The subscripts 1, 2 refer to the polarizing plate and scatterer, respectively; $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$ are the propagation vectors of the incident and scattered wave; and r is the distance from the scatterer to the point of observation.

The best experimental conditions are obtained when

$$\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}}{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}|}\cdot(\boldsymbol{\mu}_1)_a=\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}}{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}|}\cdot(\boldsymbol{\mu}_2)_a=0.$$

Under these circumstances, the intensity is given by

$$I=N_2\frac{I_0}{r^2}e^{-n\sigma x_1}\cosh\beta_1\Bigg(a^2-\frac{4M^2\mu_n{}^2}{\hbar^4}\left[\left(\frac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}}{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}|}\cdot\boldsymbol{\mu}\right)_2^2\right]_a+\frac{4M^2\mu_n{}^2}{\hbar^4}[|\boldsymbol{\mu}|_2{}^2]_a\Bigg)-N_2\frac{I_0}{r^2}e^{-n\sigma x_1}\frac{4\mu_nMa}{\hbar^2|(\boldsymbol{\mu}_1)_a|}\sinh\beta_1(\boldsymbol{\mu}_1)_a\cdot(\boldsymbol{\mu}_2)_a. \quad (60)$$

The asymmetry, defined in this case as the difference in intensity between antiparallel and parallel orientation of magnetizations divided by the average intensity, is:

$$\frac{8\mu_nMa|(\boldsymbol{\mu}_2)_a|}{\hbar^2}\frac{\tanh\beta_1}{\left(a^2-\dfrac{4M^2\mu_n{}^2}{\hbar^4}\left[\left(\dfrac{\boldsymbol{\kappa}_0-\boldsymbol{\kappa}}{|\boldsymbol{\kappa}_0-\boldsymbol{\kappa}|}\cdot\boldsymbol{\mu}\right)_2^2\right]_a+\dfrac{4M^2\mu_n{}^2}{\hbar^4}[|\boldsymbol{\mu}|_2{}^2]_a\right)}. \quad (61)$$

It is interesting to note that the maximum intensity occurs with antiparallel orientation of magnetizations, in agreement with what one would expect by elementary considerations.

If both polarizer and scatterer are saturated, $x_1=0.7$ cm, and $\Theta_1=30°$, the asymmetry is 81 percent. With given values of $|(\mu_1)_a|$, $|(\mu_2)_a|$ and x_1, the maximum asymmetry is obtained at $\Theta_1=90°$. For example, under the above conditions, but with $\Theta_1=90°$, the asymmetry, as calculated from Eq. (61), is 92 percent.

There is still a fourth possible type of experiment in which a neutron beam is polarized by scattering, and then allowed to pass through a magnetized iron plate. If the iron plate is of such dimensions that it is permissible to neglect the fact that the scattered waves are spherical and not plane waves, the intensity is given by a formula identical with Eq. (59).

In conclusion, the author wishes to express his indebtedness to Professor I. I. Rabi and Professor E. Fermi for helpful discussions and suggestions, and to Professor F. Bloch for an interesting conversation on this subject.

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Photoelectric Cross Section of the Deuteron

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Photoelectric cross section curves for a Majorana-Heisenberg potential of the type $V=-V_0e^{-r^2/a^2}$ and a velocity dependent potential determined by $J_0=-(2B/a)e^{-(r+\rho)/a}$ are compared with a cross section curve for a square hole Majorana force calculated by Breit, Condon and Stehn. In each case the values of the constants used are those which have been determined as the best for accounting for the binding energies of H^2, H^3, and He^4. Results show that the cross section values for the first two potentials differ considerably from the third but very little from each other. A general formula for the area under the cross section curve, which holds for exchange as well as for ordinary forces is derived. For exchange forces $\int\sigma(\nu)d(h\nu)\cong(\pi e^2h/2Mc)(1+a\alpha)$ and this depends only on a, the range of interaction, α being defined by $\alpha^2\hbar^2/M=\epsilon$, the binding of energy the deuteron. The addition of a long range repulsive force to the velocity dependent interaction is found to decrease the cross section for this potential type considerably. The classical equivalent of the velocity dependent potential operator is determined.

