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Inelastic Collision of Deuteron and Deuteron*

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Certain symmetry properties of the two deuteron-deuteron reactions are developed in a rigorous manner, and are shown to require a distribution of the reaction products for moderate incident deuteron energies of the form $A(1+B\cos^2\theta)$ in the coordinate system in which the center of mass is at rest. This is shown to be in qualitative agreement with recent experimental data. The quantities A and B are evaluated approximately using a modification of Born's method and neglecting polarization of the deuterons; the quantitative discrepancies between theory and experiment are discussed. The theoretical relative yield from a "thick" deuterium target agrees well with the results of several experimenters; this quantity is influenced mainly by the Gamow factor, and is relatively independent of the details of the theory. The theoretical absolute yield is too large to agree with experiment.

INTRODUCTION

THE calculation of the transmutation function and angular distribution of the products of the reactions

$$H^2+H^2\rightarrow He^3+n, \tag{A}$$

$$H^2+H^2\rightarrow H^3+p \tag{B}$$

is of interest because of the recent experimental work[1] available for comparison with the predictions of the theory. The computations in this paper will be carried through using a potential between all pairs of particles of the form[2, 3]

$$V(r)=J(r)[(1-g)P^M+gP^H]$$
$$=J(r)P^M[1-\tfrac{1}{2}g+2g\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2], \tag{1}$$

$$J(r)=De^{-4r/r_0}-2De^{-2r/r_0}, \tag{2}$$

in addition to the Coulomb repulsion between protons. We shall write all quantities in nuclear units of energy (506,000 ev) and of length (8.97×10^{-13} cm). A suitable value for r_0 which gives good agreement with experiment in computing a number of nuclear quantities[4, 5] is 0.3 (2.7×10^{-13} cm). Together with the value 4.35 (2.2 Mev) for the binding energy of the deuteron,[6] this fixes D at 71.2. Assuming that there is no bound singlet state of the deuteron, the elastic cross section 15 (12×10^{-24} cm^2) of protons for thermal neutrons[7] fixes the depth of the singlet unlike particle interaction at 28.9; so that $g=0.3$.

SYMMETRY PROPERTIES

In this section we shall present in a rigorous manner certain symmetry properties of the

* Preliminary report presented at the Cambridge meeting of the New England section of the American Physical Society, February 6, 1937.
[1] Kempton, Browne and Maasdorp, Proc. Roy. Soc. **A157**, 386 (1936); referred to here as KBM.
[2] Morse, Fisk and Schiff, Phys. Rev. **50**, 748 (1936).
[3] Breit and Feenberg, Phys. Rev. **50**, 850 (1936).
[4] Fisk, Schiff and Shockley, Phys. Rev. **50**, 1090 and 1191 (1936).
[5] Breit, Condon and Present, Phys. Rev. **50**, 825 (1936); Fisk and Morse, Phys. Rev. **51**, 54 (1937); Morse, Fisk and Schiff, Phys. Rev. **51**, 706 (1937).
[6] Feenberg and Share, Phys. Rev. **50**, 253 (1936).
[7] Amaldi and Fermi, Phys. Rev. **50**, 899 (1936).

problem that are important in predicting the angular distribution of the products. All equations refer to the coordinate system in which the center of mass is at rest (rest coordinate system).

Since the interactions are of the spin-dependent type, the initial state of reactions (A) and (B) describing two deuterons far apart must be an eigenstate of the operator $\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2$ in order to remove degeneracy. Here $\boldsymbol{\mu}_1=\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2$, $\boldsymbol{\mu}_2=\boldsymbol{\sigma}_3+\boldsymbol{\sigma}_4$, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin operators of the two particles of deuteron 1, and $\boldsymbol{\sigma}_3$ and $\boldsymbol{\sigma}_4$ are the spin operators of the two particles of deuteron 2; particles 1 and 3, and particles 2 and 4, are alike. Since the deuterons are normally in triplet states, the spin parts of the eigenfunctions of $\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2$ divide into singlet, triplet and quintet states which we denote by S_s, S_t and S_q, respectively. We shall mean by $(+--+)$ for example, that spin state in which the components of the spins of particles 1 and 4 have eigenvalues $+\frac{1}{2}$ along some arbitrary axis, and those of particles 2 and 3 have eigenvalues $-\frac{1}{2}$ along the same axis. Then

