his Fig. 3 and our Fig. 1), even when the errors in his paper are corrected.[18]

The last point of comparison between theory and experiment is in the matter of absolute yield.[19] Choosing reasonable values for n and β in Eq. (15), obtained by converting α-particle to deuteron ranges,[20] one obtains for the yield at 100 kv from a thick D_3PO_4 target about 3×10^{-4}. The absolute experimental yield from such a target varies with the experimenter over a range of about 10^{-6} to 10^{-9}; thus the theoretical yield is too large by a factor of about 10^3 to 10^5.

In conclusion it is well to repeat that that portion of the theory which concerns itself with the symmetry properties of the situation, leading to Eqs. (11) and (12), is quite rigorous, while the detailed evaluation of the quantities appearing therein is very crude, and serves as little more than an indication of the general nature of the results to be expected. Again, the excellent agreement of relative yields (see Figs. 1 and 2) is due primarily to the Gamow factor, and not to the rest of the theory. Experimental data on absolute cross section as a function of energy, and on angular distribution for higher energies, both obtained with gas targets (to eliminate the effect of penetration into the target), would be very useful at this time.

I wish to express my deep appreciation to Professor Philip M. Morse for his constant encouragement and for his help at many points of the theory. I also wish to thank Professor J. H. Van Vleck for criticizing the manuscript.

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[18] Dolch, Zeits. f. Physik **104**, 473 (1937). It is difficult to understand why Dolch's results should deviate so much more from experiment than ours, when both calculations are based on the Gamow factor. In any case, the symmetry properties developed in the present paper, while important for angle distribution, have a small effect on the yield function.

[19] I am indebted to Professor Robley D. Evans for discussion of this point.

[20] Mano, J. de phys. et rad. **5**, 628 (1934).

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On the Nuclear Two-, Three- and Four-Body Problems*

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The simplest nuclear Hamiltonian satisfying all present requirements includes a Majorana-Heisenberg interaction $\{(1-g)P+gPQ\}V(r)$ between unlike particles and an attractive singlet interaction between like particles which is equal to that for unlike particles. The experimental mass defects of H^2 and H^3 together with the cross section σ for slow neutron-proton scattering will determine the range b and depth B of the triplet well and the proportion g of Heisenberg force (we use throughout the potential $Be^{-2r/b}$). An exact analytic expression relating σ, b, B and g is derived for this potential and g is found to be very insensitive to σ. An exact solution of H^2 gives the relation between B and b. The final relation which fixes the parameters is furnished by a Ritz-Hylleraas variational treatment of H^3 with the above Hamiltonian and the wave function:

$$\psi=2^{-\frac{1}{2}}\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2)\phi_1+6^{-\frac{1}{2}}(\alpha_1(\alpha_2\beta_3+\alpha_3\beta_2)-2\beta_1\alpha_2\alpha_3)\phi_2$$

where ϕ_1 and ϕ_2 each represents an exponential times a power series in the interparticle distances of proper symmetry (ϕ_2 is brought in by the Heisenberg term; the Breit-Feenberg operator is used for the small triplet like-particle interaction). The convergence of energies obtained from successive improvements in ψ is rapid and the eigenvalue may be closely estimated. After a relativistic correction is made we obtain: $b=1.73\times 10^{-13}$ cm; $B=242\ mc^2$ and $g=0.215$. The binding energy of He^3 is obtained by the same method and the H^3-He^3 difference is found to be $1.48\ mc^2$, agreeing well with experiment. The proton-proton scattering depth is checked to within 1 percent. When applied to He^4, our potential gives approximately 20 percent too much binding energy. Parallel calculations with the Gaussian and Morse curves lead to essentially the same result. No reasonable modification of the experimental data can explain more than a small fraction of the discrepancy.

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* The contents of this paper form part of a thesis submitted by William Rarita to the Faculty of Pure Science at Columbia University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The Nuclear Hamiltonian

KNOWLEDGE of the nature and magnitude of nuclear forces has undergone rapid development within the past few years. Recent articles by Feenberg and Knipp,[1] Bethe and Bacher[2] and Feenberg and Share[3] have extended our conception of the nuclear Hamiltonian to make it consistent with all the known facts derived from two particle scattering experiments and the binding energies of the hydrogen and helium isotopes. These calculations were of an approximate character, as was justified by the novelty and unreliability of some of the experimental results, and the tentative stage of the theory. However, the agreement between the calculations and experiment was well within the uncertainty of the former. Since a fairly definite formulation of the nuclear Hamiltonian has been arrived at (with the exception that the exact analytic character of the potential functions is not known, but only their general form), the question arises whether the experimental data are now sufficiently accurate to justify a more refined theoretical treatment of these problems. An affirmative answer is given for two reasons: (1) By curiously fortunate circumstances both magnitude and range of the nuclear forces are very insensitive to uncertainties in the experimental data to be used, (2) there exist just enough problems that can be treated in an exact or nearly exact fashion to determine all the essential constants that appear in the Hamiltonian and to verify most of the fundamental assumptions. Taking a definite form for the nuclear Hamiltonian, we have carried out this program and determined the constants in such a way that the probable error arising from both experimental and theoretical uncertainties is of the order of one percent.

The spin-dependence and saturation properties of the neutron-proton forces have been taken as evidence for the mixed Majorana-Heisenberg type of interaction. The existence of an interaction between like particles in the singlet state is established by the experiments on proton-proton scattering[4] and by the failure of neutron-proton forces alone to account for the binding energies of light nuclei.[5] It is consistent with all the experimental evidence to suppose that the non-Coulomb proton-proton forces are identical with the neutron-neutron forces. Breit, Condon and Present have shown that the singlet interaction between two protons, as determined from the experiments of Tuve, Heydenburg and Hafstad,[6] agrees to within one percent with the singlet interaction between neutron and proton as determined from the experiments of Fermi and Amaldi,[7] if the Coulomb force is taken to act at all distances. This reduces the number of arbitrary constants in the nuclear Hamiltonian to four: the radius of interaction and singlet state depth of the potential well for all pairs of particles, the triplet state depth for neutron-proton interaction and the triplet depth for like-particle interaction. Of these constants, only the first three enter effectively into any of the problems under consideration.[8] The triplet depth for like-particle interaction has a small effect on the three and four body calculations which will be discussed later. Two possible forms for this interaction have been proposed, respectively, by Van Vleck[9] and by Breit and Feenberg.[10] An elegant formulation of the nuclear Hamiltonian which is consistent with the facts mentioned above, has been given by Breit and Feenberg. They include the possibility that an ordinary force and a spin exchange force, both small in comparison with unity, may appear in the Hamiltonian. There appears to be no necessity for introducing these modifications at present and in any case they would give rise only to negligible third-order effects in our calculations.[11]

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[1] Feenberg and Knipp, Phys. Rev. **48**, 906 (1935).