Cross Section Curves

THREE hypotheses about the law of interaction between neutrons and protons have been shown to be consistent with the observed binding energies of H^2, H^3, and He^4. The requirement that the interaction account for these binding energies establishes a best value for its depth and range. It is interesting therefore to compare the photoelectric cross sections predicted by the best values for these three types of interactions to see if they are sufficiently different so that an experimental determination could furnish a criterion for deciding in favor of one of the interactions, and to consider further what general information about the neutron-proton interaction the cross section can give.

The three types of interaction considered are (1) the square hole ordinary force, (2) the bell-shaped Majorana-Heisenberg force, and (3) a velocity dependent exchange force.

(1) Mohr and Massey[1] show that for the square hole ordinary force the masses of H^2, H^3, and He^4 are reasonably consistent if the range of interaction is between 1.7 and 2.2×10^{-13} cm, and the attraction between like particles between 0.2 and 0.3 of that between unlike particles. In what follows the width 2.0×10^{-13} cm has been used.

(2) Bethe and Bacher[2] find that for a potential $V=-V_0e^{-r^2/a^2}$, the values $a=2.32\times 10^{-13}$ cm, and $V_0=34.1$ Mev are consistent with the observed binding energies of H^2, H^3, and He^4. The depth of the like-particle interaction is 21.0 Mev and that of the unlike-particle force in singlet states 20.5 Mev.

(3) Way and Wheeler[3] show that a velocity dependent force[4] whose potential is defined by $V\psi=\int J(\mathbf{x},\boldsymbol{\varepsilon})\psi(\boldsymbol{\varepsilon})d\boldsymbol{\varepsilon}$, where $J_0=-(2B/a)e^{-(r+\rho)/a}$ will account for the same binding energies if $a=1.17\times 10^{-13}$ cm and the depth $B=48.3$ Mev for ^3S states and 26.2 Mev for ^1S states. They assume equivalence for ^1S states of interaction between like and unlike particles.

Morse, Fisk, and Schiff[5] consider a potential of the form $-D[2e^{2(r_1-r)/r_0}-e^{4(r_1-r)/r_0}]$ with essentially four parameters r_1, r_0, D_s (singlet), D_t (triplet). D_s and D_t are determined in terms of r_0 and r_1 so that they will give a fit for the singlet and triplet levels of the deuteron. Since the value of the other parameters necessary to account for the binding energies of H^3 and He^4 have not been determined, this type of interaction is not included.

[1] Mohr and Massey, Proc. Roy. Soc. **A156**, 634 (1936).
[2] Bethe and Bacher, Rev. Mod. Phys. **8**, 145 (1936).
[3] Way and Wheeler, Phys. Rev. **50**, 675 (1936).
[4] John A. Wheeler, Phys. Rev. **50**, 643 (1936) for notation and discussion of velocity dependent forces.
[5] Morse, Fisk, and Schiff, Phys. Rev. **50**, 748 (1936).

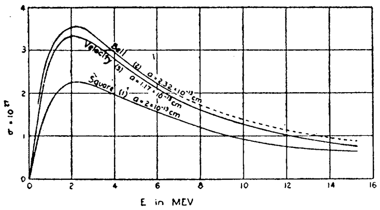

FIG. 1. Photoelectric cross section of the deuteron in units of 10^{-27} cm^2 plotted against E, the total kinetic energies of the neutron and proton after dissociation, for three different neutron-proton interactions.

The photoelectric cross sections of the deuteron for the best values of the other different interaction types are plotted against E, the total kinetic energies of the neutron and proton after dissociation, in Fig. 1. The graph shows immediately that there is very little difference for the bell-shaped Majorana-Heisenberg (2) and the velocity dependent (3) interaction. The square hole cross section curve (1), however, is considerably lower than the other two and experiment might make it possible to decide between it and either one of the others. The best energy to work at would be about $E=2$ Mev or about $h\nu=4$ Mev.

The ratios of the different cross sections at $E=15.3$ Mev and 0.48 Mev ($h\nu=2.62$ and 17.5 Mev) are given below. Here again it turns out that experimental determination of this ratio would help only to decide in favor of (1) or either (2) or (3).

