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On the Structure of Light Nuclei

E. FEENBERG AND M. PHILLIPS*
Institute for Advanced Study, Princeton, New Jersey

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Extensive calculations based on the approximation of single particle wave functions (the Hartree method) have been made for the nuclei between He^6 and O^{16} using the general symmetrical interaction operator given by Eq. (1). The Coulomb interaction is treated as a small perturbation. Secular equations are avoided by the construction of space wave functions in the normal state configuration belonging to irreducible representations of the symmetric group. These functions yield an energy matrix which is diagonal in the ordinary and Majorana interaction energies. The contributions of the spin exchange and Coulomb operators to the energy terms are found by a first-order perturbation calculation. Although the general symmetrical operator contains several parameters as yet undetermined, only those parameters which have been fixed by consideration of the two, three and four particle problems are involved in the energy differences within the group of low lying terms belonging to the normal state configuration. These term differences are identical with those recently computed for unsymmetrical interaction operators of the saturation type. New results for mass defects, excitation energies and energy relations between isobars are compared with experimental values.

I. INTRODUCTION

A THEORETICAL description of many nuclear properties is made possible by recent progress in the knowledge of nuclear forces. Extensive calculations based on the approximation of single particle wave functions (the Hartree method) have already been made[1] for the nuclei between He^6 and O^{16}. This method yields results for mass defects, excitation energies, energy relations between isobars, spins and magnetic moments which are on the whole very encouraging. In the present paper we present additional calculations based on the Hartree approximation for the same series of nuclei. Our procedure differs from that of reference 1 in two respects:

1. We employ throughout the general symmetrical interaction operator[2]

$$V=\sum_{i>j} V_{ij}=\sum_{i>j}\{(1-g-g_1-g_2)P_{ij}+gP_{ij}Q_{ij}+g_1 1+g_2Q_{ij}\}J(r_{ij}), \quad (1)$$

which does not distinguish between like and unlike particle interactions. The Coulomb repulsion between protons is treated separately as a small perturbation. Although the general symmetrical operator contains several parameters as yet undetermined, only those parameters which have been fixed by consideration of the two, three and four particle problems are involved in the energy differences within the group of low lying levels. For the low terms, in fact, the separations are identical with those given by

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* Margaret E. Maltby Fellow of the A. A. U. W.

[1] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).

[2] G. Breit and E. Feenberg, Phys. Rev. **50**, 850 (1936). The operators in Eq. (1) are defined by the equations

$$P_{ij}\psi(\cdots x_is_i\cdots x_js_j\cdots)=\psi(\cdots x_js_i\cdots x_is_j\cdots),$$
$$Q_{ij}\psi(\cdots x_is_i\cdots x_js_j\cdots)=\psi(\cdots x_is_j\cdots x_js_i\cdots).$$

the simple symmetrical Hamiltonian containing only Majorana and Heisenberg forces. From this circumstance it is apparent that the experimental excitation energies cannot be used to reduce the ambiguity which remains in the interaction operator after the restrictions imposed by the two, three and four particle problems have been taken into account.

2. We avoid secular equations by making use of the "correct" space wave functions belonging to the normal state configuration to construct an energy matrix which is diagonal in the ordinary and Majorana interaction energies. Because there is no spin-orbit interaction in Eq. (1) the total spin and the total orbital angular momentum are separately constants of the motion. If the Coulomb and spin exchange forces are neglected (Wigner's "first approximation")[3] the Hamiltonian commutes also with the permutation operators which interchange the space coordinates of two or more particles. Consequently, in this approximation, the space part of an eigenfunction belongs to an irreducible representation of the symmetric group and is characterized by a "partition" quantum number in addition to the orbital angular momentum quantum numbers.

The presence of Coulomb and spin exchange forces spoils the "partition" quantum number by mixing together functions belonging to different partitions. The contributions of the Coulomb and spin exchange energy operators to the energy matrix are small, however, and are given without sensible error by a first-order perturbation calculation in which the "partition" functions are used as zero-order wave functions. Ordinarily there is only one wave function belonging to the normal state configuration with a particular set of partition and angular momentum quantum numbers. Each set of quantum numbers is associated in this approximation with an energy term whose value is the expectation value of the Hamiltonian operator corresponding to the appropriate zero-order wave function. The resulting expressions for the term values are extremely simple. All the excitation energies computed from Eq. (1) can be expressed in terms of two integrals with simple rational (in most cases integral) coefficients.

In reference 1 the sets of functions with definite pairs of spin and orbital angular momentum quantum numbers were used to construct secular equations, often of the second and third order, for the energy terms. The linear combinations of these functions which diagonalize the energy matrix are identical with the partition wave functions so long as the Coulomb and spin exchange forces are neglected. If the contributions from these forces are included in the secular equations the correct linear combinations differ only slightly from the partition wave functions, and the energy values obtained from the two methods of calculation coincide except for negligible corrections.

II. Matrix Elements of the Interaction Energy Within the P Shell

That part of the energy arising from interactions between particles in the p shell can be expressed as a linear combination

$$E_p = \omega L + \gamma K \tag{2}$$

of the two integrals

$$\begin{aligned} L &= \int \cdots \int x_1^2 x_2^2 R_p(r_1)^2 R_p(r_2)^2 J(r_{12}) d\tau_1 d\tau_2, \\ K &= \int \cdots \int x_1 x_2 y_1 y_2 R_p(r_1)^2 R_p(r_2)^2 J(r_{12}) d\tau_1 d\tau_2, \end{aligned} \tag{3}$$

in which $xR_p(r)$, $yR_p(r)$, $zR_p(r)$ are the normalized single particle wave functions. There exists a useful approximation in which the range of the forces is taken to be large in comparison with the size of the nucleus. In this approximation

$$L = J(0), \quad K = 0. \tag{4}$$

Since for a given number of particles in the p shell the coefficient of L in the expression for a term value depends only on the quantum numbers, it is identical with the corresponding coefficient of $J(0)$ in the approximation of long range forces. This latter quantity can be determined directly from the number of particles and the quantum numbers without explicit use of the wave functions. For a nucleus with n particles in the p shell:

$$\begin{aligned} \omega(1) &= \tfrac{1}{2}n(n-1), \\ \omega(P) &= \tfrac{1}{2}n(n-1)\chi(P)/\chi(\mathcal{E}), \\ \omega(Q) &= \tfrac{1}{2}n(n-1)\chi'(Q)/\chi'(\mathcal{E}), \\ \omega(PQ) &= -\tfrac{1}{2}n(n-1)\chi''(R)/\chi''(\mathcal{E}). \end{aligned} \tag{5}$$

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[3] E. Wigner, Phys. Rev. 51, 106 (1937).

Here $\chi(P)$ and $\chi(\mathcal{E})$ are the characters corresponding to a transposition and the identity, respectively, in the irreducible representation of the symmetric group to which the space part of the wave function belongs. This also serves as a definition of the primed symbols if *spin* is substituted for *space;* for the double primed symbols substitute *isotopic spin* for *space.* The antisymmetry of the wave functions in the complete set of space, spin and isotopic spin coordinates[3–5] implies the relation $PQR=-1$ or $PQ=-R$. This relation has been used in writing the last line of Eq. (5).

The derivation of Eq. (5) for ordinary forces is trivial since there are $n(n-1)/2$ interaction terms and each contributes $J(0)$ to the total. For the exchange forces the derivation given in reference 1 (Eqs. (4)–(8)) is applicable to all three types of exchange if $J(r_{12})$ is replaced by $J(0)$ in Eqs. (4) and (5) and the coordinates $x_1, \cdots, x_n$ suitably interpreted as space, spin or isotopic spin coordinates as the case requires.