[2] Bethe and Bacher, Rev. Mod. Phys. **8**, 82 (1936).

[3] Feenberg and Share, Phys. Rev. **50**, 253 (1936).

[4] Breit, Condon and Present, Phys. Rev. **50**, 825 (1936).

[5] Present, Phys. Rev. **50**, 635 (1936).

[6] Tuve, Heydenburg and Hafstad, Phys. Rev. **50**, 806 (1936).

[7] Amaldi and Fermi, Phys. Rev. **50**, 899 (1936). See this paper for complete references to the Italian journals.

[8] The triplet like-particle interaction enters only into the partial waves of odd angular momentum l in the proton-proton scattering, but for the experimental range of energies only the $l=0$ wave is distorted. The ground states of H^3, He^3 and He^4 are all singlet states in the like-particle interaction, if second-order mixing effects due to the Heisenberg operator are neglected.

[9] Van Vleck, Phys. Rev. **48**, 367 (1935).

[10] Breit and Feenberg, Phys. Rev. **50**, 850 (1936).

[11] For the ^3S and ^1S states of H^2, the operator P reduces to unity, whereas in H^3 and He^4 Majorana and ordinary forces give results differing only slightly (because of incomplete saturation of the forces and near symmetry of the Hamiltonian). It is thus a good approximation to

We write the nuclear Hamiltonian in the simplified Breit-Feenberg form using nuclear units (mc^2 for energy, $\hbar(Mm)^{-\frac{1}{2}}c^{-1}=8.97\times 10^{-13}$ cm for length):

$$H=\sum_i -\tfrac{1}{2}\nabla_i{}^2+\sum_{i>j}\{(1-g)P_{ij}+gP_{ij}Q_{ij}\}J(r_{ij})+\sum_{i_p>j_p}\frac{\kappa}{r_{i_pj_p}}. \qquad (1)$$

P_{ij} is a permutation operator exchanging the space coordinates of particles i and j; Q_{ij} exchanges spin coordinates; g is an arbitrary parameter; $J(r)$ is a potential function of the familiar "well" type; and κ is the constant $(\hbar c)^{-1}e^2(M/m)^{\frac{1}{2}}=0.313$. The first summation gives the kinetic energy of all particles; the second summation, taken over all pairs of particles, represents the specific nuclear forces; the last summation, taken over all pairs of protons, gives the Coulomb energy. Throughout the following, we assume that $J(r)=-Be^{-2r/b}$. Previous calculations, together with one to be mentioned subsequently, show that substantially equivalent results are to be expected from the potential $Ae^{-(r/a)^2}$. It is well known from the work of Massey and Mohr,[12] Dolch,[13] and Morse, Fisk, Schiff and Shockley[14] that the properties of the light nuclei are very insensitive to the exact analytic character of the potential.

We propose to determine the parameters b, B and g from exact treatments of the deuteron and of the scattering of slow neutrons in hydrogen, and by a precise variational treatment of H^3. Verification of fundamental assumptions will be obtained by applying the potentials so determined in order to calculate the proton-proton scattering and the Coulomb energy of He^3. Finally a crucial test is provided by the alpha-particle.

Neutron-Proton Scattering

The known binding energy of the deuteron ($4.35\pm 0.10\ mc^2$)[15] determines the triplet neutron-proton depth B in terms of the range b of the forces. For the exponential potential $Be^{-2r/b}$ the relations are: $J_\lambda[b(B)^{\frac{1}{2}}]=0$, $\lambda^2=-b^2E$, $E=$ total energy of H^2 in ground state. Details may be found elsewhere.[2]

The relation between b and the singlet state depth $B'=(1\text{-}2g)B$ may be obtained from the scattering of either slow neutrons or fast protons in hydrogen. We choose the former, for which a simple exact relation may be found in the case of the exponential potential. The experiments of Fermi and Amaldi[7] have shown that the cross section for scattering of slow *nonthermal* neutrons on protons is about 13×10^{-24} cm², corresponding to a mean free path in paraffin of 1 cm. They have also shown that the large scattering is due to a positive logarithmic derivative of the wave function at the boundary of the well, i.e., that no stable ^1S level exists.

The Schrödinger equation for the relative motion

$$\{d^2/dr^2+E+Ve^{-2r/b}\}u(r)=0$$

is transformed into Bessel's equation by the substitution $y=b(V)^{\frac{1}{2}}e^{-r/b}$. For a positive incident energy E, the general solution is given by

$$u=c_1J_{ip}(y)+c_2J_{-ip}(y),$$

where $p=bk=b(E)^{\frac{1}{2}}$. The boundary condition at $r=0$ requires that

$$c_1J_{ip}[b(V)^{\frac{1}{2}}]+c_2J_{-ip}[b(V)^{\frac{1}{2}}]=0.$$

At infinity the solution is normalized to a sine wave of unit amplitude:

$$\lim_{y\doteq 0}\{c_1J_{ip}(y)+c_2J_{-ip}(y)\}=\sin(kr+\delta_0).$$

Imposing the two boundary conditions and allowing E to approach zero (very slow neutrons), one obtains the relation

$$\frac{1}{\alpha}=b\left\{\frac{\pi}{2}\frac{N_0[b(V)^{\frac{1}{2}}]}{J_0[b(V)^{\frac{1}{2}}]}-\log\left[\frac{b(V)^{\frac{1}{2}}}{2}\right]-C\right\}, \qquad (2)$$

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replace the ordinary force by a Majorana force and the spin exchange force by a Heisenberg force. More precisely, the near symmetry of the zero-order wave functions for H^3 and He^4 leads to the result that both spin exchange and ordinary forces can produce separate effects only in the third-order approximation of the perturbation theory (cf. reference 10).

[12] Massey and Mohr, Proc. Roy. Soc. **A152**, 693 (1935).

[13] Dolch, Zeits. f. Physik **100**, 401 (1936).

[14] Morse, Fisk and Schiff, Phys. Rev. **50**, 748 (1936); Fisk, Schiff and Shockley, Phys. Rev. **50**, 1090 (1936).

