On the Density of Energy Levels of Heavy Nuclei*

JOHN BARDEEN†
Harvard University, Cambridge, Massachusetts
(Received March 15, 1937)

The present calculation of the density of energy levels of a heavy nucleus is based on the statistical model of Van Vleck. As in Bethe's calculation, the particles are assumed to move in a simple potential hole, but the depth of the hole varies with the velocity of the particle. If exchange forces act, the interaction energy of a given particle with the remainder of the nucleus decreases as the velocity of the particle increases. This results in a lower density of states of the individual particles at the top of the Fermi distribution. Bethe's formula for the density of excited levels of the nucleus as a whole may be applied to the present situation if this change in the density of the individual particle states is taken into account. The spacing between the levels is over a hundred times larger than that found by Bethe, and, if one uses the Gamow value for the radius of a radioactive nucleus ($\sim 9\times 10^{-13}$ cm), is much too large to be reconciled with the frequent occurrence of resonance levels for the capture of slow neutrons. If one uses the new value for the radius suggested by Bethe ($\sim 13\times 10^{-13}$ cm), the present theory gives more reasonable values.

I. INTRODUCTION

An estimate of the average spacing between the energy levels of heavy nuclei is of interest in explaining the frequency of occurrence of resonance levels for the capture of slow neutrons.[1] Bethe[2] has recently made an approximate calculation which is based on a statistical model in which the particles are assumed to move freely in a simple potential hole (constant potential inside the nucleus). If the interaction between nuclear particles is of the exchange type, the interaction energy between a given particle and the remainder of the nucleus depends on the velocity of the particle, and decreases as the velocity of the particle increases. This effect, which is neglected by Bethe, is here roughly taken into account by using the statistical model of Van Vleck.[3] The particles are again assumed to move in a simple potential hole, but the depth of the hole varies with the velocity of the particle. This causes a lower density of states of the individual particles, and a much lower density of states of the nucleus as a whole.

Let us suppose that a slow neutron is absorbed by a nucleus of mass number $A-1$ and angular momentum I_0. The resonance level[1] of the nucleus of mass number A into which the neutron is captured lies at an energy Q above the ground state. Since the neutron will have no orbital angular momentum ($l=0$), the angular momentum of the excited nucleus will be $I_0\pm\frac{1}{2}$ (the $\frac{1}{2}$ arising from the spin of the neutron). We are therefore interested in computing the number of excited levels $\rho(Q, I)dQ$, of given angular momentum I, which have energies between Q and $Q+dQ$.

The lowest state of the nucleus as a whole is that in which all the lowest states of the individual particles are filled up to some maximum energy, ϵ_m, the remaining states being empty. The excited states are those in which particles are taken from the filled levels into some of the previously unoccupied levels, leaving behind unoccupied states or "holes" in the filled band. The energy required to take a particle from one level to another is equal to the difference in the individual particle energies of the corresponding levels. These energies will depend to some extent on the excitation of the nucleus as a whole; but since this effect is small, we will neglect it in our work. To a first approximation, we will not distinguish between the proton and neutron levels. We then suppose that each level may be occupied by four particles, two protons and two neutrons. The total number of particles A, and the number of protons, Z, are fixed.

Bethe[2] has shown that under these conditions,

* Presented at the American Physical Society, Atlantic City meeting, December 30, 1936.

† Society of Fellows.

[1] The theory of the resonance capture of slow neutrons has been given by Breit and Wigner, Phys. Rev. **49**, 516 (1936). For experimental material, see Goldsmith and Rasetti, Phys. Rev. **50**, 328 (1936); Amaldi and Fermi, Phys. Rev. **50**, 899 (1936) where further references to the literature may be found.

[2] H. A. Bethe, Phys. Rev. **50**, 332 (1936). The equations of Bethe mentioned in the text refer to this paper.

[3] J. H. Van Vleck, Phys. Rev. **48**, 367 (1935). This work is based on that of P. A. M. Dirac, Proc. Camb. Phil. Soc. **26**, 376 (1930).

the density of levels of the nucleus as a whole depends on the density of the levels of the individual particles, and on their mean square angular momentum, $[j(j+1)]_{\text{Av}}$. Both refer to energies, ϵ_m, in the neighborhood of the top of the Fermi distribution. His formula, with some minor modifications,[4] may be written

$$\rho(Q,I)=9(2I+1)[j(j+1)]_{\text{Av}}^{-\frac{3}{2}}(\pi^3/216x^2Q)e^x, \quad (1)$$

where
$$x=\pi(8N(\epsilon_m)Q/3)^{\frac{1}{2}} \quad (2)$$

and the number of levels of the individual particles between the energies ϵ and $\epsilon+d\epsilon$ is $N(\epsilon)d\epsilon$. This formula applies only to levels of low angular momentum, I.