The values of $\chi(P)/\chi(\mathcal{E})$ for $2\leqq n\leqq 6$ are to be found in Table II, reference 1. To compute $\omega(P)$ for $6<n\leqq 12$ one may use the relation connecting the energy values corresponding to n particles in the p shell with those corresponding to $12-n$ particles. This relation is given by Eq. (9) of the present paper. The coefficients $\omega(Q)$ may be found directly by a straightforward computation:[6]

$$\begin{aligned}\omega(Q)&=\sum_{i>j} Q_{ij}=\tfrac{1}{2}\sum_{i>j}(1+\sigma_i\cdot\sigma_j)\\&=\tfrac{1}{4}n(n-1)-\tfrac{1}{4}\{\textstyle\sum\sigma_i^2-(\sum\sigma_i)^2\}\\&=S(S+1)+\tfrac{1}{4}n(n-4).\end{aligned}\tag{6}$$

Similarly

$$\omega(PQ)=\sum_{i>j} P_{ij}Q_{ij}=-T(T+1)-\tfrac{1}{4}n(n-4).\tag{7}$$

Here S is the total spin and T the total isotopic spin. The possible spin and isotopic spin values which can be associated with a given partition in the space coordinates are limited by the exclusion principle. The allowed quantum numbers can be found by a systematic procedure[3] which is merely an extension of the usual one for atomic spectra. This procedure is described in Appendix 2.

The coefficients of K vary from term to term within a partition. There exist, however, two relations connecting the coefficients of L and K which are independent of the quantum numbers:

$$\begin{aligned}\gamma(1)-\gamma(P)&=3\{\omega(1)-\omega(P)\},\\\gamma(Q)-\gamma(PQ)&=3\{\omega(Q)-\omega(PQ)\}.\end{aligned}\tag{8}$$

To prove Eq. (8) we examine the integral

$$\sum_1^3 a_{ij}b_{mn}\int\cdots\int x_{1i}x_{2j}R_p(r_1)R_p(r_2)$$
$$\times J(r_{12})(1-P_{12})x_{1m}x_{2n}R_p(r_1)R_p(r_2)d\tau_1 d\tau_2,$$

in which $x_{\alpha 1}=x_\alpha$, $x_{\alpha 2}=y_\alpha$, $x_{\alpha 3}=z_\alpha$, α takes the values 1 and 2 and the summation extends over i, j, m, n. The general quadratic tensor ($\sum_{ij} a_{ij}x_{1i}x_{2j}$ for example) may be written as the sum of two terms, one symmetric and the other antisymmetric in the subscripts 1 and 2. It is clear that the integral vanishes if either tensor is symmetric, so that the two tensors may be replaced by their antisymmetrical parts and the operator $(1-P_{12})$ by the factor 2. Thus the integral takes the form

$$\text{Constant}\times\int\cdots\int(x_1y_2-x_2y_1)x_1y_2R_p(r_1)^2$$
$$\times R_p(r_2)^2J(r_{12})d\tau_1 d\tau_2=\text{Constant}\times(L-3K).$$

This suffices to prove both parts of Eq. (8) since the presence of a spin exchange operator does not affect the argument.

The energy matrix for a configuration consisting of a closed p shell minus n particles is related to the corresponding matrix for the configuration p^n by the equations

$$\begin{aligned}(i|1|j)_{12-n}&=(j|1|i)_n+(6-n)(11L-18K)\delta_{ij},\\(i|P|j)_{12-n}&=(j|P|i)_n+(6-n)(L+12K)\delta_{ij},\\(i|Q|j)_{12-n}&=(j|Q|i)_n+(6-n)(4L-12K)\delta_{ij},\\(i|PQ|j)_{12-n}&=(j|PQ|i)_n\\&\quad+(6-n)(-4L+12K)\delta_{ij}.\end{aligned}\tag{9}$$

The states designated by the quantum numbers i and j may be any states of a p shell configuration subject to the conditions that their wave

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[4] J. H. Bartlett, Phys. Rev. **49**, 102 (1936).

[5] B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936).

[6] P. A. M. Dirac, *Quantum Mechanics* (Oxford University Press, 1930) p. 215.

functions are orthogonal for $i \neq j$ and that the exclusion principle is satisfied.

The complete set of matrix elements of the interaction within the p shell for the "low" partitions together with the spin and isotopic spin quantum numbers is contained in Table I. Table II gives the same quantities for the partitions consisting of fours and two additional addends. The "partition" wave functions used in computing the energy matrix are discussed in Appendix 1.

Very often the matrix elements of Q and PQ can be found without explicit use of the spin functions. The interchange of ordinary and isotopic spin variables in the wave functions simply interchanges the spin and isotopic spin quantum numbers. Therefore the matrix elements of $PQ = -R$ are just the negatives of the corresponding matrix elements of Q with spin and isotopic spin interchanged. The low terms of the configuration p^5 will serve as an example. Since both quantum numbers have the value 1/2

TABLE I. *Matrix elements and quantum numbers within the group of low terms.*

						$\omega L+\gamma K$			
n	PARTITION	T	NUCLEUS	S	STATE	P	1	PQ	Q
2	2	0	Li^6	1	^3S	$L+2K$	$L+2K$	$L+2K$	$L+2K$
					^3D	$L-K$	$L-K$	$L-K$	$L-K$
2	2	1	He^6, Li^6, Be^6	0	^1S	$L+2K$	$L+2K$	$-L-2K$	$-L-2K$
					^1D	$L-K$	$L-K$	$-L+K$	$-L+K$
3	3	1/2	Li^7, Be^7	1/2	^2P	$3L+2K$	$3L+2K$	0	0
					^2F	$3L-3K$	$3L-3K$	0	0
4	4	0	Be^8	0	^1S	$6L+4K$	$6L+4K$	0	0
					^1D	$6L+K$	$6L+K$	0	0
					^1G	$6L-6K$	$6L-6K$	0	0
5	4+1	1/2	Be^9, B^9	1/2	^2P	$5L+9K$	$10L-6K$	$-2L+6K$	$2L-6K$
					^2D	$5L+7K$	$10L-8K$	$-2L+6K$	$2L-6K$
					^2F	$5L+4K$	$10L-11K$	$-2L+6K$	$2L-6K$
					^2G	$5L$	$10L-15K$	$-2L+6K$	$2L-6K$
6	4+2	0	B^{10}	1	^3S	$5L+16K$	$15L-14K$	$-3L+29K/2$	$5L-19K/2$
					$^3D_{I}$	$5L+13K$	$15L-17K$	$-3L+13K$	$5L-11K$
					$^3D_{II}$	$5L+13K$	$15L-17K$	$-3L+53K/5$	$5L-67K/5$
					^3F	$5L+10K$	$15L-20K$	$-3L+10K$	$5L-14K$
					^3G	$5L+6K$	$15L-24K$	$-3L+12K$	$5L-12K$
6	4+2	1	Be^{10}, B^{10}, C^{10}	0	^1S	$5L+16K$	$15L-14K$	$-5L+19K/2$	$3L-29K/2$
					$^1D_{I}$	$5L+13K$	$15L-17K$	$-5L+11K$	$3L-13K$
					$^1D_{II}$	$5L+13K$	$15L-17K$	$-5L+67K/5$	$3L-53K/5$
					^1F	$5L+10K$	$15L-20K$	$-5L+14K$	$3L-10K$
					^1G	$5L+6K$	$15L-24K$	$-5L+12K$	$3L-12K$
7	4+3	1/2	B^{11}, C^{11}	1/2	^2P	$6L+21K$	$21L-24K$	$-6L+18K$	$6L-18K$
					^2D	$6L+19K$	$21L-26K$	$-6L+18K$	$6L-18K$
					^2F	$6L+16K$	$21L-29K$	$-6L+18K$	$6L-18K$
					^2G	$6L+12K$	$21L-33K$	$-6L+18K$	$6L-18K$
8	4+4	0	C^{12}	0	^1S	$8L+28K$	$28L-32K$	$-8L+24K$	$8L-24K$
					^1D	$8L+25K$	$28L-35K$	$-8L+24K$	$8L-24K$
					^1G	$8L+18K$	$28L-42K$	$-8L+24K$	$8L-24K$
9	4+4+1	1/2	C^{13}, N^{13}	1/2	^2P	$6L+38K$	$36L-52K$	$-12L+36K$	$12L-36K$
					^2F	$6L+33K$	$36L-57K$	$-12L+36K$	$12L-36K$
10	4+4+2	0	N^{14}	1	^3S	$5L+50K$	$45L-70K$	$-15L+50K$	$17L-46K$
					^3D	$5L+47K$	$45L-73K$	$-15L+47K$	$17L-49K$
10	4+4+2	1	C^{14}, N^{14}, O^{14}	0	^1S	$5L+50K$	$45L-70K$	$-17L+46K$	$15L-50K$
					^1D	$5L+47K$	$45L-73K$	$-17L+49K$	$15L-47K$
11	4+4+3	1/2	N^{15}, O^{15}	1/2	^2P	$5L+60K$	$55L-90K$	$-20L+60K$	$20L-60K$
12	4+4+4	0	O^{16}	0	^1S	$6L+72K$	$66L-108K$	$-24L+72K$	$24L-72K$

