Only the odd powers of $t^{\frac{1}{2}}$ will have nonvanishing coefficients. The coefficient of $t^{\frac{1}{2}}$ itself is zero, and if we consider values of t for which $t^2 \ll 1$, then

$$\begin{aligned}\sqrt{2}ZI &\cong t^{\frac{3}{2}}\int_0^{2\pi}(\cos\varphi+\sin\varphi)d\varphi(\tfrac{5}{16}\cos^3\varphi-\tfrac{3}{16}\cos^2\varphi\sin\varphi-\tfrac{1}{16}\cos\varphi\sin^2\varphi-\tfrac{1}{16}\sin^3\varphi)\\ &=t^{\frac{3}{2}}\int_0^{2\pi}d\varphi(\tfrac{5}{16}\cos^4\varphi+\tfrac{1}{8}\cos^3\varphi\sin\varphi-\tfrac{1}{4}\cos^2\varphi\sin^2\varphi-\tfrac{1}{8}\cos\varphi\sin^3\varphi-\tfrac{1}{16}\sin^4\varphi)\\ &=\pi/8t^{\frac{3}{2}}.\end{aligned}$$

(b) As $t\rightarrow 1$, I becomes infinite, as may be seen from inspection. The infinity is due to the behavior of the integrand near $\varphi=0$, and it is therefore obvious that I becomes infinite in the same way as

$$J=\int_0^{2\pi}\cos\varphi d\varphi(1-t^{\frac{1}{2}}\cos\varphi)^{-\frac{1}{2}}=\text{const. }t^{\frac{1}{2}}F(\tfrac{3}{4},\tfrac{5}{4},2,t). \qquad \text{(See "G".)}$$

If $t\rightarrow 1$, J behaves as $-t^{\frac{1}{2}}\log(1-t)$, which is positive.

(c) Numerical integration yields positive values of I for values of t between 0 and 1. The following values of cI (c a constant) show the run of the function:

$$t=0.09,\quad cI=0.15;\quad t=0.25,\quad cI=0.80;\quad t=0.64,\quad cI=5.5;\quad \text{and}\quad t=0.95\quad cI=23.$$

It is therefore to be concluded that I is positive throughout the interval $0\leqq t\leqq 1$, and that the integral $\int_0^1 t^{\frac{1}{2}}dtI(t)$ cannot possibly vanish.

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On the Connection Formulas and the Solutions of the Wave Equation

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Part 1 gives a general discussion of asymptotic representations of the solutions of the one-dimensional wave equation. The forms ordinarily used in the so-called W. K. B. method are multiple valued and consequently necessitate a consideration of the Stokes' phenomenon, in any region about a turning point, i.e., a point in which the kinetic energy changes sign. Except under restrictive hypotheses they give no description of the solutions near the turning points. The author's method for representing the solutions of such differential equations by means of single valued functions is discussed, and the formulas applicable to the wave equation are given. These formulas are usable over the whole of an interval which includes a turning point. The Stokes' phenomenon is not involved. It need be considered only if expressions of the older type are desired, and then the connection formulas of the W. K. B. method are immediately evolved. An appropriate formal development of the solutions of the wave equation as power series in $\hbar$ is given.

Part 2 deals with the radial wave equation for motion in a central field of force. Both the attractive and repulsive Coulomb field are considered. It is shown that the application of the W. K. B. analysis to this equation as it has generally been made is uncritical and in error. The solution commonly identified thereby as the wave function is in fact not the wave function. The "failure" of the W. K. B. formulas, and the apparent necessity for modifying them by replacing the number $l(l+1)$ by $(l+\frac{1}{2})^2$, has been noted by many investigators. This is traced to the misapplication of the theory. When correctly applied the theory naturally yields the formulas which have been found to be called for on other grounds.

Finally the case is discussed in which a turning point lies too near the point $r=0$ for the W. K. B. method to be effectively applicable. It is shown how the solutions are describable in this case, the formulas given specializing, when the field is an attractive field and the energy is zero, to formulas which were given for that special case by Kramers.