$$
\begin{aligned}
S_s&=(1/\sqrt{3})[(++--)-\tfrac{1}{2}(+-+-)\\
&\quad-\tfrac{1}{2}(+--+)-\tfrac{1}{2}(-++-)\\
&\quad-\tfrac{1}{2}(-+-+)+(--++)],\\
S_t^1&=\tfrac{1}{2}[(+++-)+(++-+)\\
&\quad-(+-++)-(-+++)],\\
S_t^2&=(1/\sqrt{2})[(++--)-(--++)],\\
S_t^3&=\tfrac{1}{2}[(+---)+(-+--)\\
&\quad-(--+-)-(---+)],\\
S_q^1&=(++++),\\
S_q^2&=\tfrac{1}{2}[(+++-)+(++-+)\\
&\quad+(+-++)+(-+++)],\\
S_q^3&=(1/\sqrt{6})[(++--)+(+-+-)\\
&\quad+(+--+)+(-++-)\\
&\quad+(-+-+)+(--++)],\\
S_q^4&=\tfrac{1}{2}[(--+-)+(---+)\\
&\quad+(+---)+(-+--)],\\
S_q^5&=(----).
\end{aligned}
\tag{3}
$$

The final configuration in which we are interested will consist of a three-body system and a single particle (number 4, say) moving away from each other. Making the reasonable assumption that the three-body system (He^3 or H^3) is formed in its normal ^2S state, we can conveniently divide the final spin states when the reaction products are far apart into singlets and triplets. Denoting these by R_s and R_t, we obtain

$$
\begin{aligned}
R_s&=\tfrac{1}{2}[(++--)-(-++-)\\
&\quad-(+--+)+(--++)],\\
R_t^1&=1/\sqrt{2}[(++-+)-(-+++)],\\
R_t^2&=\tfrac{1}{2}[(++--)-(-++-)\\
&\quad+(+--+)-(--++)],\\
R_t^3&=1/\sqrt{2}[(+---)-(--+-)].
\end{aligned}
\tag{4}
$$

The transition from initial to final state occurs under the influence of a perturbing energy whose most general form is

$$
\begin{aligned}
M=A_0+B_0\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+C_0\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3+D_0\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_4\\
+E_0\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3+F_0\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_4+G_0\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_4,
\end{aligned}
\tag{5}
$$

where $A_0, B_0, \cdots G_0$ may be permuting functions of the space coordinates of the various particles. It is readily seen by direct expansion that all matrix elements of the form RMS vanish except

$$
\begin{aligned}
R_sMS_s&=\sqrt{3}/2[A_0-\tfrac{3}{4}(C_0+F_0)\\
&\quad+\tfrac{1}{4}(B_0+G_0-D_0-E_0)],\\
R_tMS_t&=1/\sqrt{2}[A_0-\tfrac{3}{4}C_0\\
&\quad+\tfrac{1}{4}(B_0+F_0+G_0-D_0-E_0)],
\end{aligned}
\tag{6}
$$

where the R and S functions are any of those with the same subscript given in (3) and (4). Thus only singlet-singlet and triplet-triplet transitions can occur.

Now the Pauli principle requires that the wave function be antisymmetric in the interchange $1\rightarrow3$ and in the interchange $2\rightarrow4$. We therefore choose for our initial state wave function $\psi_i(1234)$, and our final state wave function $\psi_f(1234)$, forms that are symmetric in the interchange of the pair (12) with the pair (34). Then if we later make these antisymmetric in 2 and 4, say, we will have satisfied the Pauli principle. ψ_i always has the asymptotic form

$$
\psi_i(1234)\underset{\rho\to\infty}{\longrightarrow}\phi_0(r_{12})\phi_0(r_{34})\times\sum_l\chi_l(\rho)P_l(\cos\theta')S_l(1234). \tag{7}
$$

Here, $\boldsymbol{\rho}$ is the coordinate joining the centers of mass of the two deuterons, θ' is the angle between the initial momentum vector $\mathbf{k}_0$ and $\boldsymbol{\rho}$, and ϕ_0 is the wave function for the (symmetric) normal state of the deuteron. Likewise, ψ_f has the asymptotic forms

$$
\begin{aligned}
\psi_f(1234)&\underset{r_4\to\infty}{\longrightarrow}\psi_0(123)\sum_l F_l(r_4)P_l(\cos\theta_4)R_l(1234)\\
&\underset{r_2\to\infty}{\longrightarrow}\psi_0(134)\sum_l G_l(r_2)P_l(\cos\theta_2)R_l(1432).
\end{aligned}
\tag{8}
$$

