest values of ϵ which do not contribute much to (1)). We can therefore use the evaluation of the complex Γ function:[5]

$$|\Gamma(s+i\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}})|^2=|\Gamma(1+x+iy)|^2\sim\left[\frac{\pi^2(x^2+y^2)}{\sin^2\pi x+\sinh^2\pi y}\cdot\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right]^{\frac{1}{2}},$$

which, if terms of the order x^2 are neglected, reduces to

$$\frac{2\pi|y|(1-2x)}{|e^{\pi y}-e^{-\pi y}|}.$$

Approximately, therefore,

$$F(Z,\epsilon)=\frac{2^{2s}}{(2s!)^2}\frac{2\pi|\gamma|(1+\gamma^2)}{|1-e^{-2\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}}|}\left(\frac{mcR}{\hbar}\right)^{2s-2}\times\epsilon(\epsilon^2-1)^{s-\frac{1}{2}}. \tag{4}$$

For not too small ϵ the denominator can be developed

[4] We take $R=1.5\times 10^{-13}A^{\frac{1}{3}}$ (A = atomic weight). The new slightly larger values of H. A. Bethe, Phys. Rev. **50**, 977 (1936), would increase our final values for the matrix elements by a factor 1.2 for the heaviest nuclei, which is not important.

[5] Compare Jahnke-Emde, second edition (Teubner, 1933), p. 87.

$$|1-e^{-2\pi\gamma\epsilon(\epsilon^2-1)^{-\frac{1}{2}}}|^{-1}$$

$$\sim\frac{1}{|1-e^{-2\pi\gamma}|}\left[1-\frac{2\pi|\gamma|}{|e^{2\pi\gamma}-1|}\cdot\frac{1}{2\epsilon^2}\right], \quad (5)$$

and (4) and (5) can be used for the whole range of γ, i.e., Z.

(4) is easily seen to give a factor ~ 50 between heavy and light elements, as, e.g., for $Z\sim 90$, the factors become

$$\frac{2^{2s}}{(2s!)^2}\sim 1.6, \quad \frac{2\pi\gamma(1+\gamma^2)}{|1-e^{-2\pi\gamma}|}\sim 6, \quad \left(\frac{mcR}{\hbar}\right)^{2s-2}\sim 5.$$

With the help of (4) and (5) one can integrate (1) over the energy to obtain the average lifetime

$$\frac{1}{\tau}=\int_1^{\epsilon_0} w d\epsilon=\frac{|M|^2}{\tau_0}f(\epsilon_0), \quad (6)$$

where

$$f(\epsilon_0)=\frac{2^{2s}}{(2s!)^2}\frac{2\pi|\gamma|(1+\gamma^2)}{|1-e^{-2\pi\gamma}|}\left(\frac{mcR}{\hbar}\right)^{2s-2}\times$$

$$\int_1^{\epsilon_0}\left[1-\frac{2\pi|\gamma|}{|e^{2\pi\gamma}-1|}\cdot\frac{1}{2\epsilon^2}\right](\epsilon_0-\epsilon)^{2n}(\epsilon^2-1)^{s-1}\epsilon^2 d\epsilon. \quad (7)$$

Replacing the slowly varying factor $(\epsilon^2-1)^{s-1}$ by $(\bar{\epsilon}^2-1)^{s-1}$ where $\bar{\epsilon}$ is the mean value of ϵ, we obtain for the integral in (7) for $n=1$ (Fermi)

$$(\bar{\epsilon}^2-1)^{s-1}\left[\frac{\epsilon_0^5}{30}-\frac{\epsilon_0^2}{3}+\frac{\epsilon_0}{2}-\frac{1}{5}\right.$$

$$\left.-\frac{2\pi|\gamma|}{|e^{2\pi\gamma}-1|}\frac{(\epsilon_0-1)^3}{6}\right] \quad (8a)$$

and for $n=2$ (Konopinski-Uhlenbeck)

$$(\bar{\epsilon}^2-1)^{s-1}\left[\frac{\epsilon_0^7}{105}-\frac{\epsilon_0^4}{3}+\epsilon_0^3-\frac{6}{5}\epsilon_0^2+\frac{2}{3}\epsilon_0-\frac{1}{7}\right.$$

$$\left.-\frac{2\pi|\gamma|}{|e^{2\pi\gamma}-1|}\frac{(\epsilon_0-1)^5}{10}\right]. \quad (8b)$$

As τ_0 is assumed to be constant, the product $1/\tau f(\epsilon_0)$ is proportional to the square of the matrix element. In Table I we give the relevant data for all the elements with allowed transitions, for the Fermi as well as for the Konopinski-Uhlenbeck case. The last two columns give relative values for M^2 for the different elements.