Part 1. The One-Dimensional Wave Equation

The wave equation for a mechanical system of one degree of freedom is familiarly of the form

$$d^2u/dx^2+Q^2(x)u=0, \tag{1}$$

with

$$Q^2(x)=\frac{2m}{\hbar^2}\{E-V(x)\}.$$

m, E and $V(x)$ representing respectively the mass, the total energy and the potential energy, and $\hbar$ standing for $h/2\pi$, where h is Planck's constant. With the exception of a few special functions $V(x)$ the equation is not explicitly solvable. In general, therefore, its solutions must be studied through the medium of representative functions which in a specific sense approximate them. A well-known procedure for obtaining such representations[1] consists in effect of substituting into the differential equation a series in powers of $\hbar$. With proper adjustments the coefficients of such a series may be successively determined, at least theoretically, whenever x is restricted to an interval upon which the function $Q^2(x)$ is bounded from zero. The series so obtained, still with x restricted as noted, represent the solutions in the sense that they are approximations to them which take on an asymptotic character with respect to $\hbar$ when this symbol is regarded as an arbitrary small parameter. Retaining only the first terms of the series one finds in this way, if $Q^2(x)$ is positive the real functions

$$W_+(x;\gamma,\delta)=\delta Q^{-\frac{1}{2}}(x)\cos\left\{\int_{x_1}^{x} Qdx+\gamma\right\},\quad (2)$$

and if $Q^2(x)$ is negative the real functions

$$W_-(x;\alpha,\beta)=|Q(x)|^{-\frac{1}{2}}\left\{\alpha\exp\left[\int_{x_1}^{x}|Q|dx\right]+\beta\exp\left[-\int_{x_1}^{x}|Q|dx\right]\right\}.\quad (3)$$

The letters α, β, γ, δ stand for entirely arbitrary constants, while x_1 may, in each case, be any constant for which the integral concerned is convergent.

Upon any interval on which $Q^2(x)$ is positive, the form (2) with each choice of its constants represents some solution of the equation (1). If there is also an interval upon which $Q^2(x)$ is negative that same solution is represented thereon by a form (3), with specific constants. The property that they represent one and the same solution thus correlates each form (2) with a form (3) and *vice versa*, the correlation being fixed by the association of their respective constants. It is easily seen that to determine this association much more is necessary than, for instance, the mere substitution of $i|Q|$ in the place of Q and the transcription of (2) into the form (3). For, in the first place, each of the forms (2) or (3) becomes infinite at any point where $Q^2(x)$ is zero, and hence neither form can be retained during the transit from one to another interval between which $Q^2(x)$ changes sign. In the second place the differential equation of which the functions (2) and (3) are solutions is

$$d^2W/dx^2+\{Q^2(x)-\omega(x)\}W=0\quad (4)$$

with

$$\omega(x)=3[Q'(x)]^2/4Q^2(x)-Q''(x)/2Q(x).$$

For this equation any point where $Q^2(x)$ vanishes is obviously a singular point. The functions (2) and (3) are, therefore, multiple valued in the region about such a point (i.e., a so-called turning point) whereas the solutions of the equation (1) are single valued. Since the approximation of a single valued function by a multiple valued one can maintain only in a restricted region, it is clear, that the mere existence of a pair of relations

$$\begin{aligned}u(x)&\sim W_+(x;\gamma,\delta),\\ u(x)&\sim W_-(x;\alpha,\beta),\end{aligned}\quad (5)$$

valid, respectively, in intervals on opposite sides of a turning point, cannot be used as a basis for inferring that the right-hand members are one and the same solution of the equation (4) simply because the left-hand members are the same solution of the equation (1). The contrary is in fact the case; the so-called Stokes' phenomenon.

The possibility of inferring either of the relations (5) from the other depends, therefore, upon the so-called "connection formulas," which associated the values α, β with the γ, δ, and which are, therefore, the quantitative analysis of the Stokes' phenomenon. Such formulas have been deduced from various considerations. Thus they have been obtained for such cases as permit the assumption that in a suitable interval including the turning point the function $Q^2(x)$ is representable with a sufficient degree of accuracy by a linear function.[2] They have also been found for

[1] J. Horn, Math. Annalen **52**, 271 (1899); G. D. Birkhoff, Trans. Amer. Math. Soc. **9**, 219 (1908); O. Blumenthal, Archiv. d. Math. u. Physik **19**, 136 (1912); G. Wentzel, Zeits. f. Physik **38**, 518 (1926).