TABLE II. *Matrix elements and quantum numbers for the partitions consisting of 4's and two additional addends.*

n	PARTITION	T	NUCLEUS	S	STATE	$\omega L+\gamma K$			
						P	1	PQ	Q
2	1+1	1	He^6, Li^6, Be^6	1	^3P	$-L+3K$	$L-3K$	$-L+3K$	$L-3K$
2	1+1	0	Li^6	0	^1P	$-L+3K$	$L-3K$	$L-3K$	$-L+3K$
3	2+1	3/2	He^7, Li^7, Be^7, B^7	1/2	^2P	$5K$	$3L-4K$	$-3L+4K$	$-5K$
					^2D	$3K$	$3L-6K$	$-3L+6K$	$-3K$
3	2+1	1/2	Li^7, Be^7	3/2	^4P	$5K$	$3L-4K$	$5K$	$3L-4K$
					^4D	$3K$	$3L-6K$	$3K$	$3L-6K$
3	2+1	1/2	Li^7, Be^7	1/2	^2P	$5K$	$3L-4K$	0	0
					^2D	$3K$	$3L-6K$	0	0
4	3+1	1	Li^8, Be^8, B^8	1	^3P	$2L+7K$	$6L-5K$	$-2L+6K$	$2L-6K$
					^3D	$2L+5K$	$6L-7K$	$-2L+6K$	$2L-6K$
					^3F	$2L+2K$	$6L-10K$	$-2L+6K$	$2L-6K$
4	3+1	1	Li^8, Be^8, B^8	0	^1P	$2L+7K$	$6L-5K$	$-2L+13K/2$	$\frac{1}{2}K$
					^1D	$2L+5K$	$6L-7K$	$-2L+K$	$-5K$
					^1F	$2L+2K$	$6L-10K$	$-2L+4K$	$-2K$
4	3+1	0	Be^8	1	^3P	$2L+7K$	$6L-5K$	$-\frac{1}{2}K$	$2L-13K/2$
					^3D	$2L+5K$	$6L-7K$	$5K$	$2L-K$
					^3F	$2L+2K$	$6L-10K$	$2K$	$2L-4K$
5	3+2	3/2	Li^9, Be^9, B^9, C^9	1/2	^2P	$2L+12K$	$10L-12K$	$-5L+39K/4$	$2L-45K/4$
					^2D	$2L+10K$	$10L-14K$	$-5L+25K/2$	$2L-17K/2$
					^2F	$2L+7K$	$10L-17K$	$-5L+11K$	$2L-10K$
5	3+2	1/2	Be^9, B^9	3/2	^4P	$2L+12K$	$10L-12K$	$-2L+45K/4$	$5L-39K/4$
					^4D	$2L+10K$	$10L-14K$	$-2L+17K/2$	$5L-25K/2$
					^4F	$2L+7K$	$10L-17K$	$-2L+10K$	$5L-11K$
5	3+2	1/2	Be^9, B^9	1/2	^2P	$2L+12K$	$10L-12K$	$-2L+6K$	$2L-6K$
					^2D	$2L+10K$	$10L-14K$	$-2L+6K$	$2L-6K$
					^2F	$2L+7K$	$10L-17K$	$-2L+6K$	$2L-6K$
6	3+3 and 4+1+1	1	Be^{10}, B^{10}, C^{10}	1	^3P	$3L+17K$	$15L-19K$	$-5L+15K$	$5L-15K$
					^3F	$3L+12K$	$15L-24K$	$-5L+15K$	$5L-15K$
6	3+3 and 4+1+1	0	B^{10}	0	^1P	$3L+17K$	$15L-19K$	$-3L+9K$	$3L-9K$
					^1F	$3L+12K$	$15L-24K$	$-3L+9K$	$3L-9K$
7	4+2+1	3/2	Be^{11}, B^{11}, C^{11}, N^{11}	1/2	^2P	$3L+24K$	$21L-30K$	$-9L+87K/4$	$6L-93K/4$
					^2D	$3L+22K$	$21L-32K$	$-9L+49K/2$	$6L-41K/2$
					^2F	$3L+19K$	$21L-35K$	$-9L+23K$	$6L-22K$
7	4+2+1	1/2	B^{11}, C^{11}	3/2	^4P	$3L+24K$	$21L-30K$	$-6L+93K/4$	$9L-87K/4$
					^4D	$3L+22K$	$21L-32K$	$-6L+41K/2$	$9L-49K/2$
					^4F	$3L+19K$	$21L-35K$	$-6L+22K$	$9L-23K$
7	4+2+1	1/2	B^{11}, C^{11}	1/2	^2P	$3L+24K$	$21L-30K$	$-6L+18K$	$6L-18K$
					^2D	$3L+22K$	$21L-32K$	$-6L+18K$	$6L-18K$
					^2F	$3L+19K$	$21L-35K$	$-6L+18K$	$6L-18K$
8	4+3+1	1	B^{12}, C^{12}, N^{12}	1	^3P	$4L+31K$	$28L-41K$	$-10L+30K$	$10L-30\mathrm{K}$
					^3D	$4L+29K$	$28L-43K$	$-10L+30K$	$10L-30K$
					^3F	$4L+26K$	$28L-46K$	$-10L+30K$	$10L-30K$
8	4+3+1	1	B^{12}, C^{12}, N^{12}	0	^1P	$4L+31K$	$28L-41K$	$-10L+61K/2$	$8L-47K/2$
					^1D	$4L+29K$	$28L-43K$	$-10L+25K$	$8L-29K$
					^1F	$4L+26K$	$28L-46K$	$-10L+28K$	$8L-26K$
8	4+3+1	0	C^{12}	1	^3P	$4L+31K$	$28L-41K$	$-8L+47K/2$	$10L-61K/2$
					^3D	$4L+29K$	$28L-43K$	$-8L+29K$	$10L-25K$
					^3F	$4L+26K$	$28L-46K$	$-8L+26K$	$10L-28K$
9	4+3+2	3/2	B^{13}, C^{13}, N^{13}, O^{13}	1/2	^2P	$3L+41K$	$36L-58K$	$-15L+40K$	$12L-41K$
					^2D	$3L+39K$	$36L-60K$	$-15L+42K$	$12L-39K$
9	4+3+2	1/2	C^{13}, N^{13}	3/2	^4P	$3L+41K$	$36L-58K$	$-12L+41K$	$15L-40K$
					^4D	$3L+39K$	$36L-60K$	$-12L+39K$	$15L-42K$
9	4+3+2	1/2	C^{13}, N^{13}	1/2	^2P	$3L+41K$	$36L-58K$	$-12L+36K$	$12L-36K$
					^2D	$3L+39K$	$36L-60K$	$-12L+36K$	$12L-36K$
10	4+3+3	1	C^{14}, N^{14}	1	^3P	$3L+51K$	$45L-75K$	$-17L+51K$	$17L-51K$
10	4+3+3	0	N^{14}	0	^1P	$3L+51K$	$45L-75K$	$-15L+45K$	$15L-45K$