For the maximum energies ϵ_0 we have taken the values extrapolated from Konopinski-Uhlenbeck plots, as far as they are known. Though it seems to be agreed now that the Konopinski-Uhlenbeck plots give too high values for ϵ_0 it seems still to be reasonable to use this method of reduction as otherwise the form of the electronic distribution agrees so well with the Konopinski-Uhlenbeck formulae. A possible error introduced by this choice of ϵ_0 would also enter similarly for all elements.

For the heavy radioactive elements, on the other hand, only Sargent's values which have been obtained by inspection are available. This means probably that the ϵ_0 used for the heavy elements are systematically too low compared with those for the lighter elements. To see the possible influence of this effect we have given in brackets other reduced values which are obtained when one increases Sargent's ϵ_0 arbitrarily by 0.2 MV $\sim 0.4\ mc^2$ which is of the order of the observed difference between the Konopinski-Uhlenbeck and the inspection values. From the table we see, besides individual fluctuations, chiefly a systematic decrease of the matrix elements with increasing mass number. M^2 decreases from the group of the light positron emitters of the type ${}_6C^{11}$ to the elements with medium weight as In and Ag by a factor ~ 10 and to the heavy elements by a factor 50 to 100.

Only the two heavy elements with very low maximum energy, Th B and Ra B fall somewhat out of line with the rest but owing to the much smaller ϵ_0 they are not directly comparable with the others.[6] The empirical corrections, introduced above, already reduce the matrix elements sufficiently to remove the anomaly. A further interesting remark is, that the values for M^2 straggle decidedly less if the Fermi coupling is taken instead of the Konopinski-Uhlenbeck one. This might suggest that the latter gives too strong a dependence of the life times on the maximum energy which again would explain the anomalous behavior of Th B and Ra B.

[6] The integral in (7) has been more accurately evaluated for these elements than in (8a) and (8b).

3. General Formulation of the Fermi Theory

The matrix element (2) is defined only for a single particle which makes the proton-neutron transition. We have to investigate how M is to be generalized for a nucleus with many particles.

According to our present knowledge, a nucleus containing k neutrons and l protons can be described by a wave function

$$\Psi(\{x_1\cdots x_k\}\{x_{k+1}\cdots x_{k+l}\})$$

which has to be antisymmetrical in the protons and neutrons separately as indicated by bracketing the corresponding variables. (x representing again space and spin variables). Similarly, after a β transformation, the nucleus in its final state can be described by a function

$$\Phi(y_1\cdots y_{k-1},\ y_k\cdots y_{k+l})$$

in which we have one neutron less and one proton more. The problem is to find the matrix element for the Fermi theory, when these two functions are known. It is clear that the theory has to be formulated so that one cannot distinguish the particle which makes the neutron-proton transition. For this reason we can give no elementary correlation between the variables x and y, and it is therefore not possible to take over the matrix element (2) simply summing it over the initial neutrons.

Considering that it has no meaning to ascribe the neutron or proton property to a definite group of particles as soon as there are processes involved in which a particle changes its character, it is evident that the theory has to be formulated completely symmetrical in all particles. Such a symmetrization[7] can be carried out most conveniently by introducing a new fifth coordinate for each particle. This coordinate determines the neutron or proton characteristics of the particle to which it refers. We designate the proper functions in the neutron or proton states by n and p, respectively. With this the totally antisymmetrized wave function (which is necessary to fulfill the Pauli principle) can be written as

$$\Psi_i=\left[\frac{k\,!l\,!}{(k+l)\,!}\right]^{\frac{1}{2}}\sum_I\Psi(\{x_1\cdots x_k\}\{x_{k+1}\cdots x_{k+l}\})\eta_I \times n(1)\cdots n(k)p(k+1)\cdots p(k+l),$$

$$\Phi_f=\left[\frac{(k-1)\,!(l+1)\,!}{(k+l)\,!}\right]^{\frac{1}{2}}\sum_I\Phi(\{x_1\cdots x_{k-1}\}\{x_k\cdots x_{k+l}\}) \times\eta_I n(1)\cdots n(k-1)p(k)\cdots p(k+l).$$

The sums are extended over all interpermutations I between the neutrons and protons, $\eta_I=\pm1$ depending whether the permutation is even or odd. We can use now in both functions the same variables x. The radicals serve for normalization, i.e., the functions Ψ_i, Φ_f are normalized provided the original Ψ and Φ were.