[2] H. A. Kramers, Zeits. f. Physik **39**, 829 (1926); also H. Jeffreys, Proc. London Math. Soc. (2) **23**, 428 (1923); also H. A. Kramers and G. P. Ittmann, Zeits. f. Physik **58**, 217 (1929); and for an extension of the method, S. Goldstein, Proc. London Math. Soc. (2) **28**, 81 (1929).

such cases as permit the assumption of a suitable mode of passage around the turning point through the complex plane.[3] This latter method unfortunately leads to no description of the solutions in the immediate neighborhood of the turning point. The problem can, however, be dealt with, and with greater effectiveness, by a method which eliminates completely the whole source of the difficulty,[4] namely the approximation of a single valued function by means of multiple valued ones.

If in an equation of the form (1) x_1 is a turning point at which the function $Q^2(x)$ has a zero, which may be of any order, say ν, the functions

$$U(x;\,\alpha,\,\beta)=S(x)\{\alpha\xi^{\mu}J_{-\mu}(\xi)+\beta\xi^{\mu}J_{\mu}(\xi)\} \quad (6)$$

are solutions of the differential equation

$$\frac{d^2U}{dx^2}+\{Q^2(x)-\theta(x)\}U=0 \quad (7)$$

with

$$\theta(x)=S''(x)/S(x). \quad (8)$$

The symbols J in (6) stand for the Bessel functions usually so denoted, while

$$\mu=\frac{1}{\nu+2}, \quad \xi=\int_{x_1}^{x}Qdx, \quad S(x)=Q^{-\frac{1}{2}}(x)\xi^{\frac{1}{2}-\mu}. \quad (9)$$

The differential equation (7) resembles the equation (1) on any range of values for which the function $\theta(x)$ is small relatively to $Q^2(x)$. It can be shown that this includes any range on which representation of the solutions of (1) by forms (2) or (3) is possible. Beyond that, however, the range need not exclude the turning point x_1, for in the neighborhood of this point, as may easily be verified, the function $S(x)$ is bounded from zero. Since the turning point is, therefore, an ordinary, nonsingular point for the equation (7), the functions (6) are single valued, and are thus evidently suited with a distinct advantage over such functions as (2) or (3), for instance, for the rôle of approximations to the single valued solutions of the equation (1). They are not restricted to yield representations of the solutions of (1) in intervals on the one or the other side of the turning point, but are adapted to do so in the whole interval including the turning point. The term "representation" is used to signify, as heretofore, the leading term of an expression which is asymptotic with respect to $\hbar$.

As is evident the representations (6) are explicitly relative to the point x_1. At any other turning point they are to be replaced by those relative to it, which are again obtainable from the formulas (6) with only the obvious modifications. It is a necessary restriction, then, that the interval may not extend across or even completely up to another turning point. Except for this it may extend even to infinity, provided in such case that the integral

$$\int|\theta(x)/Q(x)|dx \quad (10)$$

converges, when extended over the interval, with the permissible exception of a neighborhood of x_1.

Since the representation of a solution of Eq. (1) by a form (6), if it holds at all thus holds to the left, to the right and at the turning point, no question of connection formulas arises in association with it. The Stokes' phenomenon is therefore obviated, unless, to be sure, representations in terms of the more familiar functions (2) or (3) are desired. In that case it is merely necessary to replace the Bessel functions in (6) by their appropriate asymptotic expressions which are available from the literature of the Bessel functions. If that is done the connection formulas associating the forms (2) and (3) appear automatically and without resort to additional restrictions.

In the usual case the wave equation (1) involves a function $Q^2(x)$ which has a simple zero at x_1. With the positive direction of the x axis chosen so that $(x-x_1)$ has the same sign as $Q^2(x)$, the form (6) becomes more explicit, the general solution of the equation (1) being represented by the formula

$$u(x)\sim\left\{\frac{8\pi\xi}{3Q}\right\}^{\frac{1}{2}}\left\{\cos\left(\frac{\pi}{3}+\eta\right)J_{1/3}(\xi)+\cos\left(\frac{\pi}{3}-\eta\right)J_{-1/3}(\xi)\right\} \quad (11a)$$

with $\xi=\int_{x_1}^{x}Qdx$, and η an arbitrary constant on

[3] A. Zwaan, *Dissertation* (Utrecht, 1929), see also E. C. Kemble, Phys. Rev. **48**, 549 (1935).