If, following Bethe, we assume that the particles move freely in a sphere of radius R, so that we need consider only the kinetic energy of the particles, the quantities entering (1) have the following values. The kinetic energy, ζ, of a particle at the top of the Fermi distribution is

$$\zeta=\epsilon_m=(\hbar^2/2M)(9\pi A/8R^3)^{\frac{2}{3}}. \quad (3)$$

It should be noted that if the volume of the nucleus is proportional to the number of particles contained in it, ζ is independent of A. The density of levels, for $\epsilon=\epsilon_m$ is:

$$N_0(\epsilon_m)=(MR^2/\hbar^2)(A/3\pi^2)^{\frac{1}{3}}=3A/8\zeta. \quad (4)$$

Each such level may be occupied by two protons and two neutrons. We thus have

$$x=\pi(AQ/\zeta)^{\frac{1}{2}}. \quad (5)$$

The mean square angular momentum is (Cf. Bethe, Eq. (44)):

$$[(j+\tfrac{1}{2})^2]_{\text{Av}}=(4/5)MR^2\hbar^{-2}\epsilon_m=(3^{4/3}\pi^{\frac{2}{3}}/10)A^{\frac{2}{3}}. \quad (6)$$

Substituting these values in (1), we have

$$\rho(Q,\, I)=\pi^4 10^{\frac{3}{2}}(2I+1)x^{-4}\zeta^{-1}e^x/216, \quad (7)$$

which should be compared with Bethe's Eq. (49). The numerical factor is slightly different from that given by Bethe.[5]

The decrease in the potential energy of a particle as its velocity is increased (considered in the next section) has the effect of decreasing $N(\epsilon_m)$ by a factor of about two. The density of levels of the nucleus as a whole is very sensitive to the density of levels of the individual particles, since this term enters exponentially in (1). For reasonable values of the excitation energy $Q(\sim 8\text{MV})$, x is of the order 20 for a moderately heavy nucleus ($A\sim 100$). Thus a decrease in $N(\epsilon_m)$ by a factor two decreases x by about 6 and $\rho(Q,\, I)$ by a factor $(1/4)e^6$ or a little over 100. With the Gamow value for the radius of a radioactive nucleus ($R\sim 9\times 10^{-13}$ cm), it is found that the resulting spacing between the levels is too large to be reconciled with the frequent occurrence of resonance levels for the capture of slow neutrons. If, however, one uses the larger value for the radius suggested by Bethe[6] ($\sim 13\times 10^{-13}$ cm) more reasonable values are obtained (Cf. Section 3).

2. Density of States of the Individual Particles

The correct form for the interactions between nuclear particles is still rather uncertain. In the present work we shall assume that the interaction potential between any two nuclear particles is a linear combination of the Majorana and Heisenberg operators,

$$((1-g)P^M+gP^H)J(r). \quad (8)$$

The operator P^M interchanges the space coordinates of the two particles; P^H interchanges both space and spin coordinates; and r is the distance between the particles. The Coulomb repulsion between the protons is neglected.

We assume that the particles have individual wave functions, and that in the lowest level of the nucleus as a whole, each individual state is doubly occupied with like particles having opposite spins. The wave function of the nucleus as a whole is approximated by a product of determinants corresponding to the neutrons and protons. The mean value of the energy under these conditions has been computed by Breit and Feenberg.[7] They find:

---

[4] Cf. Bethe; reference 2, Eq. (41). Bethe does not consider explicitly the general case in which the density of the individual particle states is given arbitrarily, but his method may be extended to yield the result given above.

[5] Bethe does not take the exclusion principle into account in his derivation of $p(M)$ (the probability that a state has the Z component of angular momentum M), given in Eq. (34) and following. The proper correction can be made by dividing his n (Eq. (45)) by 2 log 2. There is an error of a factor of $\sqrt{2}$ in passing from Eq. (45) to Eq. (46).

[6] H. A. Bethe, Phys. Rev. **50**, 977 (1936).