[15] This value was communicated to us by Professor Bethe at the Ann Arbor summer session. Cf. Feather, Nature **136**, 467 (1935).

where $\alpha = k \cot \delta_0$, $C = 0.5772\cdots$ (the Euler constant), and N_0 is the Bessel function of second kind and of order 0 in the notation of Jahnke-Emde. α is approximately equal to the logarithmic derivative of the wave function at the boundary of the well and the sign of α is positive if V gives rise to a "virtual level" and negative if V produces a stable level.[16]

It is interesting to compare the formula (2) with the perturbation formula of reference 4 in order to test the accuracy of the latter.[17] The perturbation formula gives in this case:

$$V - V_0 = -\frac{\alpha u^2(\infty)}{\int_0^\infty u^2(r) e^{-2r/b} dr} = -\frac{2\alpha}{b} \frac{1}{J_0{}^2(bV^{*\frac{1}{2}}) + J_1{}^2(bV^{*\frac{1}{2}})}, \tag{3}$$

where V_0 is the depth that places a stationary level at $E=0$ and V^* is taken in first approximation as V_0 and in second approximation as V. For constants of the proper order of magnitude for the virtual ${}^1S$ state, formulas (2) and (3) agree within 3 percent in first and $\frac{1}{2}$ percent in second approximation.

For very slow neutrons ($E=0$, $l=0$), the cross section for elastic scattering is given by:

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 = \frac{4\pi}{E + \alpha^2} = \frac{4\pi}{\alpha^2}.$$

The complete cross section, averaged over the two possible orientations of spin, is

$$\sigma^a = \pi(1/\alpha_s{}^2 + 3/\alpha_t{}^2), \tag{4}$$

where α_s and α_t are to be obtained from formula (2) with $V = B'$ (singlet) and $V = B$ (triplet), respectively.[18] We use formula (2) to obtain α_t from B and b, then formula (4) to obtain α_s; finally formula (2) gives $B' = (1-2g)B$ in terms of b. B' is determined with great accuracy by this means, because the scattering is very sensitive to the depth of the well. A ten percent variation in σ^a will produce a variation in B' or g of less than one percent.

TABLE I.

TYPE OF TERM	NEUTRON-PROTON POTENTIAL	NEUTRON-NEUTRON POTENTIAL
$(\phi_1 \vert V \vert \phi_1)$	$(1-g/2)[J(r_{12})P_{12} + J(r_{13})P_{13}]$	$(1-2g)J(r_{23})$
$(\phi_2 \vert V \vert \phi_2)$	$(1-3g/2)[J(r_{12})P_{12} + J(r_{13})P_{13}]$	$-J(r_{23})$*
$(\phi_1 \vert V \vert \phi_2)$	$(g\sqrt{3}/2)[J(r_{12})P_{12} - J(r_{13})P_{13}]$	0

* The triplet neutron-neutron interaction given here is that which follows from the Breit-Feenberg form (1) of the nuclear Hamiltonian. The Van Vleck operator would give $-((1-2g)/3)J(r_{23})$ in place of $-J(r_{23})$, i.e., about one-fifth as much repulsion. The arbitrariness involved in this choice is not important since in the first place the entire mixing effect is small and in the second place the interaction is only large at close distances where the wave function is small due to orbital antisymmetry.

THEORY OF THE THREE-BODY PROBLEM

It has been seen that two of the three parameters occurring in the nuclear Hamiltonian can be fixed by exact relations. The three-body problem furnishes the third relation and the desired accuracy may be attained by the use of the Ritz-Hylleraas method. One of the writers[5] has applied this method to the H^3 nucleus, using a Hamiltonian with ordinary forces between unlike particles but with no forces between like particles. Such a model was found to be incompatible with the experimental facts. In the present paper we generalize the previous calculation to take account of the nuclear Hamiltonian discussed in section one. The generalization is threefold: (1) Like-particle forces are included, (2) The matrix elements for unlike particle forces are complicated by the presence of the Majorana operator, (3) The Heisenberg operator causes "mixing effects."

The Majorana-Heisenberg operator $(1-g)P + gPQ$ reduces in first approximation to the effective Majorana operator $(1-g/2)P$. The error introduced by neglecting the second order mixing effect has been estimated by Breit and Feenberg[10] and also by one of the writers.[19] The latter estimate of $0.5\ mc^2$ shows that the correction is by no means negligible. The effect can be treated in a rigorous manner by including in the wave function a term of the "borrowed" symmetry. The possible configurations of the three-particle system include one quartet and two doublet states, and it is readily seen that the operator Q connects

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[16] The symbol α differs from that used in Bethe and Bacher. In their notation $\alpha \equiv -u'(a)/u(a) \cong -k \cot \delta_0$ but in our notation $\alpha \equiv k \cot \delta_0 \cong u'(a)/u(a)$ where a is the maximum range of the forces.

[17] Reference 4, formulas (10.3), no. 2.

[18] It is worthy of note that a small error would be introduced by using $3/\epsilon_t$ in place of $3/\alpha_t{}^2$ ($\epsilon_t = 4.35\ mc^2$). For our final choice of constants $3/\epsilon_t = 0.69$ and $3/\alpha_t{}^2 = 1.19$, and the error in the derived value of $\alpha_s{}^2$ is 11 percent. However, this would cause an error of only 1 percent in B'.

[19] Present, Phys. Rev. **50**, 870 (1936).

the ground state (doublet with antiparallel neutron spins) only with the other doublet having parallel neutron spins. In terms of the proper spin functions for a total spin angular momentum of $\frac{1}{2}$ in the positive z direction, the complete wave function ψ may be written as (the subscript 1 refers to the proton)

$$\psi=2^{-\frac{1}{2}}\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2)\phi_1+6^{-\frac{1}{2}}(\alpha_1(\alpha_2\beta_3+\alpha_3\beta_2)-2\beta_1\alpha_2\alpha_3)\phi_2,$$

where ϕ_1 is a function to be specified later which is symmetrical in the space coordinates of the two neutrons, and ϕ_2 is similarly antisymmetrical. The only formal complication arising from the presence of ϕ_2 occurs in the potential energy terms which are spin-dependent. The results obtained from summing over spins, are given in Table I.