[4] R. E. Langer, Trans. Am. Math. Soc. **33**, 29 (1931), also **34**, 447 (1932), also Bull. Am. Math. Soc. **37**, 397 (1935).

the range $-\pi/2<\eta\leqq\pi/2$. This form is real when $(x-x_1)$ is positive or zero. When $(x-x_1)$ is negative it may be rewritten so as again to avoid the use of imaginaries in the form

$$u(x)\sim\left\{\frac{8|\xi|}{\pi|Q|}\right\}^{\frac{1}{2}}\left\{\pi\sin\eta\cdot I_{1/3}(|\xi|)+2\cos\left(\frac{\pi}{3}-\eta\right)K_{1/3}(|\xi|)\right\} \quad (11b)$$

with $|\xi|=\int_x^{x_1}|Q|dx$. These formulas describe $u(x)$ over a whole interval including x_1. In subintervals lying completely on one or the other side of x_1 the Bessel functions admit of asymptotic representations, the substitution of which results in the formulas

$$\begin{aligned}&\text{(a)}\ \ u(x)\sim 2Q^{-\frac{1}{2}}(x)\cos\{\xi-(\pi/4)+\eta\},\\&\text{(b)}\ \ u(x)\sim|Q(x)|^{-\frac{1}{2}}\{2\sin\eta\cdot e^{|\xi|}+\cos\eta\cdot e^{-|\xi|}\},\end{aligned} \quad (12)$$

which are appropriate, respectively, when $(x-x_1)$ is positive or negative. It is clear that in (12b) the term in the negative exponential is negligible relatively to the other, except when η is numerically very small.

The formulas (12) clearly involve the connection formulas appropriate to (2) and (3), for if the phase of the solution $u(x)$ dictates for η in (12a) a value differing appreciably from zero, this value substituted in (12b) leads to the connection formula

$$|Q(x)|^{-\frac{1}{2}}\left\{2\sin\eta\cdot\exp\left[\int_x^{x_1}|Q|dx\right]+\cos\eta\cdot\exp\left[-\int_x^{x_1}|Q|dx\right]\right\}\leftarrow 2Q^{-\frac{1}{2}}(x)\cos\left\{\int_{x_1}^{x}Qdx-\frac{\pi}{4}+\eta\right\}, \quad (13)$$

$$\eta\neq 0.$$

The case in which there is no other turning point to the left of x_1 requires special mention. For in that case the wave function, i.e., the solution $u(x)$ which remains bounded for all x can be given by (12b) only with $\eta=0$. This value in (12a) evidently leads to the connection formula

$$|Q(x)|^{-\frac{1}{2}}\exp\left[-\int_x^{x_1}|Q|dx\right]\rightarrow 2Q^{-\frac{1}{2}}(x)\cos\left\{\int_{x_1}^{x}Qdx-\frac{\pi}{4}\right\}, \quad (14)$$

which is central in the so-called W.K.B. method of dealing with the equation (1). Near the turning point the descriptions of the solutions to which either (13) or (14) apply are obtainable from the formulas (11a) or (11b) with appropriate values of η.

It must be born in mind that the formulas here in question are all in a sense approximate, and not exact. On that account it is essential that inference of the one representation from the other in the relations (13) or (14) be made only in the directions indicated by the arrows. A comparison of formula (12a) with experimental or other data can suffice only to determine the value of η *approximately*. This fact is of dominating importance when the distinction between "η equals zero" or "η *nearly* equals zero" is in question, for in the former case the first term in (12b) is absent, while in the latter it is present and then necessarily of dominant importance when x is sufficiently remote from x_1. The arrow in (14), therefore, may not be reversed. On the other hand, since an arbitrary multiplicative constant factor may be inserted in or removed from $u(x)$, the value of η is identifiable from (12b) only when the second term is not entirely overshadowed by the first, namely when $\eta=0$. Hence one may not reverse the arrow in (13).[5]