[7] G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936).

$$E=T+(1-\tfrac{1}{2}g)E_{ex}{}^{\nu\nu}+(-\tfrac{1}{2}-\tfrac{1}{2}g)E^{\nu\nu}+(1-\tfrac{1}{2}g)E_{ex}{}^{\pi\pi}+(-\tfrac{1}{2}-\tfrac{1}{2}g)E^{\pi\pi}+(2-g)E_{ex}{}^{\pi\nu}, \quad (9)$$

where T is the kinetic energy, and

$$E^{\alpha\beta}=\tfrac{1}{2}\int\int \rho_\alpha(x)J(x-x')\rho_\beta(x')dxdx',$$
$$E_{ex}{}^{\alpha\beta}=\tfrac{1}{2}\int\int \rho_\alpha(x,x')J(x-x')\rho_\beta(x',x)dxdx', \quad (10)$$
$$\alpha,\beta=\nu \text{ or } \pi;\quad \rho_\alpha(x)=\rho_\alpha(x,x).$$

The index π refers to protons and ν to neutrons. The Dirac density matrix $\rho_\alpha(x, x')$ is obtained by summing over all space states, each doubly occupied state occurring twice.

The individual particle energies are given by:

$$\epsilon_{\alpha i}=\int\psi_{\alpha i}{}^*(x)\left(-\frac{\hbar^2}{2M}\Delta\right)\psi_{\alpha i}(x)dx$$
$$+(1-\tfrac{1}{2}g)\int\int\psi_{\alpha i}{}^*(x)\psi_{\alpha i}(x')J(x-x')\rho_\alpha(x',x)dxdx'$$
$$+(-\tfrac{1}{2}-\tfrac{1}{2}g)\int\int\psi_{\alpha i}{}^*(x)\psi_{\alpha i}(x)J(x-x')\rho_\alpha(x')dxdx'$$
$$+(1-\tfrac{1}{2}g)\int\int\psi_{\alpha i}{}^*(x)\psi_{\alpha i}(x')J(x-x')\rho_\beta(x',x)dxdx'. \quad (11)$$

If a particle is excited from a state i to a state j the excitation energy of the nucleus is given to a first approximation by $\epsilon_j-\epsilon_i$. We neglect the effect of the excitation of the nucleus on the density matrices, and therefore on the individual particle energies. The excitation energy, Q, of a nucleus with several particles excited will then be the sum of the energies of the excited particles minus the sum of the energies of the holes left behind in the filled band. In this approximation states of different multiplicities will have the same energy. However, in counting the number of levels, the only degeneracy we assume is that resulting from the angular momentum of the nucleus.

For the present calculation we use the statistical model[3] of the nucleus; i.e., we take the plane wave functions

$$\psi_{\alpha\mathbf{k}}=V^{-\frac{1}{2}}e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (12)$$

in which V is the volume of the nucleus and $\mathbf{k}$ is the propagation vector. The density of states in k space is then

$$n(\mathbf{k})d\mathbf{k}=(2\pi)^{-3}Vd\mathbf{k}. \quad (13)$$

The density of states in energy, with which we are concerned, is

$$N(\epsilon)d\epsilon=(2\pi^2)^{-1}Vk^2(dk/d\epsilon)d\epsilon \quad (14)$$

if it be assumed that ϵ depends only on the magnitude of $\mathbf{k}$ and not on its direction. Each such state may be occupied by two like particles of opposite spins. The occupied states are assumed to fill a sphere in k space of radius k_α ($\alpha=\nu$ or π), where

$$k_\pi=(9\pi Z/4R^3)^{\frac{1}{3}};\quad k_\nu=(9\pi N/4R^3)^{\frac{1}{3}}; \quad (15)$$

and Z and N are the numbers of protons and neutrons, respectively. Again, R is the radius of the nucleus. In actual nuclei k_π and k_ν are very nearly equal, and we will later obtain certain simplifications by assuming that they are both equal to:

$$k_m=(9\pi A/8R^3)^{\frac{1}{3}}, \quad (16)$$

where A is the mass number. If the volume of the nucleus is proportional to the mass, k_m will be independent of A.

The Dirac density matrix corresponding to a sphere of particles in k space is:[8]

$$\rho_\alpha(r)=(\sin k_\alpha r-k_\alpha r\cos k_\alpha r)/\pi^2r^3;\quad r=|\mathbf{x}-\mathbf{x}'|. \quad (17)$$

Substitution in (11) gives the following expression for the energy of an individual particle in the statistical model:

$$\epsilon_{\alpha k}=(\hbar^2/2M)k^2+(1-\tfrac{1}{2}g)(\zeta(k,k_\nu)-\zeta(k,k_\pi))-(\tfrac{1}{2}+\tfrac{1}{2}g)(4k_\alpha{}^3/3\pi)\int_0^\infty J(r)r^2dr \quad (18)$$

where $\zeta(k,k_\alpha)=\int\exp(i\mathbf{k}\cdot\mathbf{r})J(r)\rho_\alpha(r)d\mathbf{r}. \quad (19)$

The last term on the right-hand side of (18) is independent of k, and so represents an additive constant which may be neglected for the present considerations.