The natural generalization of the matrix element (2) will then be

$$M=\sum_m\int\Phi_f^*[F_m\Psi_i]dx_1\cdots dx_{k+l} =(k+l)\int\Phi_f^*[F_k\Psi_i]dx_1\cdots dx_{k+l}, \quad (9)$$

where the sum is to be extended over all particles. The transformation operator F_m signifies that in Ψ_i $n(m)$ has to be replaced by $p(m)$. (The terms of Ψ_i which already contain $p(m)$ give automatically no contribution to M). All terms in the sum over m give the same contribution due to the complete symmetrization so that one obtains the above final form. In this we attribute the $n\to p$ transition to the definite particle k and multiply then by the total number $k+l$ of all the particles. The summation over the character coordinates can be carried out with the help of the orthogonality and normalization relations

$$\sum n^2(m)=\sum p^2(m)=1,\qquad \sum p(m)n(m)=0. \quad (10)$$

Introducing now the full expressions Ψ_i and Φ_f in (9) we see with the help of (10): Firstly, all interpermutations in $F_k\Psi_i$ which involve the particle k are orthogonal to Φ_f as the number of protons is different. Secondly, all interpermutations in Φ_f which involve the particle k are orthogonal to $F_k\Psi_i$ as they contain $n(k)$. Therefore all interpermutations which involve k drop out. The remaining interpermutations are then the same for both wave functions and only corresponding terms give a contribution owing to (10). As these terms differ only by the designation of the integration variables, all the integrals are identical, their total number being the number of inter-

[7] G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936); B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936).

permutations with the exception of k, i.e., $(k+l-1)!/(k-1)!l!$, and we obtain finally

$$M=(k+l)\frac{(k+l-1)!}{(k-1)!l!}\left[\frac{k!l!}{(k+l)!}\right]^{\frac{1}{2}}\left[\frac{(k-1)!(l+1)!}{(k+l)!}\right]^{\frac{1}{2}}$$
$$\times\int\Phi^*\Psi dx_1\cdots dx_{k+l}$$
$$=[k(l+1)]^{\frac{1}{2}}\int\Phi^*(\{x_1\cdots x_{k-1}\}\{x_k\cdots x_{k+l}\})$$
$$\times O_k\Psi(\{x_1\cdots x_k\}\{x_{k+1}\cdots x_{k+l}\})dx_1\cdots dx_{k+l}, \quad (11)$$

where Ψ and Φ are the original wave functions for the initial and final state for the total nucleus which do not contain the character variables. In the final form we have reintroduced an operator O_k which might act on the spin or space variable of the particle k in case it is assumed to do so in the original interaction.

Naïvely one would perhaps have expected the result (11) only with the numerical factor k(=number of initial neutrons) instead of the factor $[k(l+1)]^{\frac{1}{2}}$ which is symmetric in the number of neutrons in the initial and protons in the final state.

4. Discussion of the Matrix Element

To get an idea of the meaning of our prescription (11), especially the factor $[k(l+1)]^{\frac{1}{2}}$, we evaluate M under the assumption that Ψ and Φ are Hartree-Fock wave functions, i.e., can be written as determinants of individual particle wave functions. We develop these functions in the minors of the particle assumed to make the transition, i.e.,

$$\Psi=\frac{1}{(k)^{\frac{1}{2}}}\sum\psi_{ni}(x_k)R_{in},$$
$$\Phi=\frac{1}{(l+1)^{\frac{1}{2}}}\sum\phi_{pf}(x_k)R_{fp}, \quad (12)$$

where ψ_n and ϕ_p are the individual particle wave functions for the initial neutrons and final protons and R_n and R_p, respectively, the normalized wave functions of the rest nuclei.

Inserting (12) into (11) we get simply

$$M=\int\sum_{nipf}\phi_{pf}^*(k)(O_k\psi_{ni}(k))R_{ni}R_{pf}^*d\tau. \quad (13)$$

We obtain therefore as a prescription for the case of Hartree-Fock wave functions the following one: Take an arbitrary initial neutron state and final proton state, assign them to the particle which makes the transition, multiply them with normalized wave functions for the rest nucleus, sum over all possible combinations and integrate. The numerical factor is then exactly unity.