TABLE III. *Energy differences within the group of low terms.*

n	NUCLEUS	TRANSITION	ENERGY DIFFERENCE	
2, 10	Li^6, N^{14}	${}^3D-{}^3S$	$-3K$	4.8 mc^2
		${}^1S-{}^3S$	$-G_1(2L+4K)$	6.3
		${}^1D-{}^3D$	$-G_1(2L-2K)$	4.2
	He^6, Li^6, C^{14}, N^{14}	${}^1D-{}^1S$	$-(1-2G_1)3K$	2.7
3, 9	Li^7, Be^7, C^{13}, N^{13}	${}^2F-{}^2P$	$-(1-G_1)5K$	6.3
4, 8	Be^8, C^{12}	${}^1G-{}^1S$	$-(1-G_1)10K$	12.6
		${}^1D-{}^1S$	$-(1-G_1)3K$	3.8
5, 7	Be^9, B^9, B^{11}, C^{11}	${}^2G-{}^2P$	$-(1-G_1)9K$	11.3
		${}^2F-{}^2P$	$-(1-G_1)5K$	6.3
		${}^2D-{}^2P$	$-(1-G_1)2K$	2.5
6	B^{10}	${}^3G-{}^3S$	$-(1-3G_1/4)10K$	13.4
		${}^3F-{}^3S$	$-(1-G_1/4)6K$	9.1
		${}^3D_{\mathrm{I}}-{}^3S$	$-(1-G_1/2)3K$	4.3
		${}^3D_{\mathrm{II}}-{}^3S$	$-(1+3G_1/10)3K$	5.2
		${}^1S-{}^3S$	$-G_1(2L+5K)$	6.6
		${}^1D_{\mathrm{I}}-{}^3D_{\mathrm{I}}$	$-G_1(2L+2K)$	5.6
		${}^1D_{\mathrm{II}}-{}^3D_{\mathrm{II}}$	$-G_1(2L-14K/5)$	3.9
		${}^1F-{}^3F$	$-G_1(2L-4K)$	3.4
		${}^1G-{}^3G$	$-G_12L$	4.9
6	Be^{10}, B^{10}, C^{10}	${}^1G-{}^1S$	$-(1-5G_1/4)10K$	11.7
		${}^1F-{}^1S$	$-(1-7G_1/4)6K$	5.9
		${}^1D_{\mathrm{I}}-{}^1S$	$-(1-3G_1/2)3K$	3.2
		${}^1D_{\mathrm{II}}-{}^1S$	$-(1-23G_1/10)3K$	2.4

the matrix elements of Q and R are identical, and

$$\omega(Q)=-\omega(PQ),\quad \gamma(Q)=-\gamma(PQ).$$

According to Eq. (8) and the already determined values of ω, $2\gamma(Q)=6\omega(Q)=12$. Thus the matrix elements of Q and PQ are $2L-6K$ and $-2L+6K$, respectively. In other cases it is convenient to compute the interactions between like and unlike particles separately using the relations

$$\left.\begin{aligned} Q&=-P\\ PQ&=-1 \end{aligned}\right\}\ \text{(like particles)},\qquad \left.\begin{aligned} Q_{12}&=(1+\sigma_1\cdot\sigma_2)/2\\ P_{12}Q_{12}&=P_{12}(1+\sigma_1\cdot\sigma_2)/2 \end{aligned}\right\}\ \text{(unlike particles)}. \tag{10}$$

If the total spin of either group of like particles vanishes the terms in $\sigma_1\cdot\sigma_2$ contribute nothing to the matrix elements. When this happens only the space wave function is needed for the evaluation of the energy matrix.

III. REDUCTION OF THE NUMBER OF FREE PARAMETERS IN THE INTERACTION OPERATOR

We interrupt the discussion of the energy matrix to show that the total energies of the low terms involve the exchange parameters g, g_1, g_2 in only two linearly independent combinations and furthermore that only those parameters which have been fixed by consideration of the two, three and four particle problems enter into the expressions for the excitation energies within the group of low terms. Let

$$\begin{aligned} G_1&=g+g_2,\\ G_2&=g-g_2,\\ G_3&=1+g-5g_1-3g_2. \end{aligned} \tag{11}$$

The parameter G_1 determines the singlet-triplet splitting in the deuteron;[2] the binding energy of the deuteron and the scattering cross section of hydrogen for slow neutrons require $G_1\sim 0.22$. The inequality $G_3\geqq 0$ must be satisfied if the theory is to give the binding energy of heavy nuclei proportional to the number of particles.[2] Nothing is known about G_2. By direct substitution we find

$$\begin{aligned} (1-g-g_1-g_2)P+gPQ+g_11+g_2Q &=(1-G_1)P+G_1PQ+G_1(1-P)Q/2\\ &\quad+(1-G_1-G_3)(1-P)/5\\ &\qquad+G_2(1-P)(4-5Q)/10. \end{aligned} \tag{12}$$

The matrix elements of $\sum_{i>j}(1-P_{ij})(4-5Q_{ij})J(r_{ij})$ vanish for the group of low terms. The truth of this statement is obvious for the contributions from the interactions within the s shell. For the interaction between shells it follows from Eq. (16). Finally the reader can readily verify by the use of Table I that the statement also holds for the interaction within the p shell. Thus, so long as only the low terms are considered, the operator in Eq. (1) may be replaced by an effective interaction operator

$$V'=\sum_{i>j}\{(1-G_1)P_{ij}+G_1P_{ij}Q_{ij}+\tfrac{1}{5}(1+G_1-G_3)(1-P_{ij})\}J(r_{ij}). \tag{13}$$

For an s^4p^n configuration the matrix elements of the last term in Eq. (13) depend only on the partition quantum number and can be computed at once from the ω's and Eq. (8). Consequently the separations of the low terms in the Hartree approximation depend only on the simple interaction operator

$$V''=\sum_{i>j}\{(1-G_1)P_{ij}+G_1P_{ij}Q_{ij}\}J(r_{ij}). \tag{14}$$

This symmetrical operator containing only Majorana and Heisenberg interactions has already been discussed in reference 1.

In Table III the separations of the low levels are given in terms of the single parameter G_1. The numerical results are computed from the values of L and K given in Section V.

These results (as deduced from the secular equations) have already been discussed and compared with experiment in reference 1. Some additional remarks supplementing the earlier discussion will be found in Section VI of this paper.

IV. Remaining Contributions to the Total Energy Matrix

The diagonal matrix elements of the operator V for interactions within the s shell are simply $6(1-G_1)B$ where

$$B=\int\cdots\int R_s(r_1)^2R_s(r_2)^2J(r_{12})d\tau_1d\tau_2, \quad (15)$$

the average value of $J(r)$ in the s shell. From the interaction between a single p particle and the closed s shell we obtain the matrix elements

$$\begin{aligned} E_{sp}(1)&=4C-D, & E_{sp}(Q)&=2C-2D,\\ E_{sp}(P)&=4D-C, & E_{sp}(PQ)&=2D-2C, \end{aligned} \quad (16)$$

where

$$\begin{aligned} C&=\int\cdots\int x_1{}^2R_p(r_1)^2R_s(r_2)^2J(r_{12})d\tau_1d\tau_2,\\ D&=\int\cdots\int x_1x_2R_p(r_1)R_p(r_2)R_s(r_1)\\ &\qquad\times R_s(r_2)J(r_{12})d\tau_1d\tau_2. \end{aligned} \quad (17)$$

The contribution to the energy terms from the interaction between the s and p shells is then

$$E_{sp}(V)=n\{3(1-G_1)D+G_3(D-C)\}. \quad (18)$$

The Coulomb energy matrix can be expressed in terms of the integrals already defined with the interpretation of $J(r_{12})$ as e^2/r_{12}. We shall denote this particular choice of J by adding the subscript c to the symbols used for the general integrals. It is clear that the Coulomb interaction within the s shell gives rise to the energy B_c, and that between the closed s shell and a single p proton to $2C_c-D_c$. The matrix elements of the Coulomb interaction within the p shell are given in Table IV for the group of low terms.