[8] J. H. Van Vleck, reference 3, or E. Wigner and F. Seitz, Phys. Rev. **43**, 804 (1933).

If we use for $J(r)$ the Gaussian function $A_s e^{-\alpha r^2}$, the integral on the right-hand side of (19) may be evaluated.[9] The result is:

$$\zeta(k, k_\alpha) = A_s\{\mathrm{erf}\,(\omega_+) - \mathrm{erf}\,(\omega_-) + (4\alpha/\pi k^2)^{\frac{1}{2}}(\exp\,(-\omega_+^2) - \exp\,(-\omega_-^2))\}, \quad (20)$$

where $\omega_\pm = (k \pm k_\alpha)/2\alpha^{\frac{1}{2}}$

$$\mathrm{erf}\,\omega = 2\pi^{-\frac{1}{2}}\int_0^\omega e^{-x^2}dx.$$

In order to find the density of states in energy, we need to compute $d\epsilon/dk$. With the approximation $k_\nu = k_\pi = k_m$, we have, for $k = k_m$,

$$k_m(d\epsilon/dk)_{k=k_m} = \hbar^2 k_m^2/M - (2-g)A_s(\pi y)^{-\frac{1}{2}}f(y), \quad (21)$$

where $y = k_m^2/\alpha$

and $f(y) = e^{-y}(y+2) + y - 2.$

The first term on the right-hand side of (21) comes from the kinetic energy, the second term from the potential energy, of the particle. Omission of the potential energy term leads to Bethe's formula (7) for the density of levels, $\rho(Q, I)$. This term is, however, of the same order of magnitude as the kinetic energy term. It is convenient to express our result as a ratio of the slope of the total energy (kinetic plus potential) to the slope of the kinetic energy at the top of the Fermi distribution ($k = k_m$). This ratio, which we call γ is equal to the ratio of the density of states (in energy), $N_0(\epsilon_m)$, of the free particle model[10] (kinetic energy only included) to the density of states, $N(\epsilon_m)$, of our statistical model, as follows from (14). We have:

$$\gamma = N_0(\epsilon_m)/N(\epsilon_m) = 1 - (2-g)A_s(\alpha l^2)^{-1}(\pi y^3)^{-\frac{1}{2}}f(y), \quad (22)$$

TABLE I. *Values of $\gamma = N_0(\epsilon_m)/N(\epsilon_m)$ for the symmetric potential, $(0.8P^M + 0.2P^H)A_s e^{-\alpha r^2}$, as computed from Eq. (22).*

$\alpha^{-\frac{1}{2}}$(cm)	$A_s(mc^2)$ \ R(cm)	γ: 9×10^{-13}	11×10^{-13}	13×10^{-13}
2.8×10^{-13}	-53	2.34	2.40	2.35
2.0×10^{-13}	-81	2.07	1.94	1.78
1.6×10^{-13}	-105	1.76	1.60	1.45

[9] C. H. Fay, Phys. Rev. **50**, 560 (1936).
[10] Given by Eq. (4).

where A_s is expressed in units of mc^2 and $l = (\hbar^2/Mmc^2)^{\frac{1}{2}}$ is Feenberg's unit of length (8.97×10^{-13} cm). Values of γ for different values of α and R, computed from (22), are given in Table I. The value $g = 0.2$ was used. The values of A_s for the corresponding values of α were estimated from the work of Feenberg and Share.[11] These values bear out the statement, made earlier, that the dependence of the potential energy of a particle on the velocity of the particle (or on k) reduces the density of states of the individual particles by a factor of about two at the top of the Fermi distribution.

In order to see how the ratio depends on the type of interactions assumed, the corresponding calculations have been carried out with the omission of like-particle interactions. The analysis shows that in this case

$$\gamma = 1 - (1 - \tfrac{1}{2}g)A_{\pi\nu}(\alpha l^2)^{-1}(\pi y^3)^{-\frac{1}{2}}f(y). \quad (23)$$

The values of γ as computed from (23) are given in Table II. The strength of the interaction, $A_{\pi\nu}$, is somewhat larger than the strength for the symmetric potential, A_s, in order to obtain the proper binding energies for the light particles. It is seen that the values of γ are not extremely sensitive to the type of interaction involved. The value of γ would, however, be greatly reduced if the interactions were not of the exchange type.