The deduction of power series in $\hbar$ which are asymptotic to $u(x)$, and of which the forms (2) or (3) are the leading terms is familiar, as has already been mentioned. It is to be noted that such series with the forms (6) as leading terms may also be formally derived. Let the expression

$$U'(x)A(x,\hbar^2)+U(x)B(x,\hbar^2),$$

[5] The case in which two turning points lie too close together for application of the formulas (13) and (14) requires independent consideration. Cf. S. Goldstein, Proc. London Math. Soc. (2) **33**, 246 (1932); W. Voss, Zeits. f. Physik **83**, 581 (1933); R. E. Langer, Trans. Am. Math. Soc. **36**, 90 (1934), and Bull. Am. Math. Soc. **40**, 574 (1934).

in which

$$A(x, \hbar^2)=\sum_{j=0}^{\infty}\alpha_j(x)\hbar^{2j},$$

$$B(x, \hbar^2)=\sum_{j=0}^{\infty}\beta_j(x)\hbar^{2j},$$

be substituted for $u(x)$ in the equation (1). It is found then, if

$$Q^2(x)=\frac{1}{\hbar^2}q_1^2(x)+q_0(x),$$

that

$$U(x)\left\{\left(2\theta-\frac{2q_1^2}{\hbar^2}-2q_0\right)A' +\left(\theta-\frac{q_1^2}{\hbar^2}-q_0\right)'A+B''+\theta B\right\} +U'(x)\{2B'+A''+\theta A\}=0.$$

If the coefficient of each power of $\hbar^2$ which multiplies either $U(x)$ or $U'(x)$ is set equal to zero, the choice $\alpha_0=0$, $\beta_0=1$, is seen to be permissible, while the subsequent coefficients must satisfy the relations

$$2q_1^2\alpha_j'+2q_1q_1'\alpha_j=\beta_{j-1}''+\theta\beta_{j-1} +(2\theta-2q_0)\alpha_{j-1}'+(\theta'-q_0')\alpha_{j-1},$$

$$2\beta_j'=-\alpha_j''-\theta\alpha_j, \qquad j=1, 2, 3, \cdots.$$

Thus, one obtains formally the expansion

$$u(x)=U(x)+U'(x)\sum_{j=1}^{\infty}\alpha_j(x)\hbar^{2j} +U(x)\sum_{j=1}^{\infty}\beta_j(x)\hbar^{2j}, \tag{15}$$

in which the coefficients are given in succession by the formulas

$$\alpha_j(x)=\frac{1}{2q_1(x)}\int^x\frac{2(\theta-q_0)\alpha_{j-1}'+(\theta'-q_0')\alpha_{j-1}+\beta_{j-1}''+\theta\beta_{j-1}}{q_1}dx, \tag{16}$$

$$\beta_j(x)=-\frac{1}{2}\int^x(\alpha_j''+\theta\alpha_j)dx,$$

with any chosen constants as the lower limits of integration. It is to be noted that the formulas (16) are convergent even when x is the turning point.

Part 2. The Radial Wave Equation

In the case of motion in a central field of force the general wave equation in spherical polar coordinates permits a separation of the variables.[6] If the radial component of the motion is designated by $\psi(r)/r$, the wave equation for $\psi(r)$ is thus found to be

$$d^2\psi/dr^2+Q_0^2(r)\psi=0 \tag{17}$$

with

$$Q_0^2(r)=\frac{2m}{\hbar^2}\left\{E+\frac{Ze^2}{r}\right\}-\frac{l(l+1)}{r^2}. \tag{18}$$

With the customary interpretation of the symbols the formula (18) implies the field to be an attractive field, since the potential energy is given by the term $-Ze^2/r$. The case of a repulsive field may be included in the discussion by the simple expedient of admitting for the charge Z negative values as well as positive ones. The variable r representing radial distance is, of course, restricted to be positive.

Structurally, the equation (17) bears evident features of resemblance to the equation (1). Insofar as the writer is aware, the degree of this resemblance has hitherto been regarded without exception as sufficient for assuming the out-and-out applicability to the equation (17) of the formulas deduced in the foregoing discussion, especially of the formula (14). The first point at which $Q_0^2(r)$ is zero is found directly to be

$$\text{(a)}\quad \rho_1=\frac{l(l+1)\hbar^2}{mZe^2\left\{1+\left(1+\dfrac{2El(l+1)\hbar^2}{mZ^2e^4}\right)^{\frac{1}{2}}\right\}}, \qquad \text{if } Z>0,\ l\neq0,\ E\gtreqless0,$$

$$\text{(b)}\quad \rho_1=\frac{Ze^2}{-2E}\left\{1+\left(1+\frac{2El(l+1)\hbar^2}{mZ^2e^4}\right)^{\frac{1}{2}}\right\}, \qquad \text{if } Z<0,\ E>0. \tag{19}$$

[6] Cf., for instance, T. Sexl, Zeits. f. Physik 99, 751 (1936).