Here, $\mathbf{r}_4$ is the coordinate joining the center of mass of (123) with 4, $\mathbf{r}_2$ the coordinate joining the center of mass of (143) with 2, θ_4 and θ_2 are the angles between $\mathbf{r}_4$ and $\mathbf{k}_0$ and between $\mathbf{r}_2$ and $\mathbf{k}_0$, respectively, and ψ_0 is the wave function for the (symmetric) normal state of the three-body system. In (7), the symmetry requirement [i.e., that ψ_i be symmetric in the interchange of the pair (12) with the pair (34)] and (3) show that $S_l = S_t$ for odd l, and that S_l is a linear combination of S_s and S_q for even l (except for $l=0$, when $S_0 = S_s$; this is because the Pauli principle excludes a spherically symmetric quintet state). Again in (8), the symmetry requirement and (4) show that $F_l(r) = -G_l(r)$ for all l. Since (7) and (8) are symmetric in the interchange of the pair (12) with the pair (34), the final wave function which satisfies the Pauli principle has the asymptotic form for large r_4

$$\psi_f(1234) - \psi_f(1432) \sim 2\psi_0(123) \sum_l F_l(r_4) P_l(\cos\theta_4) R_l(1234). \quad (9)$$

To determine the R_l's, we note that the reactions (A) and (B) are generally thought to be radiationless; any radiation that is given off must be so weak that it seems very unlikely that the emission of photons can play an important part in the transitions. Therefore we can assume that total angular momentum is conserved; since total spin is conserved, it follows that orbital angular momentum (l value) is also conserved. The allowed transitions are then ${}^1S \rightarrow {}^1S$, ${}^3P \rightarrow {}^3P$, ${}^1D \rightarrow {}^1D$, ${}^3F \rightarrow {}^3F$, etc. Thus in the final states given by (9), $R_l = R_s$ for even l, and $R_l = R_t$ for odd l.

In (9) we can always put

$$|F_l(r_4)| = f_l(k_0, k)/r_4, \quad (10)$$

giving for the differential cross section in the rest coordinate system

$$I_0(\theta) = (2k/k_0)[2\sum_l f_l(k_0, k) P_l(\cos\theta) R_l(1234)]^2,$$
$$k^2 = 3(W + W_1)/2, \quad k_0^2 = 2W = W_0, \quad (11)$$

where $\mathbf{k}$ is the final momentum vector, and θ is the angle between $\mathbf{k}$ and $\mathbf{k}_0$. W is the kinetic energy of the incident deuterons in the rest coordinate system, W_0 this kinetic energy in the laboratory coordinate system, and W_1 is the energy given up in the reaction; the factor 3/2 comes from a consideration of the reduced masses involved in the final state. The best values for W_1 appear to be[6] 6.26 for reaction (A), and 7.82 for reaction (B). The coefficient $(2k/k_0)$ in (11) is the ratio of the final neutron (or proton) velocity to the incident deuteron velocity. Because of the orthogonality of the R_l's for even l and the R_l's for odd l, (11) has the symmetry about 90° required by the general physical picture. If we restrict ourselves to incident deuteron energies small enough so that the contribution of the terms for $l \geqq 2$ can be neglected, (11) takes on the particularly simple form

$$I_0(\theta) = A(1 + B\cos^2\theta),$$
$$A \equiv (8k/k_0) f_0^2(k_0, k), \quad (12)$$
$$B \equiv f_1^2(k_0, k)/f_0^2(k_0, k).$$

The total cross section then becomes

$$\sigma = 4\pi A(1 + B/3). \quad (13)$$

There will be an extra factor of 3 appearing in B which is due to the relative statistical weights of the P and S parts of the initial wave.

Results

The accurate evaluation of the various quantities appearing in (12) is very difficult. A modification of the Born approximation method has been used here; the details of the calculations will be omitted. We shall consider first under what circumstances we should expect the results to be reliable. The method is that outlined by Mott and Massey,[8] using for the approximate initial wave function that appears inside the integral, variational deuteron functions[4] times the solution in the Coulomb field,[9] rather than the simple plane wave as in Born's method. This neglects in the zero-order solution both the effect of the nonelectric field and the polarization of the deuterons at short distances. There is no doubt that both of these effects will be important in a quantitative evaluation of A and B, except perhaps at very high energies. But unless both can be accounted for, and it is not clear how this

[8] Mott and Massey, *Theory of Atomic Collisions*, p. 110.
[9] Reference 8, pp. 33 and 39.

can be done simply, the additional effort required by using a better initial state function connecting the two deuterons, without considering polarization, hardly seems worthwhile. Moreover, it seems possible that at low energies, the Coulomb repulsions might tend to keep the deuterons far enough apart so that the other shorter range effects would not play an important part.