Table IV. *Matrix elements of the Coulomb interaction within the p shell.*

State				
	Be^6	C^{14}	N^{14}	O^{14}
S	L_c+2K_c	$6L_c-8K_c$	$10L_c-20K_c$	$15L_c-30K_c$
D	L_c-K_c	$6L_c-11K_c$	$10L_c-20K_c$	$15L_c-30K_c$
	Be^7	C^{13}	N^{13}	
P	$L_c+2K_c/3$	$6L_c-28K_c/3$	$10L_c-20K_c$	
F	L_c-K_c	$6L_c-11K_c$	$10L_c-20K_c$	
	Be^8	C^{12}		
S	$L_c+2K_c/3$	$6L_c-28K_c/3$		
D	$L_c+K_c/6$	$6L_c-59K_c/6$		
G	L_c-K_c	$6L_c-11K_c$		
	Be^9	B^9	B^{11}	C^{11}
P	$L_c+2K_c/3$	$3L_c-14K_c/3$	$3L_c-14K_c/3$	$6L_c-28K_c/3$
D	$L_c+2K_c/5$	$3L_c-76K_c/15$	$3L_c-76K_c/15$	$6L_c-48K_c/5$
F	L_c-K_c	$3L_c-14K_c/3$	$3L_c-14K_c/3$	$6L_c-11K_c$
G	L_c-K_c	$3L_c-6K_c$	$3L_c-6K_c$	$6L_c-11K_c$
	Be^{10}	B^{10}	C^{10}	
S	$L_c+3K_c/2$	$3L_c-19K_c/4$	$6L_c-17K_c/2$	
DI	$L_c+K_c/2$	$3L_c-5K_c$	$6L_c-19K_c/2$	
DII	$L_c-3K_c/10$	$3L_c-23K_c/5$	$6L_c-103K_c/10$	
F	L_c-K_c	$3L_c-5K_c$	$6L_c-11K_c$	
G	L_c-K_c	$3L_c-6K_c$	$6L_c-11K_c$	

The matrix of the Coulomb interaction is diagonal in the two D states of B^{10}, but not of Be^{10} and C^{10}. For the latter nuclei the non-diagonal matrix element connecting the two D states has the value $(21/20)^{\frac{1}{2}}K_c$, which is small in comparison with the separation of the D states resulting from the spin exchange forces, and can therefore be neglected.

The correct kinetic energy operator is the difference between the sum of the single particle operators and the kinetic energy of the center of gravity:

$$-\tfrac{1}{2}\sum\Delta_i+\frac{1}{2(n+4)}(\sum\nabla_i)^2=$$

$$-\tfrac{1}{2}\left(1-\frac{1}{n+4}\right)\sum\Delta_i+\frac{1}{n+4}\sum_{i>j}\nabla_i\cdot\nabla_j. \quad (19)$$

Here, as in reference 1, the unit of length is $h/(Mm)^{\frac{1}{2}}c$ and the energy is measured in units of mc^2. The last term of Eq. (19) is well known in atomic spectra.[7] It has nonvanishing matrix elements only when i and j refer to like particles having parallel spins and "azimuthal" quantum numbers differing by one unit. The contribution to the energy is therefore one term per particle in the p shell, so that the total effect is proportional to $n/(n+4)$.

V. Numerical Calculations

The numerical analysis is most conveniently carried through using the familiar oscillator functions

$$R_s(r)=ce^{-\alpha\sigma r^2}, \quad R_p(r)=c'e^{-\alpha\tau r^2},$$

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[7] D. S. Hughes and C. Eckart, Phys. Rev. **36**, 694 (1930). J. H. Bartlett and J. J. Gibbons, Phys. Rev. **44**, 538 (1933).

together with $J(r)=-72e^{-\alpha r^2}$, $\alpha^{-\frac{1}{2}}=2.25\times 10^{-13}$ cm, as in reference 1. A variational calculation using these functions was made for the ground state energies of the series of nuclei Li^6 to O^{16}. In this calculation the terms proportional to G_3 were omitted. The energy has a broad minimum for values of σ and τ ranging from 0.55 to 0.70; no sensible improvement results from varying both σ and τ independently. It thus seems reasonable to take $\sigma=\tau=0.6$ throughout. With this choice of parameters the integrals occurring in the potential energy have the following values:

$$\begin{array}{llll} B=-16.53\ mc^2, & & B_c=1.10\ \ mc^2, & \\ C=-11.37\ \text{``}\ , & & L_c=0.89\ \ \text{``}\ , & \\ D=-\ \ 5.17\ \text{``}\ , & & K_c=0.055\ \text{``}\ , & (20) \\ L=-11.04\ \text{``}\ , & & 2C_c-D_c=1.64\ \ \text{``}\ . & \\ K=-\ \ 1.61\ \text{``}\ , & & & \end{array}$$

The general expressions for these integrals in terms of σ and τ are to be found in reference 1. The kinetic energy is also discussed there, with the exception of the last term in Eq. (19). This term makes only a small contribution to the total energy, being equal to $9.6n/(n+4)\ mc^2$ if $\sigma=\tau=0.6$.

It is, however, not necessary to use separate wave functions for the individual particles within a shell. An attempt was made to improve the energy calculation by the use of functions of a more general character involving a greater number of parameters. A symmetrical linear combination of oscillator functions with different values of τ may be substituted for the simple product functions without affecting the validity of the general analysis. For example, the radial wave function for Li^6 may be

$$e^{-\alpha\sigma(r_1{}^2+r_2{}^2+r_3{}^2+r_4{}^2)}\{e^{-\alpha\tau(r_5{}^2+r_6{}^2)}+\lambda e^{-\alpha\tau'(r_5{}^2+r_6{}^2)}\}.$$

A variational calculation for the total energy of Li^6 using this function gave a small improvement, in no case greater than 4 mc^2. This result is in agreement with the work of Inglis[8] who finds that about half the binding energy of Li^6 is accounted for by the effect of excited configurations. It thus appears necessary to consider the interaction with excited configurations to obtain an appreciable increase in the theoretical binding energies. One result of interest which emerged from the calculations with the more general wave function was the essential constancy of the integral K over a wide range of σ, τ, τ' values.

VI. DISCUSSION

We add a few remarks here supplementing the discussion in reference 1. The theoretical total energies calculated by the Hartree method are all so small that a direct comparison with the experimental values is useless. To obtain a significant comparison the theoretical values are modified by the addition of a linear function of n chosen to fit the experimental binding energies of He^4 and O^{16}. Results obtained in this way are shown in Fig. 1. The ground state of the most stable nucleus of each mass number is plotted for two values of G_3, namely $G_3=0$ and $G_3=1$, represented by the two solid curves. The curve labeled "coefficient" of G_3 is constructed by subtracting a suitably chosen linear function of n from the actual computed coefficient of G_3. Unquestionably the whole procedure is arbitrary; nevertheless we venture to state the plausible conclusion that G_3 must be small in comparison with unity.