When $Z<0$, $E<0$, $Q_0^2(r)$ does not vanish at all, and when $Z>0$, $E<0$, it vanishes a second time, namely at the point which is given precisely by (19b). The case $l=0$ for an attractive field stands out as exceptional, for the pole of $Q_0^2(r)$ at $r=0$ is then of a different sort than otherwise, while the zero otherwise given by (19a) is then entirely absent. With the point ρ_1 used in the place of x_1, and with r and $Q_0(r)$ replacing x and $Q(x)$ the relation (14) has been commonly regarded as applicable to the equation (17), and to describe the wave function for that case.[7] This procedure, however, soon leads to a difficulty. The resulting formulas were found by Kramers to give an incorrect phase for the solution unless the number $l(l+1)$ is replaced by $(l+\frac{1}{2})^2$, a modification which is tantamount to raising the potential barrier. Young and Uhlenbeck found the same modification requisite, both if the Balmer formula is to be obtained, and if the wave function is to vanish to the proper degree at $r=0$. This "failure" of the W.K.B. method, i.e., to the extent that the change in question is requisite, has been generally verified both in studies of attractive and repulsive fields. No explanation of it seems to have been given, though that can be done very simply to the following effect. The fault lies not in the method but in the application of it. The commonly accepted assumption that the equation (17) is of the form (1) as it stands is, namely, incorrect. The formula (14) is at best only restrictedly applicable to the equation (17), and when it is applicable it describes a solution which is not bounded, i.e., which is not the wave function.

The functions which constitute the several members of the relations (14) and (13) are solutions of the differential equation

$$d^2\Psi/dr^2+\{Q_0^2(r)-\omega_0(r)\}\Psi=0 \qquad (20)$$

with

$$\omega_0(r)=\frac{3[Q_0'(r)]^2}{4Q_0^2(r)}-\frac{Q_0''(r)}{2Q_0(r)}. \qquad (21)$$

The resemblance between this equation and the equation (17), and hence the possibility of representing the solutions of the latter by means of those of the former, fails, of course, near the points where $Q_0^2(r)$ becomes zero. It fails also, though this seems to have been overlooked, near $r=0$, for at that point the function $\omega_0(r)$ becomes infinite similarly to $Q_0^2(r)$. The relation (14), therefore, though it can represent a solution of the equation (17) in suitable intervals, cannot do so in an interval which actually reaches up to the point $r=0$. It follows at once from this that the vanishing of the left-hand member of (14) at $r=0$, gives no ground for concluding that the solution of (17) which is elsewhere represented by (14) also vanishes at $r=0$. That is in fact not so. The solution elsewhere represented by (14) becomes infinite at $r=0$, and so is not the wave function.

Let the variables in the equation (17) be changed by the substitution

$$r=e^x, \quad \psi=e^{x/2}u. \qquad (22)$$

The domain of the variable becomes then $-\infty<x<\infty$, and the resulting equation is found to be

$$d^2u/dx^2+Q^2(x)u=0, \qquad (23)$$

with

$$Q^2(x)=\frac{2m}{\hbar^2}\{E\cdot e^{2x}+Ze^2\cdot e^x\}-(l+\tfrac{1}{2})^2. \qquad (24)$$

Now this equation is obviously of the form (1). Its turning points computed from (24) but expressed in terms of r, are found to be

$$\text{(a)}\quad r_1=\frac{(l+\frac{1}{2})^2\hbar^2}{mZe^2\left\{1+\left(1+\dfrac{2E(l+\frac{1}{2})^2\hbar^2}{mZ^2e^4}\right)^{\frac{1}{2}}\right\}},\quad \text{if } Z>0,\ E\gtreqless 0, \qquad (25)$$

$$\text{(b)}\quad r_1=\frac{Ze^2}{-2E}\left\{1+\left(1+\frac{2E(l+\frac{1}{2})^2\hbar^2}{mZ^2e^4}\right)^{\frac{1}{2}}\right\},\quad \text{if } Z<0,\ E>0,$$

there being no turning point if $Z<0$, $E<0$, and a second one given by (25b) if $Z>0$, $E<0$. The formulas (22) and (24) are found, moreover, to