If we assume furthermore that the individual particle wave functions are practically the same for neutrons and protons and in both nuclei, and that they form an orthogonal set, (13) will give unity for M if we have in the initial and final nucleus just one corresponding neutron proton state with the rest nuclei identical. In all other cases we would obtain zero. According to this oversimplified picture we would get the result that M would be ~ 1 for the radioactive nuclei of the type ${}_6C^{11}$ in which just one surplus neutron or proton would be present which could then go over into the corresponding state of opposite character. In all other cases M would be very small.

This result shows again that the Hartree-Fock approximation is inadequate for heavier nuclei. According to the considerations of Feenberg and Wigner[8] and Bethe and Rose[9] we can, however, expect to obtain reasonable approximations to the true many-body wave functions by the superposition of a number of such configurations. In using such a development

$$\Psi_i=\sum a_r\psi_r, \quad \sum a_r^2=1,$$
$$\Phi_f=\sum b_s\phi_s, \quad \sum b_s^2=1 \quad (14)$$

the matrix element M will be

$$M=\sum a_r b_r^*, \quad (15)$$

where the sum is over all identical spin and space configurations which occur both in the initial and final nucleus. (15) will give again unity for the light positron emitters of the group ${}_6C^{11}$, as here initial and final nuclei are built up with the same configurations and approximately the same coefficients. However, we have to expect, firstly that the number of configurations necessary for a good approximation increases considerably with the complexity of the nucleus; secondly, that in

[8] E. Feenberg and E. Wigner, Phys. Rev. **51**, 195 (1937).
[9] H. A. Bethe and M. E. Rose, Phys. Rev. **51**, 205 (1937).

case the numbers of neutrons and protons differ by more than 1, only a fraction of the configurations will occur in both Ψ_i and Φ_f with large coefficients.

We can illustrate this behavior by the following simple consideration. Let us assume that for the function Ψ_i a set of f configurations and for the function Φ_f another set of g configurations is important so that only part of them, in number h, occur in both sets. In the average over all possible distributions of the coefficients, that means all directions of the unit vectors Ψ and Φ in their respectively f and g dimensional spaces, we obtain

$$(M^2)_{Av}=(a\cdot b)^2=h(a_x^2)_{Av}(b_x^2)_{Av}=h/fg, \quad (16)$$

where x designates an arbitrary component of the corresponding vector.

It seems not possible at the present stage to obtain estimates of the number of configurations which are necessary to give a good approximation for higher nuclei. But we think that the empirical behavior of the matrix elements as found in §2 gives a definite indication about the nature of the wave functions in heavy nuclei. From the crude estimate (16) one would conclude that the number of configurations required for a fair approximation will increase more than linearly with the mass number. A factor of the order 1/100 for the heavy elements would be given by some such values as $f\sim g\sim 20$ of which $h\sim 4$ would coincide, which means that a rather large number of configurations would be necessary as would seem to be required by our other knowledge of the constitution of heavy nuclei.

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On the Structure of Nuclei Beyond Oxygen

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An attempt is made to correlate the kinks in the mass defect curve with the energy differences between isobars, both as obtained from direct measurements and also from the shift of the isotopic number to higher values with increasing number of particles. Since the single-particle picture is known to be an insufficient approximation, the symmetry property of the wave function, resulting from the use of a symmetric Hamiltonian is utilized. The average interaction between symmetrically and antisymmetrically coupled particles ($L+L'$ and $L-L'$) is determined mainly from the kinks in the mass defect curve and enables one to calculate the energy differences between isobars. The energy change at the end of the shell is obtained from experimental data. It should enable one to get some idea of the probabilities with which the particles are in excited configurations. For heavier elements, the formula obtained here should naturally be identical with Weizsäcker's semi-empirical formula and the connection between both is discussed.

(1)

THE extension of the calculations of E. Feenberg and the present author[1] on the spectroscopic characteristics of the normal state and the low excited states to higher elements encounters great computational difficulties. With the one-particle picture, after the $2p$ shell is completed at O^{16}, the $3d$ and $2s$ shells begin to be built up probably simultaneously. Even the normal states of these elements will contain wave functions from several configurations ($3d^n$, $3d^{n-1}2s$, $3d^{n-2}2s^2$, $\cdots$) with about equal coefficients. But even the d^n configuration alone gives rise to a large number of terms with the lowest partition and the explicit calculation of all the matrix elements between these states becomes increasingly difficult. Table I shows[2] the "low terms" of the d^n configurations up to d^4, together

[1] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937); also H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936), and F. Hund, Zeits. f. Physik, to appear soon.

[2] For the preparation of Table I, cf. E. Wigner, Phys. Rev. **51**, 106 (1937), F. Hund, ref. 1. The terms for Table I have been first determined by E. Feenberg (private communication).