The nuclear Hamiltonian (1) for H^3 is written in variational form in terms of the elliptic coordinates $s=r_{12}+r_{13}$, $t=r_{12}-r_{13}$, $\rho=r_{23}$. The procedure and the notation are the same as in the previous paper by one of us.[5] The functions ϕ_1 and ϕ_2 are each taken to be an exponential in s multiplied by a power series in s, t, ρ of proper symmetry.

$$\begin{aligned}\phi_1&=e^{-k_1(s/2)}\sum_{p,\,r,\,\text{even }q}c^{(1)}{}_{pqr}s^pt^q\rho^r,\\ \phi_2&=e^{-k_2(s/2)}\sum_{p,\,r,\,\text{odd }q}c^{(2)}{}_{pqr}s^pt^q\rho^r.\end{aligned}\tag{5}$$

For convenience we take $k_1=k_2=k$ and adjust k to make the most important terms of ϕ_1 as good as possible. The summations are to be extended over all positive values of the indices, subject to the above restrictions, and as many terms taken as result in an appreciable lowering of the total energy. It is convenient to make the transformations $s'=ks$, $t'=kt$, $\rho'=k\rho$ (and drop the primes for the remainder of this section). Then the energy matrix element between any terms m and n in the double series is given by:

$$H_{mn}=k^2(P_{mn}+Q_{mn})+L_{mn}(k).$$

The kinetic energy integrals P_{mn}, Q_{mn} and the unity matrix element N_{mn} are different from zero only if m and n are in the same series and in this case they are given by the formulas of reference 5. The matrix elements of potential energy L_{mn} are obtained from the following formulas (Latin subscripts refer to terms in φ_1; Greek subscripts to φ_2).

$$\begin{aligned}L_{mn}&=-(1-g/2)Bl_{mn}{}^{+}-(1-2g)Bl_{mn}{}^{*},\\ L_{\mu\nu}&=-(1-3g/2)Bl_{\mu\nu}{}^{+}+Bl_{\mu\nu}{}^{*},\\ L_{m\nu}&=-((g\surd 3)/2)Bl_{m\nu}{}^{-},\end{aligned}$$

where

$$l_{ij}{}^{\pm}=\int_0^\infty ds\int_0^s d\rho\int_0^\rho dt\rho(s^2-t^2)e^{-s/2}s^{p_i}t^{q_i}\rho^{r_i}\{e^{-\beta(s+t)}P_{12}\pm e^{-\beta(s-t)}P_{13}\}e^{-s/2}s^{p_j}t^{q_j}\rho^{r_j}$$

$$l_{ij}{}^{*}=\int_0^\infty ds\int_0^s d\rho\int_0^\rho dt\rho(s^2-t^2)e^{-s}s^{p_i+p_j}t^{q_i+q_j}\rho^{r_i+r_j}e^{-2\beta\rho},\quad \beta=1/bk.$$

It is readily shown that $l_{ij}{}^{\pm}=l_{ji}{}^{\pm}$. Using the relations

$$P_{12}\begin{Bmatrix}s\\ t\\ \rho\end{Bmatrix}=\begin{Bmatrix}(s+t)/2+\rho\\ (s+t)/2-\rho\\ (s-t)/2\end{Bmatrix},\quad P_{13}\begin{Bmatrix}s\\ t\\ \rho\end{Bmatrix}=\begin{Bmatrix}(s-t)/2+\rho\\ -(s-t)/2+\rho\\ (s+t)/2\end{Bmatrix},$$

we may evaluate the integrals $l_{ij}{}^{\pm}$ in terms of

$$\begin{matrix}K_{acb}{}^{(12)}\\ \\ K_{acb}{}^{(13)}\end{matrix}=\int_0^\infty dse^{-(\beta+3/4)s}s^a\int_0^s d\rho e^{-\rho/2}\rho^c\int_0^\rho dte^{\mp(\beta+1/4)t}t^b\tag{6}$$

and $l_{ij}{}^{*}$ in terms of

$$K^*{}_{acb}=\int_0^\infty dse^{-s}s^a\int_0^s d\rho e^{-2\beta\rho}\rho^c\int_0^\rho dtt^b.$$

Details of evaluation of the K's are reserved for the appendix. Using the optimum value of β, tables of K's are calculated for a wide range of the indices. Variation of the constants c_{pqr} determines the lowest energy obtainable from a given combination of the terms in (5) (Ritz-Hylleraas method).

TABLE II. *Coefficients of the H^3 and He^3 wave functions normalized to unity.*

Terms	$H^3(E=-15.880)$	$He^3(E=-14.431)$
[000]	0.450 422	0.433 685
[001]	−0.142 982	−0.135 8180
[002]	0.013 9300	0.012 9308
[003]	−0.000 49357	−0.000 45451
[100]	0.012 1361	0.012 9370
[010]	−0.006 0689	−0.006 29724

DETAILS OF H^3 CALCULATIONS

The range b of the forces must be found which will give the observed binding energy of H^3 ($16.55\pm0.25\ mc^2$).[15] In preliminary calculations to determine this range approximately we have neglected the second-order mixing effects and also the relativistic correction. Suitable values for k and β must be selected at the outset and this is done by computing the energy given by the single (and most important) term $\varphi_1=e^{-k(s/2)}$.

For a radius $b=0.25$ nuclear unit the best values are $k=6.5$, $\beta=0.61$. At $b=0.20$ the lowest energy of $-5.39\ mc^2$ is obtained for $k=7.4$; with $\beta=0.61$ we would have $k=8.2$ and $E=-5.13$. Since the optimum k increases as b decreases it is satisfactory to use the same β for several nearby radii. If $\beta=0.61$ is used then the energy for $b=0.20$ misses its best value by only $0.26\ mc^2$ and it is clear that other terms in the wave function will automatically make up for this deficiency. Since the optimum value of k undergoes modification as successive terms in ψ are added, the use of a fixed β is probably within the uncertainty of this method of estimating k. As has been pointed out previously, the use of an unsuitable k or β will not affect the final energy to be obtained from this type of calculation, but may result in a poor convergence of the energy contributions of successive terms. The advantage of a fixed β is that only one set of potential energy integrals need be calculated. The value of $\beta=0.61$ was chosen for the initial computation at $b=0.2508$ and used for subsequent calculations at $b=0.200$ and 0.193.

The previous work by Feenberg and Knipp and by Bethe and Bacher indicated a radius for the Gaussian potential of 2.25×10^{-13} cm. (0.2508 nuclear unit). Our first calculation, made at this radius ($b=0.2508$), gave excellent convergence with six terms in ϕ_1 to an energy of $-12.32\ mc^2$. Clearly a smaller b was required. Repeating this calculation for $b=0.200$, we obtained -15.33 from an eight term function. Before attempting an accurate estimation of b, it was necessary to obtain an approximate idea of the corrections due to the relativistic and mixing effects. We have estimated the former according to the method of Feenberg,[20] for a range of the forces approximately determined from the preceding calculations. A brief calculation, mentioned in the appendix, gives $-0.47\ mc^2$ for the relativistic correction to the kinetic energy. According to one of the writers[19] the mixing effect contributes about $-0.5\ mc^2$ to the energy. Then the proper range of the forces, determined from a calculation with ϕ_1 alone, should lead to an energy of $-15.6\ mc^2$ approximately. It was found that for $b=0.193$ the ϕ_1 series gave energies converging to -15.7.