The experimental results of KBM[1] (see their Figs. 6 and 11, which refer to the rest coordinate system) appear to be of the form (12). This is consistent with the theory presented here, since an estimate of the $l\geqq 2$ terms indicates that they should not come in strongly before about 2 Mev, giving a considerable factor of safety above the maximum of 0.2 Mev used by these experimenters. However, a theoretical evaluation of the quantity B in (12) shows that it is much too small: 0.03 from the theory as compared with 0.5 required by the KBM experiments. One might say that this discrepancy was due to the inaccuracy of the calculations, were it not for the following considerations. When one assumes that the departure from spherical symmetry in the final distribution is due to the $l=1$ part of the initial wave, it seems reasonable that such asymmetry should depend strongly on incident deuteron energy, and should be small until the classical distance of closest approach $1/k_0$ for angular momentum $h/2\pi$ is of the order of magnitude of the dimensions of the deuterons. The experiments, on the other hand, show a very marked asymmetry as low as 0.1 Mev, where $1/k_0$ is about ten times the deuteron "radius," and show practically no change in shape of the distribution when the energy is doubled. There is some possibility that more accurate calculation of B and its dependence on initial deuteron energy will remove these difficulties. If not, we may have to look for some way in which an appreciable final state $l=1$ wave can arise from an initial state $l=0$ wave, apparently without radiation; this would be necessary to make the magnitude of the final P wave independent of incident particle energy. Such a transition appears to be inexplicable, using nuclear interactions such as are in vogue at the present time.

Further experimental data on angle distribution have been obtained very recently by Neuert.[10] His results are in agreement with those of KBM over the range covered by the latter (0° to 90°). However, when his data are transformed to the rest coordinate system, the resulting distribution is not symmetrical about 90° within the indicated experimental error. That such symmetry exist is required by *any* reasonable theory (since there should be no distinction between incident and target deuterons in the rest coordinate system); indeed, this symmetry was used by KBM as one criterion for the validity of their results.

There seems to be no experimental evidence on the variation of total cross section with energy; such evidence would require the use of a very thin unbacked solid target or of a gas target. However, several experimenters have measured the *yield* from "thick" targets. The yield Y in neutrons (or protons) per incident deuteron is given by

$$Y=\int_0^{x_0} n(x)\sigma[E(x)]dx. \tag{14}$$

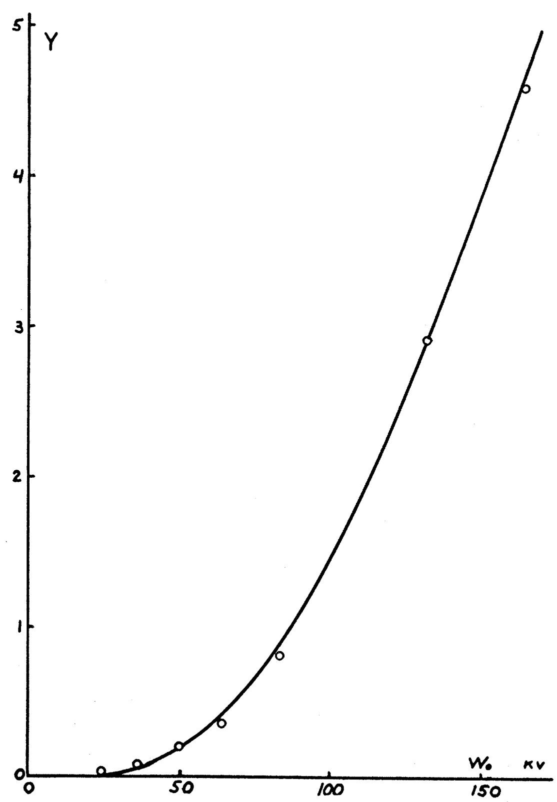

FIG. 1. Relative theoretical yield from infinitely thick target, with experimental points of reference 11. Low energy range, ordinate scale arbitrary.

[10] Neuert, Physik. Zeits. **38**, 122 (1937).