[7] H. A. Kramers, Zeits. f. Physik **39**, 836 (1926); L. A. Young and G. E. Uhlenbeck, Phys. Rev. **36**, 1158 (1930); E. C. Kemble, Phys. Rev. **48**, 560 (1935); F. L. Yost, J. A. Wheeler and G. Breit, Phys. Rev. **49**, 180 (1935); T. Sexl, Zeits. f. Physik **99**, 771 (1936).

give the identities

$$e^{x/2}Q^{-\frac{1}{2}}(x)\equiv Q_1^{-\frac{1}{2}}(r),$$
$$\int_{x_1}^{x}Q(x)dx\equiv\int_{r_1}^{r}Q_1(r)dr, \tag{26}$$

in which

$$Q_1^2(r)=\frac{2m}{\hbar^2}\left\{E+\frac{Ze^2}{r}\right\}-\frac{(l+\frac{1}{2})^2}{r^2}. \tag{27}$$

The formulas (11), (13) and (14) are directly applicable to the equation (23). If they are so applied, and the original variables are reintroduced, the results are found to be those which are formally obtainable by the direct substitution of r, ψ for u and x, and of the r_1 and $Q_1^2(r)$ of (25) and (27), for x_1 and $Q^2(x)$. The difference between this procedure and the incorrect one of substituting the ρ_1 and $Q_0^2(r)$ of (19) and (18) for x_1 and $Q^2(x)$, is seen to amount formally to precisely the replacement of $l(l+1)$ by $(l+\frac{1}{2})^2$. It will be noted, moreover, that for the formula (25a) the case $l=0$ is not exceptional as it was for (19a).

Since the applicability of the formulas as described right up to the point $r=0$ depends upon their applicability to the equation (23) over the infinite interval to $x=-\infty$, the validity of the procedure is still questionable. It is, however, easily established, for from the formulas (24), (8) and (9) it is seen that as $x\rightarrow-\infty$

$$Q(x)=O(1),\quad \theta(x)=O(x^{-2}).$$

Thus the integrand in (10) is of the order of x^{-2}, and for such the integral to $x=-\infty$ converges. The formulas relative to the right-hand turning point (which may, of course, be the only turning point) are similarly seen to be applicable even to $x=+\infty$, for from the formulas (24), (8) and (9) it is seen that as $x\rightarrow+\infty$

$$Q(x)=O(e^{-x}),\quad \theta(x)=O(1),$$

whence the integrand in (10) is of the order of e^{-x}, and the integral to $x=+\infty$ is convergent.

Whenever the number l is small the turning point (25a) lies very near to the point $r=0$, and this is true also in the case of the turning point (25b) whenever the energy is very large. The use of the formulas (13) or (14) under such circumstances is questionable, and is generally less satisfactory than otherwise. The formulas are of course entirely inapplicable when there is no turning point at all, namely when $E<0$ with a repulsive field. In such cases the following analysis of the wave equation may be resorted to.

The change of variables

$$r=s^2,\quad \psi=s^{\frac{1}{2}}v, \tag{28}$$

gives to the equation (17) the form

$$(d^2v/ds^2)+\{\lambda^2\varphi^2(s)+(\tfrac{1}{4}-\beta^2)/s^2\}v=0 \tag{29}$$

with

$$\lambda^2=8mZe^2/\hbar^2,\quad \beta=2l+1,\quad \varphi^2(s)=1+Es^2/Ze^2 \tag{30}$$

The solutions of an equation of this type are known,[8] however, to be represented by the functions

$$\left\{\frac{\int_0^s\varphi ds}{\varphi(s)}\right\}^{\frac{1}{2}}C_\beta\left(\lambda\int_0^s\varphi ds\right), \tag{31}$$

the symbol C_β signifying any Bessel function of the order $\pm\beta$. The representations are valid from $s=0$ over an interval which does not reach up to a zero of the function $\varphi^2(s)$.