3. Density of Levels of the Nucleus as a Whole

From the density of states of the individual particles, one can easily obtain the density of levels of the nucleus as a whole from Bethe's expression (1). The value of x is now:

$$x = \pi(AQ/\gamma\zeta)^{\frac{1}{2}} \quad (24)$$

with γ given by (22) or (23) and ζ by (3). The

TABLE II. *Values of $\gamma = N_0(\epsilon_m)/N(\epsilon_m)$ for neutron-proton interaction as computed from Eq. (23).*

$\alpha^{-\frac{1}{2}}$(cm)	$A_s(mc^2)$ \ R(cm)	γ: 9×10^{-13}	11×10^{-13}	13×10^{-13}
2.8×10^{-13}	-71	1.89	1.93	1.90
2.0×10^{-13}	-108	1.71	1.63	1.52
1.6×10^{-13}	-140	1.51	1.40	1.30

[11] E. Feenberg and S. S. Share, Phys. Rev. **50**, 253 (1936).

TABLE III. *Average spacing,* Δ_0, between nuclear levels of zero angular momentum, for $\gamma=2$.*

R (cm) for $A=230$	QA (MV) / ζ (MV)	Δ_0 200	400	600	800	1000	1200	1400
9×10^{-13}	21.5	6.7×10^6	1.8×10^6	4.3×10^5	1.3×10^5	5.7×10^4	1.38×10^4	5250
11×10^{-13}	14.4	2.4×10^6	3.2×10^5	4.5×10^4	9000	2100	5200	152
13×10^{-13}	10.3	7.5×10^5	5.3×10^4	5000	660	109	18	3.3

* Units of Δ_0 are ev.

mean value of $(j+\frac{1}{2})^2$ will again be given by (4). We thus have:

$$\rho(Q, I)=10^{\frac{3}{2}}\pi^4(2I+1)x^{-4}(\gamma\zeta)^{-1}e^x/216. \quad (25)$$

The mean spacing between the levels is now:

$$\Delta=1/\rho(Q, I)=\Delta_0/(2I+1) \quad (26)$$

with $$\Delta_0=216\times10^{-\frac{3}{2}}\pi^{-4}\gamma\zeta x^4e^{-x}. \quad (27)$$

Values of Δ_0 computed from Eq. (27) are given in Table III for $\gamma=2$, and for different assumed values of the nuclear radius, R. The values of the nuclear radius listed are for a nucleus of mass number $A=230$; it is assumed that the volumes of all nuclei are proportional to A. The Gamow value is $R=9\times10^{-13}$ cm, while the value recently suggested by Bethe is 13×10^{-13} cm. This table should be compared with Bethe's Table I.

The energy Q set free in the capture of a slow neutron is of the order $7-10$ MV. The relevant values of QA thus range from about 800 MV for $A\sim100$ to about 1400 MV for very heavy nuclei ($A\sim200$). From the table, one sees that the mean spacing of levels of zero angular momentum ($I=0$) ranges from 150,000 to 5000 volts if $R=9\times10^{-13}$ cm and from 1000 to 4 volts if $R=13\times10^{-13}$ cm. An average spacing of some tens of thousands of volts seems to be much too large to be reconciled with the frequent occurrence of resonance levels; an average spacing of some hundreds of volts would be more reasonable. The results of the present calculation thus give evidence in favor of the larger radius. A radius as large as 13×10^{-13} cm is not required, however, as the values of Δ_0 listed in Table III for this radius are probably somewhat too small, and they would be even smaller if we had taken into account the decrease of γ with R which is given in Table I.

It is questionable how much one can rely on a calculation of the density of nuclear levels which is based on a statistical model. Actually very few particles are excited. The total number of excited particles and holes is

$$n=\frac{6}{\pi}\left(\frac{AQ}{\gamma\zeta}\right)^{\frac{1}{2}}\log 2,$$

which, for $AQ=1000$ MV, $\zeta=20$ MV and $\gamma=2$ is only 6 or 7. The number is increased to 9 if $\zeta=10$ MV (corresponding to $R=13\times10^{-13}$ cm) for the same values of AQ and γ. The fluctuations in the density of levels among the individual nuclei are probably quite large. The calculation almost certainly leads to a too high density of levels of high angular momenta. If the computed value of the density of levels of low angular momenta is also too high, the conclusion that the frequency of occurrence of resonance levels indicates a larger radius than the Gamow value would not be invalidated.