The final calculation with the complete wave function ψ including ϕ_2, was made at $b=0.193$. Let us designate by $[abc]$ a single term $s'^a t'^b \rho'^c$ in the ϕ_1 or ϕ_2 series. Then the most important terms in ϕ_1 are the following [000], [001], [002], [003], [100] which give in combination an energy of -15.56. When combined with this basic function, the terms [200], [300] and [004] contribute, respectively, 0.05, 0.06, and 0.02 mc^2. All other terms examined including [400], [020], [101] contribute less than 0.01 mc^2, from which it may be concluded that the lowest energy obtainable from a function with the symmetry of ϕ_1 may be less than -15.70 by only a few hundredths of 1 mc^2. This follows from the satisfactory convergence of energy contributions and the completeness of the function system.[20a] The energy is still further decreased by the mixing effect when terms in ϕ_2 are included. The terms investigated were [010], [011], [110] and [012], of which the first two lowered the energy by 0.43 mc^2 when inserted with the eight most important terms of ϕ_1 and the last two were altogether negligible. Then the ground state eigenvalue for the nuclear Hamiltonian (1) with $b=0.193$ may be said to occur at $-16.15\ mc^2$, the difference between this value and the experimental one of $-16.55\pm0.25\ mc^2$ being accounted for by the relativistic correction.

This determination of the nuclear force constants yields the following results: $b=0.193$ nuclear unit (about 1.73×10^{-13} cm), $B=241.8\ mc^2$ (triplet neutron-proton depth), $B'=137.6\ mc^2$

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[20] Feenberg, Phys. Rev. **50**, 674 (1936).

[20a] The completeness of the Hylleraas function system for the general three-body problem and the convergence of the successive approximations to the energy upon the lowest characteristic value of the differential equation have been proven by H. M. James and A. S. Coolidge in a paper to be published soon.

(singlet depth)[21] and $g=0.215$. The reliability of these results in the light of possible modifications of the experimental data will now be discussed. The cross section for scattering of slow nonthermal neutrons on protons seems to have been determined within a probable error of 10 percent.[7] The uncertainty in B' is then slightly less than 1 percent and this will produce an uncertainty in the calculated H^3 energy of 0.15 mc^2. The H^2 binding energy appears to be subject to an uncertainty of 3 percent and the H^3 binding energy, calculated from the deuteron-deuteron transmutation, has a corresponding uncertainty of 1.5 percent. Now an increase of ϵ in the measured H^2 binding energy would cause an increase of 2ϵ in the experimental H^3 binding energy and an increase of approximately 2ϵ in the calculated value. Hence no reasonable modification of the value 4.35 mc^2 will affect the values of b and B' given above and B will vary by 0.6 percent for a 3 percent modification. In the neighborhood of $b=0.193$ the H^3 binding energy increases by roughly 0.10 mc^2 for a decrease in b of 0.001 nuclear unit.

The wave function for H^3 is readily calculated from the secular determinant evaluated at the energy for which it vanishes.[22] Because of theoretical uncertainties in the nuclear model, there would be no point to evaluating the coefficients c_{pqr} for the best 10 term function, which gives an energy of -16.12 mc^2. However, there exists a simple six-term combination giving only 1.5 percent less binding energy. Coefficients for this function, normalized in the coordinates s', t', ρ' may be found in Table II.[23]

FURTHER APPLICATIONS

Having completed the calculations for H^3, it was a relatively simple matter to work out the binding energy of He^3. This was done *directly* by computing the matrix elements of the Coulomb energy, adding them to the H^3 matrix elements, and solving the secular equations. The He^3 energy matrix element H_{mn} includes the Coulomb term C_{mn} where

$$H_{mn}=k^2(P_{mn}+Q_{mn})+kC_{mn}+L_{mn}(k),$$
$$C_{mn}=\kappa[J(200)-J(020)].$$

Here κ is the constant of Eq. (1) and $J(abc)$ is the integral defined in reference 5. The matrix elements C_{mn} vanish between m and n of different symmetry. Those ten terms were included in the He^3 wave function which had been found most important for H^3. Their relative weight was the same as before and the convergence of energies was slightly better. Coefficients for the same six-term wave function used with H^3 are to be found in Table II. The ten term function for He^3 gives 14.645 mc^2 for the binding energy as compared with 16.124 mc^2 for H^3 with the same ten terms (with slightly different coefficients of course). The difference of 1.48 mc^2 represents the Coulomb energy of He^3 with great accuracy; since the corrections for convergence, relativistic effect, inaccuracy in experimental data, and uncertainty in the analytic character of the potential would all act as small perturbations affecting the two problems in nearly equal fashion. This value for the Coulomb energy is then somewhat better than the theory behind it. The difference between our value and that obtained by Share[24] (1.37 mc^2) is principally due to the use of quite different radii of interaction for the forces. For the Gaussian analog to our radius (about 0.217 n.u., see below), Share's method would give 1.45 mc^2. As first pointed out by Goldsmith and fully discussed in Bethe and Bacher, the difference in binding energies of H^3 and He^3 is directly measurable as the difference in reaction energies of the two deuteron-deuteron transmutations. Bethe and Bacher give the value 1.49 ± 0.27 mc^2; a later estimate by Bethe, quoted by Share is 1.58 ± 0.18 mc^2. It is seen that our calculated value lies well within the range of error of both estimates; hence we may conclude that the fundamental assumption of the equality of neutron-neutron and non-Coulomb proton-proton forces is verified quantitatively by experiment.

The other explicit assumption used in this work is that the singlet neutron-proton and proton-

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[21] For these values of b and B' the "virtual level" comes at 128 kv.

[22] Cf. James and Coolidge, J. Chem. Phys. **1**, 825 (1933).

[23] It is worthy of note that the coefficients of the successive powers of ρ' alternate in sign and diminish rapidly in size. They diminish much more rapidly than would be expected from the exponential function $e^{-\rho'/2}$ or from any simple exponential. From this we may infer that the convergence would not have been much improved by using a symmetrical exponential $e^{-k/2(s+\rho)}$ in φ (Eqs. 5).

[24] Share, Phys. Rev. **50**, 488 (1936).

proton interactions are identical. Confirmation of this assumption is provided by using the proton-proton scattering data of Tuve, Heydenburg and Hafstad to determine the potential depth corresponding to the radius of $b=0.193$ for the simple exponential function. In accordance with the methods described in reference 4, this depth was found to be 135.8 mc^2 as compared with the value of 137.6 mc^2 determined from the neutron-proton scattering. Both figures have a probable error of 1 percent arising from the uncertainties in their experimental determination. As pointed out in reference 4, the excellence of the agreement is seriously impaired if the Coulomb force is taken to act only outside the well. This result is consistent with the agreement of our calculated Coulomb energy for He^3 with experiment, since it was there assumed that the Coulomb potential extended all the way to the origin.