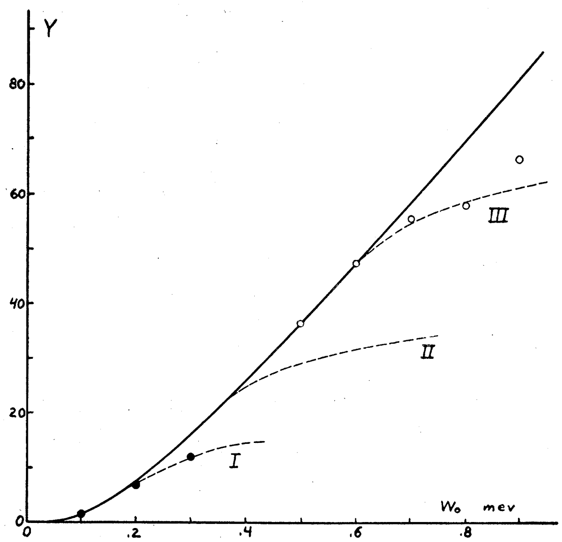

FIG. 2. Relative theoretical yield from infinitely thick target (solid line), with experimental points of references 12 and 13. Higher energy range, ordinate scale same as in Fig. 2. Dotted lines are for finite target thicknesses; I corresponds to thinnest target and III to thickest target. See reference 14.

Here, $n(x)$ is the number of deuterons per cc of target at the depth x below its surface, x_0 is the target thickness, and $E(x)$ is the energy of a deuteron of initial energy W_0 after it has penetrated a distance x into the target. (14) assumes quite reasonably that no deuterons are destroyed as the beam penetrates into the target (since deuterons, like protons and α-particles, have range properties), and that no emitted neutrons (or protons) are lost in coming out of the target. For definiteness in evaluating (14) we assume that $n(x)$ has the constant value n and that $E(x)=W_0-\beta x$. The linear relation between range and energy is a fairly good approximation for moderately small initial energies, by analogy with the range energy data for protons and α-particles. For a target thick enough to stop all the deuterons, (14) then becomes

$$Y=(n/\beta)\int_0^{W_0}\sigma(E)dE. \qquad (15)$$

n/β may be treated as an adjustable parameter, as far as relative yields are concerned. In Fig. 1, the computed curve of Y is plotted in arbitrary units, together with the low energy data of Oliphant, Harteck and Rutherford,[11] the points and the curve being fitted at 130 kv. The agreement is excellent. In Fig. 2 the data of Ladenburg and Roberts[12] and of Bonner and Brubaker,[13] for higher energies, are plotted to the same ordinate scale as Fig. 1, both being fitted to the curve at the lowest energy. When this is done, the theoretical yield at higher energies is too large in both cases. It seems plausible that the low energy deuterons (Fig. 1) are completely stopped in the thin layer of deuterium salt that constitutes the target, while the higher energy deuterons (Fig. 2) penetrate into the backing material where the only deuterium present is that driven in by the beam. The effect of this can be estimated by assigning a thickness x_0 to the deuterium salt layer, and performing the integration in (15) from $(W_0-\beta x_0)$ to W_0. The dotted curves in Fig. 2 correspond to $\beta x_0=0.15$ Mev for I, $\beta x_0=0.30$ Mev for II, and $\beta x_0=0.55$ Mev for III. The data of Ladenburg and Roberts fit well on I (thin target layer), while the data of Bonner and Brubaker fit well[14] on III (relatively thick target layer). In addition to the above, Burhop[15] has given data for extremely low energies (<20 kv) which would be unobservable on Fig. 1, while Döpel[16] has investigated the range of 5 to 150 kv. Over the range in which Döpel's results overlap those of Oliphant, Harteck and Rutherford, they are proportional to the latter, and a factor of about 10^{-3} smaller. For the lower energies, an analytic evaluation of (15) agrees well with both Burhop's and Döpel's results.

It must be emphasized that the excellent agreement between theoretical and experimental relative yields is due primarily to the effect of the Gamow factor, arising from the Coulomb repulsion between incident deuterons, and has very little to do with the details of the remainder of the calculation. Similar calculations of yield have been performed by Dolch.[17] His calculated *absolute* yields agree well with ours (see below), although his relative yields are in much poorer agreement with experiment than ours (compare

[11] Oliphant, Harteck and Rutherford, Proc. Roy. Soc. **A144**, 692 (1934).

[12] Ladenburg and Roberts, Phys. Rev. **50**, 1190 (1936).