	Square (1)	Bell (2)	Velocity (3)
$\frac{\sigma_{2.62}}{\sigma_{17.5}}$	1.55	1.95	2.07

Details of Calculations of Curves

Curve 1

Although Mohr and Massey consider a square hole interaction of an ordinary type, the cross section curve given (curve 1, Fig. 1) is for a square hole potential of Majorana type since this seems the more logical type and its use would probably not have greatly altered the binding energy calculations. Moreover, for the width $a = 2 \times 10^{-13}$ cm, the cross section curves for the two types differ very little. For energies up to $E = 8.8$ Mev the points on this curve have been calculated by Breit, Condon, and Stehn[6] from a formula given by Breit and Condon.[7] It should be noted that the corrected values of the Majorana curves are given in the first paper referred to. The point at $E = 15.3$ Mev has been added. Breit *et al.* take the value of ϵ, the binding energy of the deuteron as 2.2 Mev. For the calculation of the other curves the value used was 2.14 Mev. The lower value of ϵ would slightly increase the ordinates of Breit's curves.

Curve 2

Four points on this curve were calculated by numerical integration of the wave equations and subsequent numerical integration of the matrix element. The point at $E = 15.3$ Mev is from extrapolation with the help of the theorem about the areas under the curves given below. In the course of the work it was noticed that the wave functions obtained were almost identical with those for a Majorana square hole of width 1.5 times that of the $1/e$-width of the bell-shaped hole, *i.e.*, 3.48×10^{-13} cm. It would be interesting to know if this were true in general.

Curve 3

The points on this curve can be obtained by direct integration from the well-known equation for the cross section

$$\sigma = \frac{8\pi^3\nu}{c}\frac{4}{hv}|M|^2,$$

where

$$M = \frac{e}{2}\int \psi_0{}^* z \psi_E d\tau.$$

The notation used is the same as that of Breit *et al* and is as follows: $\alpha^2\hbar^2/M = \epsilon$, the binding energy of the deuteron; M = mass of proton; $h\nu$ = energy of photon; N_0 = normalization factor for ground state; $k^2\hbar^2/M = E$, sum of kinetic energies of neutron or proton after dissociation; a = "width" of interaction, either $1/e$-width or end of square hole; v = relative velocity of proton and neutron after dissociation.

When $l = 0$ the radial wave equation for the velocity dependent force considered by Wheeler is

$$\frac{\hbar^2}{M} f_0{}''(r) - \epsilon f_0(r) = \frac{-2B}{a}\int e^{-(r+\rho)/a} f_0(\rho) d\rho$$

and

$$f_0(r) = N_0[e^{-\alpha r} - e^{-r/a}].$$

For other values of l, $J_l(r,p)$ is assumed equal to zero, so that the wave functions for the upper states have the simple form

$$f_E(r) = \sin kr/kr - \cos kr.$$

Integration of the matrix element then gives for the cross section

$$\sigma = \frac{8\pi}{3}\frac{e^2}{\hbar c}\frac{a\alpha(1+a\alpha)}{(1-a\alpha)^2}(a^2\alpha^2 + a^2k^2)a^3k^3a^2 \times \left[\frac{1}{(a^2\alpha^2 + a^2k^2)^2} - \frac{1}{(1+a^2k^2)^2}\right]^2.$$

Increasing the value of a increases this cross section considerably. For $a = 1.17 \times 10^{-13}$ cm, the best value, and $h\nu = 2.62$ or $E = 0.48$ Mev, $\sigma = 1.57 \times 10^{-27}$ cm². At this same energy, with the same orders of magnitude, $\sigma = 2.66$ for $a = 1.87$ and $\sigma = 5.31$ for $a = 2.80$.