The experimental binding energy differences $(\mathrm{C}^{11}-\mathrm{B}^{11})$ and $(\mathrm{N}^{13}-\mathrm{C}^{13})$ are very nearly equal while the corresponding quantities for other adjacent pairs of isobars with odd mass differ appreciably.[9] This fact finds a simple explanation in the Eqs. (21) and (22) which are constructed from the normal state energies listed in Table IV:

$$(\mathrm{N}^{13}-\mathrm{C}^{13})-(\mathrm{C}^{11}-\mathrm{B}^{11})=L_c-6K_c, \qquad (21)$$

$$\begin{aligned}(\mathrm{O}^{15}-\mathrm{N}^{15})-(\mathrm{N}^{13}-\mathrm{C}^{13})&\\ &=(\mathrm{C}^{11}-\mathrm{B}^{11})-(\mathrm{B}^{9}-\mathrm{Be}^{9}) \qquad (22)\\ &=L_c+\tfrac{2}{3}K_c.\end{aligned}$$

In Eq. (21) the Coulomb exchange integral K_c

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[8] D. R. Inglis, to appear in the Phys. Rev.

[9] Reference 1, Table VIII.

Added in proof: All the "experimental" energy differences in Table VIII except that for $\mathrm{H}^3-\mathrm{He}^3$ are incorrect. The correct values are 1 mc^2 larger than those given in the table. The discrepancies between the calculated electrostatic and the corrected experimental energy differences average about thirty-five percent. This discrepancy can be reduced by using more concentrated wave functions, but at the expense of exaggerating the four-shell structure and the $\mathrm{Li}^8-\mathrm{Be}^8$ energy difference. The corrected experimental energy differences, however, reduce materially the disagreement between the theoretical and observed singlet-triplet separations in the ten and fourteen particle systems.

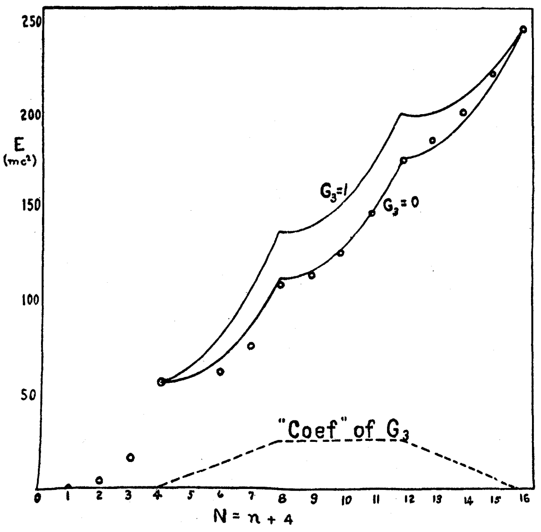

FIG. 1. Binding energy against nuclear mass. The calculated values lie on the solid curves; the circles mark experimental points.

effectively cancels an appreciable part of the ordinary Coulomb repulsion, while in Eq. (22) the exchange integral has the opposite effect. The equality stated in the first line of Eq. (22) is not verified experimentally,[9] but the apparent failure of the theory is probably closely connected with the small binding energy of the particle outside of the two alpha-particle groups in Be^9 and B^9. The small binding energy implies a large volume and consequently an exceptionally small Coulomb energy difference between Be^9 and B^9.

In a recent discussion of the β-ray spectrum of Li^8, Breit and Wigner[10] reach the tentative conclusion that the normal state of Li^8 is ${}^3P_2$. An examination of the theoretical ${}^3P$, ${}^1P$ energy difference reveals that the order of the levels depends primarily on the sign of the parameter g_2. From Eq. (12) and Table II the p shell potential energies are

$$\begin{aligned}E({}^3P)&=\tfrac{1}{5}(1-G_1)(14L+23K)\\&\quad-\tfrac{2}{5}(G_2+2G_3)(L-3K),\\E({}^1P)&=\tfrac{1}{5}(1-G_1)(14L+23K)\\&\quad-\tfrac{1}{2}G_1(2L-7K)+\tfrac{1}{5}(3G_2-4G_3)(L-3K).\end{aligned}\qquad(23)$$

The triplet-singlet energy difference is

$$E({}^3P)-E({}^1P)=2g_2(L-3K)-\tfrac{1}{2}G_1K\sim-12.4g_2+0.18.\qquad(24)$$

A sufficient condition for the triplet level to lie below the singlet is $g_2>0.015$ or $G_2\leqq 0.19$. The necessity of this condition is weakened somewhat by the existence of small spin-orbit interactions which lower the ${}^3P_2$ level while leaving the singlet level unchanged.[11, 12]

For the $\mathrm{Li}^8({}^3P)$, $\mathrm{Be}^8({}^1S)$ mass difference we find

$$\begin{aligned}&-\tfrac{1}{5}(1-G_1)(16L-3K)-\tfrac{2}{5}(G_2+2G_3)(L-3K)\\&\quad-L_c-\tfrac{2}{3}K_c-2C_c+D_c+n^1-H^1\\&\quad\sim 25.8\ mc^2+2.5(G_2+2G_3)\ mc^2.\end{aligned}\qquad(25)$$

This agrees very well with the total energy release of 14 Mev found experimentally, if G_2 and G_3 are small in comparison with unity. The analysis also leads to the equation

$$\{\mathrm{Li}^8({}^3P)-\mathrm{Be}^8({}^1S)\}-\{\mathrm{B}^{12}({}^3P)-\mathrm{C}^{12}({}^1S)\}=2L_c-21K_c/4\sim 1.5\ mc^2.\qquad(26)$$

Eq. (26) cannot be expected to hold with great accuracy because L and K are not accurately independent of the number of particles in the nucleus. The observed difference between the total energy released in the 8 and in the 12 particle reactions[13–15] is not known with sufficient accuracy to provide a test for Eq. (26).

The singlet and triplet levels discussed in the preceding paragraph occur also in Be^8, but displaced upward relative to the Li^8 levels by the amount

$$L_c+\tfrac{1}{2}K_c+2C_c-D_c-(n^1-H^1)\sim 1.0\ mc^2.\qquad(27)$$

There exists a ${}^3P$ level which belongs to Be^8 alone ($\tau=0$, $s=1$), and differs from the normal state of Li^8 by the amount

$$-G_1(L-7K/2)-G_2(L-3K)+(n^1-H^1)-L_c-\tfrac{1}{2}K_c-2C_c+D_c.\qquad(28)$$

Both this state and the ${}^3D$ belonging to Be^8 alone will fall below $\mathrm{Li}^8({}^3P)$ if $G_2>0$. In this case radioactive transitions from the ground state of Li^8 to these states of Be^8 should occur. Recently long range alpha-particles have been observed in connection with the radioactive disintegration of Li^8 with about the right energy to come from the

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[10] E. Wigner and G. Breit, Phys. Rev. **50**, 1191 (1936).

[11] E. Wigner, private communication.

[12] M. E. Rose and H. A. Bethe, Phys. Rev. **51**, 205 (1937).

[13] W. B. Lewis, W. E. Burcham, W. Y. Chang, Nature **139**, 24 (1937).

[14] L. H. Rumbaugh and L. R. Hafstad, Phys. Rev. **50**, 681 (1936).

[15] W. A. Fowler, L. A. Delsasso, C. C. Lauritsen, Phys. Rev. **49**, 561 (1936).

breaking up of these highly unstable Be^8 nuclei into alpha-particles.[13] The short range alpha-particles which are observed in the same experiment probably result from the disintegration of Be^8 nuclei formed in the ^1D state by the beta decay of Li^8. Additional evidence for a ^1D state in Be^8 with an excitation energy of 2 or 3 Mev is supplied by the short range alpha-particles observed in the disintegration of B^{11} by protons.[16]

It is probably safe to assume that the order of the levels is generally given correctly by the Hartree approximation, but there is some reason for believing that exceptions do occur. The group of low terms in B^{10} contains two D levels which probably interact strongly with each other through excited configurations. For this reason, the possibility that the normal state of B^{10} is ^3D cannot be excluded.