[13] Bonner and Brubaker, Phys. Rev. **49**, 19 (1936).

[14] Fig. 1 of reference 13 is printed to such a small scale that the inaccuracy in measuring it is larger than the size of the open circles in our Fig. 2.

[15] Burhop, Proc. Camb. Phil. Soc. **32**, 643 (1936).

[16] Döpel, Ann. d. Physik **28**, 87 (1937).

[17] Dolch, Zeits. f. Physik **100**, 401 (1936).

his Fig. 3 and our Fig. 1), even when the errors in his paper are corrected.[18]

The last point of comparison between theory and experiment is in the matter of absolute yield.[19] Choosing reasonable values for n and β in Eq. (15), obtained by converting α-particle to deuteron ranges,[20] one obtains for the yield at 100 kv from a thick D_3PO_4 target about 3×10^{-4}. The absolute experimental yield from such a target varies with the experimenter over a range of about 10^{-6} to 10^{-9}; thus the theoretical yield is too large by a factor of about 10^3 to 10^5.

In conclusion it is well to repeat that that portion of the theory which concerns itself with the symmetry properties of the situation, leading to Eqs. (11) and (12), is quite rigorous, while the detailed evaluation of the quantities appearing therein is very crude, and serves as little more than an indication of the general nature of the results to be expected. Again, the excellent agreement of relative yields (see Figs. 1 and 2) is due primarily to the Gamow factor, and not to the rest of the theory. Experimental data on absolute cross section as a function of energy, and on angular distribution for higher energies, both obtained with gas targets (to eliminate the effect of penetration into the target), would be very useful at this time.

I wish to express my deep appreciation to Professor Philip M. Morse for his constant encouragement and for his help at many points of the theory. I also wish to thank Professor J. H. Van Vleck for criticizing the manuscript.

[18] Dolch, Zeits. f. Physik **104**, 473 (1937). It is difficult to understand why Dolch's results should deviate so much more from experiment than ours, when both calculations are based on the Gamow factor. In any case, the symmetry properties developed in the present paper, while important for angle distribution, have a small effect on the yield function.

[19] I am indebted to Professor Robley D. Evans for discussion of this point.

[20] Mano, J. de phys. et rad. **5**, 628 (1934).

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On the Nuclear Two-, Three- and Four-Body Problems*

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The simplest nuclear Hamiltonian satisfying all present requirements includes a Majorana-Heisenberg interaction $\{(1-g)P+gPQ\}V(r)$ between unlike particles and an attractive singlet interaction between like particles which is equal to that for unlike particles. The experimental mass defects of H^2 and H^3 together with the cross section σ for slow neutron-proton scattering will determine the range b and depth B of the triplet well and the proportion g of Heisenberg force (we use throughout the potential $Be^{-2r/b}$). An exact analytic expression relating σ, b, B and g is derived for this potential and g is found to be very insensitive to σ. An exact solution of H^2 gives the relation between B and b. The final relation which fixes the parameters is furnished by a Ritz-Hylleraas variational treatment of H^3 with the above Hamiltonian and the wave function:

$$\psi=2^{-\frac{1}{2}}\alpha_1(\alpha_2\beta_3-\alpha_3\beta_2)\phi_1+6^{-\frac{1}{2}}(\alpha_1(\alpha_2\beta_3+\alpha_3\beta_2)-2\beta_1\alpha_2\alpha_3)\phi_2$$

where ϕ_1 and ϕ_2 each represents an exponential times a power series in the interparticle distances of proper symmetry (ϕ_2 is brought in by the Heisenberg term; the Breit-Feenberg operator is used for the small triplet like-particle interaction). The convergence of energies obtained from successive improvements in ψ is rapid and the eigenvalue may be closely estimated. After a relativistic correction is made we obtain: $b=1.73\times 10^{-13}$ cm; $B=242$ mc^2 and $g=0.215$. The binding energy of He^3 is obtained by the same method and the H^3-He^3 difference is found to be 1.48 mc^2, agreeing well with experiment. The proton-proton scattering depth is checked to within 1 percent. When applied to He^4, our potential gives approximately 20 percent too much binding energy. Parallel calculations with the Gaussian and Morse curves lead to essentially the same result. No reasonable modification of the experimental data can explain more than a small fraction of the discrepancy.

* The contents of this paper form part of a thesis submitted by William Rarita to the Faculty of Pure Science at Columbia University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.