The Alpha-Particle

The triplet neutron-proton potential, determined by our calculation to be $Be^{-2r/b}(B=242\ mc^2,\ b=0.193\ \text{n.u.})$, appears to differ considerably from $Ae^{-(r/a)^2}$ ($A=71.5\ mc^2$, $a=0.250$ n.u.), which is the final potential obtained by Feenberg and Share in agreement with Bethe and Bacher who use the same method. Some calculations mentioned in reference 5 indicated a correspondence of the simple and Gaussian exponentials for $a=b$. The large difference in depths is then partly to be explained by the steeper decline of the simple exponential and partly by a genuine discrepancy in the radii.[25] Feenberg and Share have used nearly the same values of the experimental data as ourselves, except for the ^1S level of the deuteron which they assume to lie at $E=0$. They have made their calculations on H^3 and He^4 with the equivalent two-body method, which is known to give an upper limit to the binding energy and for which they have computed a first-order correction. Both of these facts would tend to explain their greater value for the range of the forces; however, neither is the principal reason for the discrepancy, which is due essentially to the determination of the forces from two different sets of problems. Whereas they have used $H^2(^3S)$, $H^2(^1S)$, H^3 and He^4 to establish the range of the forces, the triplet neutron-proton depth and the singlet neutron-proton and neutron-neutron depths, we have used $H^2(^3S)$, $H^2(^1S)$ and H^3 explicitly and the proton-proton scattering implicitly (through equality of the singlet interactions) in our determination. Therefore the essentially different results appear to imply incompatibility of the three binding energies and two scattering cross sections with the nuclear Hamiltonian in its present form. We consider this possibility further.

The like-particle depth calculated by Feenberg and Share is 41 mc^2 which is to be compared with the value of approximately 36.6 mc^2 determined from the proton-proton scattering for the same radius ($a=0.250$) of the Gaussian potential. This comparison has been taken to indicate satisfactory agreement and general confirmation of the nuclear model. As a matter of fact 41 mc^2 lies far outside the range of error of the scattering depth which is determined to within 1 percent by the present experimental data. Considering the approximations made in the treatments of H^3 and He^4, the discrepancy may be justified.

In order to determine whether the theory agrees with experiment within the limits of error of experiment, we have applied the potential determined in the preceding sections to a calculation of the binding energy of the alpha-particle. A simple variational method gives significant results. If the wave function is assumed as

$$\chi=N\exp\left[-(\nu/2)(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)\right]$$

then the variational energy to be minimized with respect to ν is given by

$$\begin{aligned}E&=9\nu-(4B^*+2B')F(\sigma),\\ F(\sigma)&=\{[2\sigma^2+1][1-\varphi(\sigma)]e^{\sigma^2}-2\sigma/\pi^{\frac{1}{2}}\},\end{aligned}\tag{7}$$

where $\sigma=1/(b(2\nu)^{\frac{1}{2}})$, $\varphi(\sigma)$ denotes the error function integral (tabulated in Peirce) and B^* and B' represent the effective neutron-proton depth $B(1-g/2)$ and the singlet like-particle depth $B(1-2g)$, respectively. Minimization of E gives

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[25] An experimental determination of the range of nuclear forces can be made by measuring the proton-proton scattering at the lowest energy for which an anomalous scattering is observable (about 100–200 kv) and again at the highest energy available. The energy dependence of the phase shift will determine the proper radius of interaction of the forces.

an energy for He^4 of $-56.5\ mc^2$. The exact energy must certainly lie below this. The experimental value is $-55.2\ mc^2$ which is subject to three corrections before comparison with the calculated value. The Coulomb energy of $1.7\ mc^2$ approximately, the mixing effect correction of the order of $-0.5\ mc^2$, and the relativistic correction of the order of $-1.5\ mc^2$ will produce a mutual cancellation, leaving $-55\ mc^2$ as the value to be compared with theory. It is seen that the simple wave function χ already gives too much binding.

The wave function may be improved by taking a linear combination of functions χ with different values of the parameter ν.[26] The matrix elements are:

$$H_{ii}=9\nu_i-(4B^*+2B')F(\sigma_i),\quad N_{ii}=1,$$

$$H_{ij}=18\nu_i\nu_jN_{ij}/(\nu_i+\nu_j)-(4B^*+2B')N_{ij}F(\sigma_{ij}),$$

$$N_{ij}=16\sqrt{2}(\nu_i\nu_j)^{9/4}/(\nu_i+\nu_j)^{9/2},\quad \sigma_{ij}=1/(b(\nu_i+\nu_j)^{\frac{1}{2}}).$$

With three terms included the energy is lowered to $-62.2\ mc^2$. In order to estimate the approximate location of the eigenvalue, we apply Feenberg's equivalent two-body method to this problem (see appendix for application of two-body method to H^3 and He^4 with an arbitrary potential function). The resulting energy is $-71.7\ mc^2$ which may be taken as a lower bound. Now Feenberg and Share have shown that if the potential is a function of only even powers of the distance, then a correction can be computed to the two-body method which is approximately twice as large for He^4 as for H^3. It is reasonable to suppose that nearly the same relation will hold for our potential as well, since its general form can be approximated by even power functions. The two-body method gives $-18.2\ mc^2$ for H^3 which is to be compared with $-15.7\ mc^2$, the value obtained previously by the Hylleraas method ignoring mixing effects. Hence the corrected He^4 will have an energy of $-66.7\ mc^2$. If we take this value as the true theoretical energy for He^4 with out potential, the discrepancy is $11.7\ mc^2$ or 20 percent of the experimental value. No sensible modification of the experimental data will suffice to explain so large a difference, since an increase in the range b of 0.001 n.u. causes a decrease in the He^4 binding energy of only 0.25 mc^2 and the uncertainty in the other problems will not justify a much greater increase in b. To explain the binding energy of He^4, it is clear that we should have to go to nearly as wide a radius as that found by Feenberg and Share.

Does this mean that the simple exponential well is not an acceptable analytic form for the nuclear potentials? First it is well to see if the same discrepancies would be encountered with other analytic forms. This will be done in an approximate manner for the Gaussian and Morse potentials. We proceed to fit the experimental data in the same order as previously done with the simple exponential function, using the corrected equivalent two-body method for H^3 and He^4. Neutron-proton scattering, the ^3S level of H^2 and the binding energy of H^3 determine the constants for the Gaussian potential $Ae^{-(r/a)^2}$ with the following values:[27] $a=0.217$ n.u. $(1.945\times 10^{-13}$ cm), $A=88.2\ mc^2$ (triplet neutron-proton depth), $A'=49.4\ mc^2$ (singlet depth), $g=0.220$. When these potentials are applied to He^4, the variational method with the function χ gives $-51.5\ mc^2$ and the corrected equivalent method $-63.2\ mc^2$. The discrepancy is 15 percent of the experimental value.