According to the first-order calculation the ground state of N^{14} is ^3S with a magnetic moment equal to that of Li^6 and H^2, 0.85 nuclear magnetons.[12, 17] Experimentally the total angular momentum is unity in agreement with theory, but the magnetic moment appears to be much smaller[18] than 0.85. The assumption that the normal state is ^3D leads to agreement with both the observed total angular momentum and the observed magnetic moment. Because of the spin-orbit interaction the level ^3D splits into three, $^3D_1$, $^3D_2$, $^3D_3$. We expect the inverted order $^3D_3<{}^3D_2<{}^3D_1$ in Li^6 (this follows from the theory of Inglis[19] and Furry[20]). The fine structure pattern is therefore normal in N^{14} (two particles missing from a closed shell) with the order $^3D_1<{}^3D_2<{}^3D_3$. The magnetic moment has the value

$$\begin{aligned}\mu=&0.5\{J(J+1)+L(L+1)-S(S+1)\}/2(J+1)\\&+0.85\{J(J+1)+S(S+1)-L(L-1)\}/2(J+1)\\=&0.33 \text{ nuclear magnetons.}\end{aligned}\tag{29}$$

These speculative considerations on the order of the levels in B^{10} and N^{14} can be tested by studying the fine structure in the energy distribution of the particles emitted in transmutations in which B^{10} or N^{14} is formed.

A noteworthy property of the term differences within a partition is revealed by Tables I and II. The Majorana and ordinary interactions together give rise to the term differences

$$\tfrac{1}{2}\{l(l+1)-l'(l'+1)\}(1-G_1)K,\tag{30}$$

in which l and l' are the orbital angular momenta of the initial and final states, respectively. The inclusion of the spin exchange forces leaves Eq. (30) still correct for those transitions in which $s=\mathrm{T}=s'=\mathrm{T}'$. The same expression holds also for the arithmetic mean of the energy differences associated with the paired transitions

$$\begin{aligned}s=s'=0,\quad \mathrm{T}=\mathrm{T}'=1,\\ s=s'=1,\quad \mathrm{T}=\mathrm{T}'=0,\end{aligned}$$

and also, in the same manner, for the pair

$$\begin{aligned}s=s'=\tfrac{1}{2},\quad \mathrm{T}=\mathrm{T}'=\tfrac{3}{2},\\ s=s'=\tfrac{3}{2},\quad \mathrm{T}=\mathrm{T}'=\tfrac{1}{2}.\end{aligned}$$

This last statement follows from the relation

$$\begin{aligned}E(PQ,l,s,\mathrm{T})&+E(PQ,l,\mathrm{T},s)\\&=E(PQ,l,s,\mathrm{T})-E(Q,l,s,\mathrm{T})\end{aligned}\tag{31}$$

since the right-hand member of Eq. (31) is independent of l.

It will be necessary to extend the Hartree calculations to the first excited configuration. One level belonging to this configuration is already known, namely an excited state of N^{15} with an excitation energy[15, 21] of 10.5 mc^2. A

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[16] P. I. Dee and C. W. Gilbert, Proc. Roy. Soc. **A154**, 279 (1936).

Added in proof: The maximum kinetic energy of the electrons produced by the beta-decay of B^{12} is at least 2 mc^2 greater[15] than the corresponding quantity in the case of Li^8. This fact is noteworthy because the Coulomb interaction reduces the $B^{12}-C^{12}$ mass difference more than the Li^8-Be^8 difference (Eq. 26) and thus would be expected to make the energy available for beta-decay smaller for B^{12} than for Li^8. There is, however, no difficulty if the reactions involved are

$$\begin{aligned}Li^8(^3P_2)&\rightarrow Be^8(^1D_2)+e^-\\ B^{12}(^3P_0)&\rightarrow C^{12}(^1S_0)+e^-\end{aligned}$$

since the $^1D-{}^1S$ energy difference is at least 4 mc^2. The selection rules for the beta-decay (G. Gamow and E. Teller, Phys. Rev. **49**, 895 (1936)) fix the total angular momentum of the Li^8, B^{12} normal states once the quantum numbers of the decay products are known. The interpretation of the beta-decay in this manner requires that the fine structure pattern in B^{12} be inverted with respect to the pattern in Li^8. An inversion is to be expected from the general theory of "holes" for systems symmetrically situated on opposite sides of the center of the shell. Professor G. Breit has pointed out that the resonance level observed when C^{12} is bombarded with protons (L. R. Hafstad and M. A. Tuve, Phys. Rev. **48**, 306 (1935)) can be identified as the ^2F level of N^{13}. From the masses of C^{12}, N^{13} and H^1 and the kinetic energy (0.0004 mass units) at which resonance occurs we calculate the excitation energy of the resonance level to be 0.0024 mass units or 4.4 mc^2. This value compares favorably with the theoretical value of 6.3 mc^2. There are also indications of a fine structure in the resonance level.

[17] J. H. Manley and S. Millman, Phys. Rev. **51**, 19 (1937).

[18] R. F. Bacher, Phys. Rev. **43**, 1001 (1933).

[19] D. R. Inglis, Phys. Rev. **50**, 783 (1936).

[20] W. H. Furry, Phys. Rev. **50**, 784 (1936).

[21] E. O. Lawrence, E. McMillan and M. C. Henderson, Phys. Rev. **47**, 273 (1935); J. D. Cockcroft and W. B. Lewis, Proc. Roy. Soc. **A154**, 261 (1936).

simple calculation, using the oscillator wave functions with $\sigma=\tau=0.6$, yields the result that the single excitation of a particle in N^{15} should increase the kinetic energy by the amount $14\alpha\sigma/15=9\ mc^2$. This value is in satisfactory agreement with the experimental excitation energy since it leaves some margin for the decrease in potential energy which is to be expected in an excited configuration.

In conclusion we wish to record our indebtedness to Professor E. Wigner for many valuable discussions.

Appendix 1

Construction of functions belonging to irreducible representations of the symmetric group

The single particle wave functions for the p shell may be classified according to the component of orbital angular momentum in an arbitrary direction:

$$(a,\ b,\ c)=\{(x+iy)/2^{\frac{1}{2}},\ z,\ (x-iy)/2^{\frac{1}{2}}\}R_p(r). \tag{32}$$

The set of product functions

$$\phi_k=P_k a(1)\cdots a(n_1)b(n_1+1)\cdots b(n_1+n_2) \times c(n_1+n_2+1)\cdots c(n_1+n_2+n_3), \tag{33}$$

in which the arguments of the various factors designate the space coordinates of the particles and P_k is a permutation operator, subtend the linear manifold $(n_1+n_2+n_3)$ in the total function space of n particles. The total space wave functions for the p^n configuration are then linear combinations of the functions

$$\psi_l=\sum_k c_{lk}\phi_k \tag{34}$$

where the summation extends over all permutations. We wish to determine the coefficients c_{lk} so that the ψ_l's belong to irreducible representations of the symmetric group.

In order to do this we introduce the symmetrical "displacement" operator

$$\begin{gathered} D_{\alpha\beta}=\sum_{i=1}^{n} D_{\alpha\beta}(i), \\ D_{\alpha\beta}(1):\ \alpha(1)\rightarrow\beta(1),\ \beta(1)\rightarrow 0,\ \gamma(1)\rightarrow 0, \\ \alpha,\ \beta,\ \gamma=a,\ b,\ c \text{ in arbitrary order,} \end{gathered} \tag{35}$$

and note that $D_{\alpha\beta}$ commutes with the elements of the permutation group. By means of the displacement operators it is possible to express any manifold as a sum of invariant subspaces, each subtended by a definite set of functions which generates an irreducible representation of the symmetric group.