Area Under Cross Section Curves

For ordinary forces it is well known that the area under the cross section curve, $\int \sigma(\nu) d(h\nu)$, is independent of the range and type of interaction and is given by the f sum rule as $\pi e^2 h/2Mc$. Several writers have pointed out that this relation does not hold for exchange forces. It is interesting to examine, therefore, the general expression for $\int \sigma(\nu) d(h\nu)$ for any type of force.[8]

$$\int_0^\infty \sigma(\nu) d(h\nu) = \int_c^\infty \frac{8\pi^3\nu}{h}\frac{4}{v}\left|\frac{e}{2}\int \psi_0{}^* z \psi_E d\tau\right|^2 d(h\nu)$$

on rearrangement

$$= \frac{-\pi e^2}{ci}\int_0^\infty \left\{\int \psi_0{}^* \dot{z} \psi_E d\tau \int \psi_E{}^* z \psi_0 d\tau - \int \psi_0{}^* z \psi_E d\tau \int \psi_E{}^* \dot{z} \psi_0 d\tau\right\} dk.$$

[6] Breit, Condon and Stehn, Phys. Rev. **51**, 56 (1937).

[7] Breit and Condon, Phys. Rev. **49**, 904 (1936).

[8] I am indebted to Dr. John A. Wheeler for pointing out the possibility of deriving the following relation.

Applying Parseval's theorem

$$=\frac{-\pi^2e^2}{2ci}\int\psi_0^*[\dot{z}z-z\dot{z}]\psi_0d\tau$$

and on substitution of $\dot{z}\psi=(i/\hbar)(Hz-zH)\psi$

$$=\frac{\pi e^2h}{2Mc}\left\{1-\frac{M}{2\hbar^2}\int\psi_0^*[Vzz-2zVz+zzV]\psi_0d\tau\right\}$$

$$=\frac{\pi e^2h}{2Mc}\{1+T_2\}$$

when V represents the potential function. If V and z are commutative operators the second term is seen to vanish leaving the familiar expression of the f sum rule.

In the general case, expressing $V\psi$ as $\int J(\mathbf{x},\boldsymbol{\varepsilon})\psi(\boldsymbol{\varepsilon})d\boldsymbol{\varepsilon}$ the second term becomes

$$T_2=-\frac{M}{6\hbar^2}\int\int\psi_0^*(\mathbf{x})J(\mathbf{x},\boldsymbol{\varepsilon})(\mathbf{x}-\boldsymbol{\varepsilon})^2\psi_0(\boldsymbol{\varepsilon})d\boldsymbol{\varepsilon}d\mathbf{x}.$$

For Majorana forces $J(\mathbf{x},\boldsymbol{\varepsilon})=V(\mathbf{x})\delta(\mathbf{x}+\boldsymbol{\varepsilon})$. For a square hole potential the integration gives for the second term

$$T_2=\frac{a\alpha}{9a^2\beta^2(1+a\alpha)}\{a^2\beta^2(a^2\alpha^2+a^2\beta^2)+3(1+a\alpha)(a^2\beta^2-a\alpha)+3a^2\beta^2a\alpha\},$$

where $\hbar^2\beta^2/M=D-\epsilon$, D being the constant depth of the hole. For the bell-shaped Majorana a numerical integration is necessary.

For the velocity dependent force for which J is given above

$$T_2=a\alpha(7+4a\alpha+a^2\alpha^2)/6(1+a\alpha).$$

Table I gives values of T_2 for these three types of potentials for different values of a.

It turns out that in all cases T_2 is very nearly equal to $a\alpha$ so that $\int\sigma(\nu)d(h\nu)\cong\pi e^2h(1+a\alpha)/2Mc$ for any of these three exchange forces. A determination of the area under the cross section curve would thus establish the width of the interaction without giving any information as to its type, provided, of course, that the true interaction is any of the three considered here. An ordinary force could be detected by the absence of the extra term, $a\alpha$. Curves (2) and (3) can thus be distinguished by the different areas under them, the ratio being $A_3/A_2=1.526/1.286\cong1.2$. For the curves shown, extrapolations can be made beyond $E=15$ Mev with fair accuracy by assuming that they fall off either as $1/E^2$ or $1/E^{3/2}$.