In the case of an attractive field ($Z>0$) the value λ^2 is positive. The choice of C_β successively as J_{2l+1} and Y_{2l+1} yields respectively the representations, which in terms of the original variables are

$$\psi_1(r)\sim\left\{\frac{\int_0^r Q_2dr}{Q_2(r)}\right\}^{\frac{1}{2}}J_{2l+1}\left(\int_0^r Q_2dr\right),$$
$$\psi_2(r)\sim\left\{\frac{\int_0^r Q_2dr}{Q_2(r)}\right\}^{\frac{1}{2}}Y_{2l+1}\left(\int_0^r Q_2dr\right) \tag{32}$$

with

$$Q_2^2(r)=\frac{2m}{\hbar^2}\left\{E+\frac{Ze^2}{r}\right\}. \tag{33}$$

Clearly $\psi_1(r)$ is the solution which vanishes at $r=0$, while $\psi_2(r)$ is a specific one which is unbounded there. Upon substituting for the

[8] R. E. Langer, Trans. Amer. Math. Soc. **37**, 397 (1935).

Bessel functions their familiar asymptotic representations, the formulas become alternatively

$$\psi_1(r)\sim\left(\frac{2}{\pi}\right)^{\frac{1}{2}}Q_2^{-\frac{1}{2}}(r)\cos\left\{\int_0^r Q_2dr-(l+\tfrac{3}{4})\pi\right\},$$
$$\psi_2(r)\sim(2\pi)^{\frac{1}{2}}Q_2^{-\frac{1}{2}}(r)\sin\left\{\int_0^r Q_2dr-(l+\tfrac{3}{4})\pi\right\}. \quad (34)$$

These formulas are usable in intervals which extend across and include the first turning point (25a). When $E<0$ there is a second turning point which limits the intervals of the present formulas, and relative to which the analysis given previously may be applied. In the special case in which $E=0$ the formulas (32) and (34) for the solution $\psi_1(r)$ reduce to formulas which were obtained for that case by Kramers.[9]

In the case of a repulsive field ($Z<0$) the value λ^2 is negative and hence the choices

$$I_{2l+1}\left(|\lambda|\int_0^s\varphi ds\right),\quad\text{and}\quad K_{2l+1}\left(|\lambda|\int_0^s\varphi ds\right)$$

for the Bessel functions in (31) will avoid the introduction of imaginaries. The resulting formulas, in terms of the original variables, are then found to be

$$\psi_1(r)\sim\left\{\frac{\int_0^r|Q_2|dr}{|Q_2(r)|}\right\}^{\frac{1}{2}}I_{2l+1}\left(\int_0^r|Q_2|dr\right),$$
$$\psi_2(r)\sim\left\{\frac{\int_0^r|Q_2|dr}{|Q_2(r)|}\right\}^{\frac{1}{2}}K_{2l+1}\left(\int_0^r|Q_2|dr\right). \quad (35)$$

The interval upon which they are usable extends from $r=0$ outward, but when $E<0$ it must not include or extend up to the turning point which then exists. The substitution of asymptotic forms gives the alternative representations

$$\psi_1(r)\sim\frac{1}{(2\pi)^{\frac{1}{2}}}|Q_2(r)|^{-\frac{1}{2}}\exp\left[\int_0^r|Q_2|dr\right],$$
$$\psi_2(r)\sim-\left(\frac{\pi}{2}\right)^{\frac{1}{2}}|Q_2(r)|^{-\frac{1}{2}}\exp\left[-\int_0^r|Q_2|dr\right]. \quad (36)$$

Finally, the substitutions

$$\rho=\frac{(2mE)^{\frac{1}{2}}}{\hbar}r,\quad k=\frac{-Z}{\hbar}\left(\frac{2me^2}{E}\right)^{\frac{1}{2}},$$

reduce the equation (17) to

$$\frac{d^2\psi}{d\rho^2}+\left\{1-\frac{k}{\rho}-\frac{l(l+1)}{\rho^2}\right\}\psi=0.$$

Analyses of the equation in this form, both with $l=0$ and $l\neq 0$, have been given.[10] By comparison with them the present analysis and its consequent formulas may be found to have at least the advantages of simplicity and of generality of method.

[9] H. A. Kramers, Zeits. f. Physik **39**, 837 (1926).

[10] T. Sexl, Zeits. f. Physik **56**, 72 (1929); F. L. Yost, J. A. Wheeler and G. Breit, Phys. Rev. **49**, 174 (1935).