To illustrate the method we give a complete discussion of the three particle system. The manifold $(3+0+0)$ contains only one function, $\psi_1=a(1)a(2)a(3)$. This function is completely symmetrical and generates the irreducible representation denoted by the partition $[3]$. The application of the operator D_{ab} to ψ_1 transforms it into the function

$$\psi_2=a(1)a(2)b(3)+a(1)b(2)a(3)+b(1)a(2)a(3)$$

which corresponds to an invariant subspace of the manifold $(2+1+0)$. The complementary invariant subspace (the space of functions orthogonal to the completely symmetrical function) is subtended by

$$\begin{aligned} \psi_3&=a(1)\{a(2)b(3)-a(3)b(2)\}, \\ \psi_4&=a(2)\{a(1)b(3)-a(3)b(1)\}. \end{aligned}$$

These functions generate the two dimensional representation $[2+1]$. Together with ψ_2 they exhaust the manifold $(2+1+0)$. The sum of $D_{ac}\cdot(2+1+0)$ and $D_{bc}\cdot(1+2+0)$ is subtended by the functions

$$\left.\begin{aligned} \psi_5=a(1)c(2)b(3)+a(1)c(3)b(2)+a(2)c(1)b(3) \\ +a(2)c(3)b(1)+a(3)c(1)b(2)+a(3)c(2)b(1) \end{aligned}\right\}\quad [3],$$

$$\left.\begin{aligned} \psi_6=b(3)[a(1)c(2)+a(2)c(1)] & \\ -b(2)[a(1)c(3)+a(3)c(1)] & \\ \psi_7=b(3)[a(1)c(2)+a(2)c(1)] & \\ -b(1)[a(2)c(3)+a(3)c(2)] & \end{aligned}\right\}[2+1],$$

$$\left.\begin{aligned} \psi_8=c(3)[a(1)b(2)+a(2)b(1)] & \\ -c(2)[a(1)b(2)+a(2)b(1)] & \\ \psi_9=c(3)[a(1)b(2)+a(2)b(1)] & \\ -c(1)[a(2)b(3)+a(3)b(2)] & \end{aligned}\right\}[2+1].$$

The subspace of the manifold $(1+1+1)$ complementary to $D_{ac}\cdot(2+1+0)+D_{bc}\cdot(1+2+0)$ is subtended by the completely antisymmetrical function

$$\psi_{10}=\begin{vmatrix} a(1) & b(1) & c(1) \\ a(2) & b(2) & c(2) \\ a(3) & b(3) & c(3) \end{vmatrix},$$

which generates the representation $[1+1+1]$. The set of linearly independent functions ψ_5 to ψ_{10} exhaust the manifold $(1+1+1)$. These results may be summarized in the equations

$$\begin{aligned} (3+0+0)&=[3], \\ (2+1+0)&=[3]+[2+1], \\ (1+1+1)&=[3]+2[2+1]+[1+1+1], \end{aligned} \tag{36}$$

which state the number of times the various irreducible representations appear in the unreduced manifolds. The Eq. (36) and the corresponding equations for other values of n can also be derived directly from the orthogonality and normalization properties of the group characters.

The values of the total orbital angular momentum which appear in the various irreducible representations of the symmetric group may be determined by an inspection of the manifolds and their subspaces. The wave functions describing definite quantized L, M_L values are easily found with the aid of the angular momentum operators $L_x\pm iL_y$.

It is often convenient for the calculation of Coulomb and spin exchange energies to identify the partition functions with linear combinations of products of neutron and proton functions such as are used in reference 1. This can be done by inspection if one constructs the states of reference 1 according to the ordinary methods familiar from the theory of complex spectra. The resulting combinations are identical with those which diagonalize the matrices of the total Majorana and ordinary interactions, as we noted in the introduction.

APPENDIX 2

Determination of the ordinary and isotopic spin values belonging to a definite partition in the space coordinates

The exclusion principle requires that the complete wave function vanish if more than one particle has a given set of orbital angular momentum, spin and isotopic spin quantum numbers. To obtain the total spin and total isotopic spin quantum numbers associated with the different partitions we first make a list, for each manifold separately, of the sets of individual particle quantum numbers compatible with the exclusion principle, classifying them according to Σm_s and Σm_T. Because of the symmetry about the origin, we may restrict ourselves to Σm_s, $\Sigma m_T \geqq 0$. Since $\Sigma m_T = M_T$ and $\Sigma m_s = M_s$, we immediately arrive at the total s and T necessary to account for these projections. To illustrate the method we again consider the case of three particles:

Manifold		m_s	m_T		m_s	m_T		m_s	m_T	Σm_s	Σm
(3)	a	$\frac{1}{2}$	$\frac{1}{2}$	a	$\frac{1}{2}$	$-\frac{1}{2}$	a	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
(2+1)	a	$\frac{1}{2}$	$\frac{1}{2}$	a	$\frac{1}{2}$	$-\frac{1}{2}$	b	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$-\frac{1}{2}$		$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$		$-\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$		$-\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$\frac{1}{2}$		$-\frac{1}{2}$	$-\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
		$\frac{1}{2}$	$-\frac{1}{2}$		$-\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

From the table it is obvious that there is only one pair of s, T values for the partition [3], namely $s=\frac{1}{2}$, $T=\frac{1}{2}$. After the partition [3] has been removed from the manifold (2+1) there remains the three sets of quantum numbers ($s=\frac{3}{2}$, $T=\frac{1}{2}$), ($s=\frac{1}{2}$, $T=\frac{3}{2}$), ($s=\frac{1}{2}$, $T=\frac{1}{2}$), associated with the partition [2+1]. A similar table for the manifold (1+1+1) together with Eq. (36) yields the result that the partition [1+1+1] has only two pairs of spin values, ($s=\frac{3}{2}$, $T=\frac{3}{2}$) and ($s=\frac{1}{2}$, $T=\frac{1}{2}$).

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The Disintegration of Cadmium with Deuterons

J. M. CORK AND R. L. THORNTON
University of Michigan, Ann Arbor, Michigan

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The bombardment of a cadmium target by deuterons of energy 6.3 Mev yields certain radioactive isotopes. Chemical analysis of the bombarded metal shows the presence of two radioactive isotopes of cadmium having half-life periods of 4.3 hr and 58 hr, respectively. Both emit negative electrons. The long-period cadmium activity gives rise to a radioactive indium of half-life 2.3 hr. This indium isotope in succession emits negatives resulting in the formation of a stable tin isotope. Successive chemical separations carried out over a period of sixty hours showed the indium to be in equilibrium with the long period cadmium. The existence of radioactive isotopes of half-life periods shorter than about 30 minutes is not precluded.

IT has been observed by Fermi[1] that cadmium is relatively inactive after bombardment with neutrons. Since cadmium is a very effective absorber of slow neutrons its large capture cross section for this process is probably due to the formation of one or more stable isotopes. However, many elements not rendered radioactive by neutrons are readily activated under bombardment with high energy deuterons, and it seemed desirable to investigate the disintegrations of cadmium under such conditions.

Targets of chemically pure metallic cadmium were exposed to a beam of deuterons of energy about 6.3 Mev. Exposures of the order of four microampere-hours rendered the bombarded specimens strongly radioactive. The deuteron beam was obtained by the multiple acceleration of deuterium nuclei in the cyclotron.

In order to determine with what elements the activity was associated, the bombarded specimen was dissolved in nitric acid. To this solution small quantities of solutions of the neighboring elements, silver, palladium, and indium were added.

The silver was first precipitated from this solution as the chloride by the addition of hydrochloric acid. This silver was redissolved in ammonium hydroxide and reprecipitated by the addition of nitric acid. The palladium was next brought down from the original filtrate as palladium dioxime by the addition of dimethylglyoxime to the slightly acid solution. Ammonium hydroxide added to the filtrate next precipitated the indium as indium hydroxide.

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[1] E. Amaldi, O. D'Agostino, E. Fermi, B. Pontecorvo, F. Rasetti, and E. Segrè, Proc. Roy. Soc. **149**, 522 (1935).