The Morse potential $V(r)=-2Ce^{-2(r-c_1)/c}+Ce^{-4(r-c_1)/c}$, discussed in reference 14, was taken to have an inner repulsive region defined by $c_1=c/2$. It might be expected that the presence of the repulsive region would tend to minimize the discrepancy, since this region would be of more importance for He^4 than for H^3 because of the narrower dimensions of the former nucleus. The neutron-proton scattering and the binding of H^2 are treated by exact relations given in reference 14; the equivalent two-body method (see appendix) is used for H^3 with a correction intermediate between those found for $Be^{-2r/b}$ and $Ae^{-(r/a)^2}$. This results in the following values for the constants: $c=0.150$ n.u., $c_1=0.075$ n.u., $C=100.8\ mc^2$ (triplet neutron-proton), $C'=56.7\ mc^2$ (singlet) and $g=0.218$. The inner repulsive region for the triplet interaction rises to a height of $+197\ mc^2$. A treatment of He^4 with these potentials by the corrected equivalent method gives

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[26] This was suggested to us by Dr. Feenberg.

[27] Since the numerical integrations needed in the derivation of A' from the Fermi-Amaldi data are somewhat tedious, we observe that the following relation holds approximately for values of a investigated: $a^2A'=2.33$.

about $-65\ mc^2$, representing an excess binding of 18 percent.

Although the calculations for the Gaussian and Morse potentials involve an uncertain approximation in the treatment of H^3, nevertheless the results are qualitatively significant and indicate that no simple potential well is likely to explain the discrepancy. A potential function falling off more steeply than the simple exponential is almost certainly excluded; on the other hand a function with a large inner repulsive region, giving rise to a predominantly repulsive proton-proton interaction, is excluded by the proton-proton scattering experiments.[4] Although very shallow long-range forces would have an inappreciable effect on the binding, they would lead to phase shifts of higher order in the proton-proton scattering which might destroy the agreement of the singlet interactions and invalidate the calculations of this paper. It is also possible, without contradicting the proton-proton scattering experiments, that an additional force of intermediate range ($<3e^2/mc^2$) or a momentum-dependence unobservable for low energies (<1 Mev) may occur to explain the discrepancy. Another possibility is that the singlet and triplet interactions have different ranges; this would imply that g is a simple function of the distance.[28] Also the binding energy of He^4 may be reduced relative to that of H^3 by the introduction of repulsive many-body forces such as those which would occur in the proposed theory of Gamow and Teller.[29]

As previously mentioned the introduction of small spin exchange and ordinary forces would have only a third-order effect on these calculations and as long as the operator is predominantly Majorana the form of interaction used by us is justified. The success of Feenberg and Wigner,[30] Wigner,[31] and Rose and Bethe [32] in explaining properties of intermediate nuclei, using the Majorana type of interaction, is good evidence for the essential correctness of the latter. Direct but somewhat uncertain evidence for a predominantly Majorana type of interaction between like particles is afforded by the proton-proton scattering experiments which seem to require for their interpretation at small angles a negative phase shift in the p wave and a consequent repulsive triplet interaction.[4] Finally it is possible that either or both of the fundamental assumptions that were verified in these calculations (cf. Section 5) may be partly incorrect. The proton-proton forces may not be identical with the neutron-neutron forces, as was assumed to be true in the H^3 calculation and verified for He^3. Also the agreement between the singlet proton-proton and proton-neutron potentials may be a coincidence; in reality they may have slightly different ranges and depths.

We conclude that it is possible to fit accurately only five out of the six experimental data by means of three constants and two assumptions. Since it is possible to fit all six with a fair degree of approximation,[33] the nuclear model, though incorrect in its present form, may still be of great usefulness. From this point of view our tabulation of H^3 and He^3 wave functions is to be justified.

It is a pleasant obligation to acknowledge the hospitality of the physics department at Ann Arbor, where this work was begun last summer.

APPENDIX I

Evaluation of the *K* integrals

Let $K_{a,c,b}{}^{\lambda,\mu,\nu}=\int_0^\infty dse^{-\lambda s}s^a\int_0^s d\rho e^{-\mu\rho}\rho^c\int_0^\rho dte^{-\nu t}t^b.$

Then

$$K_{a,c,b}{}^{(12)}=K_{a,c,b}{}^{\lambda,\mu,\nu}\ (\lambda=\beta+3/4,\ \mu=1/2,\ \nu=\beta+1/4),$$
$$K_{a,c,b}{}^{(13)}=K_{a,c,b}{}^{\lambda,\mu,\nu}\ (\lambda=\beta+3/4,\ \mu=1/2,\ \nu=-(\beta+1/4)),$$
$$K_{a,c,b}{}^{*}=K_{a,c,b}{}^{\lambda,\mu,\nu}\ (\lambda=1,\ \mu=2\beta,\ \nu=0).$$

The integrals $K_{a,c,b}{}^{\lambda,\mu,\nu}$ may be evaluated by means of two recursion formulas:

$$K_{a,c,b}{}^{\lambda,\mu,\nu}=\frac{1}{\lambda}\{R_{a+c,b}{}^{\lambda+\mu,\nu}+aK_{a-1,c,b}{}^{\lambda,\mu,\nu}\}$$

(valid for $a=0$), (1)

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[28] The potential function would have the form:

$$\left\{\frac{B+B'e^{-(\Delta\alpha)r}}{2}P+\frac{B-B'e^{-(\Delta\alpha)r}}{2}PQ\right\}e^{-\alpha r}.$$

[29] Gamow and Teller, Phys. Rev. **51**, 289 (1937). This theory appears to be inconsistent with the observed proton-proton scattering at small angles.

[30] Feenberg and Wigner, Phys. Rev. **51**, 95 (1937).

[31] Wigner, Phys. Rev. **51**, 106 (1937).

[32] Rose and Bethe, Phys. Rev. **51**, 205 (1937).

[33] For certain purposes, e.g., approximate calculations on intermediate and heavy nuclei, it may prove useful to have approximate values for the range and depth of the forces. In this case one may distribute the errors of the model more evenly by taking a radius of interaction for the Gaussian potential intermediate between the value of 1.95×10^{-13} cm determined by our method and the value of 2.25×10^{-13} obtained by Feenberg and Share. A suitable set of constants for $Ae^{-(r/a)^2}$ would be: $a=2.1\times10^{-13}$ cm, A(triplet) $=78.3\ mc^2$, A' (singlet) $=42.5\ mc^2$.

$$K_{a,c,b}{}^{\lambda,\mu,\nu}=\frac{1}{\nu}\{bK_{a,c,b-1}{}^{\lambda,\mu,\nu}-R_{a,b+c}{}^{\lambda,\mu+\nu}\},\tag{2}$$

$$K_{a,c,0}{}^{\lambda,\mu,\nu}=\frac{1}{\nu}\{R_{a,c}{}^{\lambda,\mu}-R_{a,c}{}^{\lambda,\mu+\nu}\},$$

where

$$R_{j,k}{}^{\alpha,\beta}=\int_0^\infty dxe^{-\alpha x}x^j\int^x dye^{-\beta y}y^k.$$

The integrals $R_{j,k}{}^{\alpha,\beta}$ are evaluated from the recurrence relations:

$$R_{j,k}{}^{\alpha,\beta}=\frac{1}{\alpha}\{S_{j+k}{}^{\alpha+\beta}+jR_{j-1,k}{}^{\alpha,\beta}\}\qquad(\text{valid for } j=0),\tag{3}$$

$$R_{j,k}{}^{\alpha,\beta}=\frac{1}{\beta}\{kR_{j,k-1}{}^{\alpha,\beta}-S_{j+k}{}^{\alpha+\beta}\},\tag{4}$$

$$R_{j,0}{}^{\alpha,\beta}=\frac{1}{\beta}\{S_j{}^{\alpha}-S_j{}^{\alpha+\beta}\},$$

where

$$S_n{}^{\gamma}=\int_0^\infty dse^{-\gamma s}s^n=\frac{n!}{\gamma^{n+1}}.$$

APPENDIX II

The relativistic correction for H^3

Using the method of Feenberg,[20] we find for the relativistic first-order correction to the kinetic energy of the three-body problem:

$$\Delta T=-3m/8M\int\psi\Delta_1\Delta_1\psi d\tau_1 d\tau_{23}.$$

Let $\psi=Ne^{-(\nu/2)(r_{12}{}^2+r_{13}{}^2+r_{23}{}^2)}=N'e^{-(\nu/2)(2r^2+(3\rho^2/2))}$,

where $\mathbf{r}=\mathbf{r}_1-(\mathbf{r}_2+\mathbf{r}_3)/2,\quad \boldsymbol{\rho}=\mathbf{r}_2-\mathbf{r}_3.$

In these coordinates ΔT is readily evaluated to be

$$\Delta T=-(45m/8M)\nu^2=-0.00305\nu^2.$$

The value of ν may be determined by calculating the variational energy of H^3 with the trial wave function ψ and minimizing with respect to ν. This gives:

$$E=9\nu/2-(2B^*+B')F(\sigma),\quad \sigma=1/[b(3\nu/2)^{\frac{1}{2}}],$$

where $F(\sigma)$ is the function defined in Eq. (7). The value of ν obtained by minimizing E is 12.4 and the relativistic correction is $-0.47\ mc^2$. This is considerably larger than the value $-0.2\ mc^2$ estimated by Feenberg for the Gaussian potential, the difference being due to our use of a narrower interaction radius for the forces.

APPENDIX III

The equivalent two-body method for an arbitrary potential

Let the potential function be represented by $V(r)=B\varphi(r,\ b,\ b_1)$ where $B=B^*$ (effective neutron-proton depth) or $B=B'$ (singlet like-particle depth) and b is the range of the forces and b_1 any other parameter. Let the same potential function for the equivalent two-body problem have the constants D, d and d_1. Using the Gaussian wave function $Ne^{-\mu r^2/2}$ the variational energy of the two-body problem is

$$E=\frac{3}{2}\mu-D\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty e^{-x^2}\varphi(x/\mu^{\frac{1}{2}},\ d,\ d_1)x^2dx.\tag{1}$$

The Gaussian wave functions for H^3, $Ne^{-(\nu/2)(r_{12}{}^2+r_{13}{}^2+r_{23}{}^2)}$, and for He^4, $Ne^{-(\nu/2)(r_{12}{}^2+r_{13}{}^2+r_{14}{}^2+r_{23}{}^2+r_{24}{}^2+r_{34}{}^2)}$, give the following energy expressions:

$$E(\mathrm{H}^3)=\frac{9}{2}\nu-(2B^*+B')\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty e^{-x^2}\varphi(x/(3\nu/2)^{\frac{1}{2}},\ b,\ b_1)x^2dx,\tag{2}$$

$$E(\mathrm{He}^4)=9\nu-(4B^*+2B')\frac{4}{\pi^{\frac{1}{2}}}\int_0^\infty e^{-x^2}\varphi(x/(2\nu)^{\frac{1}{2}},\ b,\ b_1)x^2dx.\tag{3}$$

The equivalent method energy is obtained by solving the two-body problem exactly for the fictitious potential $D(r,\ d,\ d_1)$, where the parameters D, d and d_1 are determined by identifying (1) with (2) or (3). Thus for H^3 we must have:

$$D\equiv 2B^*+B',\quad 3\mu/2\equiv 9\nu/2,$$
$$\varphi(x/\mu^{\frac{1}{2}},\ d,\ d_1)\equiv\varphi(x/(3\nu/2)^{\frac{1}{2}},\ b,\ b_1).$$

Substituting the second identity into the third and transforming back to r, we obtain the correspondence relations for H^3:

$$D\equiv 2B^*+B',\qquad \varphi(r/(2)^{\frac{1}{2}},\ d,\ d_1)\equiv\varphi(r,\ b,\ b_1).$$

Similarly for He^4

$$D\equiv 4B^*+2B',\quad \varphi(r/(3)^{\frac{1}{2}},\ d,\ d_1)\equiv\varphi(r,\ b,\ b_1).$$

As an illustration consider the relations for H^3 with the Morse potential: $V(r)=-2Be^{-2(r-b_1)/b}+Be^{-4(r-b_1)/b}$. The two-body constants D, d and d_1 are determined by the conditions $D\equiv 2B^*+B'$ and

$$-2e^{2d_1/d}e^{-2r/d(2)^{\frac{1}{2}}}+e^{4d_1/d}e^{-4r/d(2)^{\frac{1}{2}}}\equiv-2e^{2b_1/b}e^{-2r/b}+e^{4b_1/b}e^{-4r/b}.$$

Hence $d=b/(2)^{\frac{1}{2}},\ d_1/d=b_1/b,\ d_1=b_1/(2)^{\frac{1}{2}}.$