TABLE I. $T_2\times100$ *percent is the percentage increase in* $\int\sigma(\nu)d(h\nu)$ *for exchange forces as compared with ordinary forces.*

$a\times10^{13}$	$a\alpha$	T_2 (SQUARE)	T_2 (BELL)	T_2 (VEL)
1.17	0.267	0.242		0.286
2.32	.529	.495	0.526	.542
3.00	.684	.654		.691
4.00	.912	.892		.913

Lowering of Velocity Dependent Cross Section

The only experimental value of the cross section determined so far is that of Chadwick and Goldhaber[9] who find $\sigma=0.5\times10^{-27}$ cm^2 with a possible error of a factor of 2 for $h\nu=2.62$. It must be remembered that this value includes the magnetic dipole cross section which for low energy γ-rays (2.62 Mev) may be $\frac{1}{4}$ or $\frac{1}{2}$ the photoelectric cross section depending upon whether the ^1S level of the deuteron is stable or unstable, according to calculations of Breit and Condon in the paper already mentioned. For γ-ray energies above 4 Mev the ratio is 0.05 at most. All the calculated cross sections are thus above this experimental limit, the square hole one being nearest to it. However, more accurate experimental determinations are needed to show whether there is really a serious contradiction between theory and experiment. It seems interesting in the meantime to calculate the effect on the velocity dependent cross section of adding a long range repulsive force to the assumed interaction, and it is found that such an addition does lower the theoretical cross section for this interaction type considerably.

The interaction is assumed to be

$$J_0=-(2B_1/a)e^{-(r+\rho)/a}+(2B_2/b)e^{-(r+\rho)/b}.$$

Three of the constants B_1, B_2, a, and b can be chosen arbitrarily. The fourth is then determined by a relation with the binding energy of the deuteron. The expression for the cross section is a

[9] Chadwick and Goldhaber, Proc. Roy. Soc. **A151**, 479 (1935).

TABLE II. *Velocity dependent cross sections for $h\nu=2.62$ for attractive force alone and attractive force plus two different additional repulsive forces.*

$a\times10^{13}$	B_1 (Mev)	$b\times10^{13}$	B_2 (Mev)	$\sigma\times10^{27}$
1.17	48.3	0.0	0.0	1.57
1.17	49.3	12.0	0.268	0.386
1.17	49.3	16.6	0.307	0.187

simple extension of the one already given. In the following B_1, a and b were chosen arbitrarily, a being taken as the best value width, B_1 a depth very near to the best value depth 48.3 Mev and b a width large in comparison with a. σ for $h\nu=2.62$ is given for two different values of b in Table II.

Classical Equivalent of Velocity Dependent Potential

The velocity dependent cross section curve is very similar to those of the other interactions and the area beneath it obeys the $1+a\alpha$ rule where a is the $1/e$-width of the operator

$$J_0=-(2B/a)e^{-(r+\rho)/a},$$

although this width does not seem to be exactly comparable to that of more familiar interactions. It is interesting, therefore, to determine the classical equivalent of the velocity dependent potential operator.

Dirac has shown that the best quantum-mechanical representation of a classical interaction potential $V(\mathbf{y},\mathbf{p})$ dependent on position and momentum is given by an integral operator connected with $V(\mathbf{y},\mathbf{p})$ by the following equation

$$V(\mathbf{y},\mathbf{p})\psi=\int J(\mathbf{x},\boldsymbol{\varepsilon})\psi(\boldsymbol{\varepsilon})d\boldsymbol{\varepsilon},$$

where $\mathbf{y}=\frac{1}{2}(\mathbf{x}+\boldsymbol{\varepsilon})$. In equivalent form we have

$$V(\mathbf{y},\mathbf{p})=\int J(\mathbf{x},\boldsymbol{\varepsilon})e^{i\mathbf{p}(\mathbf{x}-\boldsymbol{\varepsilon})/\hbar}d(\mathbf{x}-\boldsymbol{\varepsilon}),$$

where $\mathbf{y}$ is to be kept constant in the integration.

For

$$J(\mathbf{x},\boldsymbol{\varepsilon})=-(2B/4\pi r\rho a)e^{-(r+\rho)/a}$$

the integration can be carried out for the case $\sin(\mathbf{y},\mathbf{p})=0$ giving

$$V(\mathbf{y},\mathbf{p})=\frac{-8Bh}{ap}e^{-2y/a}\left\{\tan^{-1}\frac{ap}{\hbar}+\left[\frac{2y}{a}+\frac{1}{2!}\left(\frac{2y}{a}\right)^2\right]\left[\tan^{-1}\frac{ap}{\hbar}-\frac{ap}{\hbar}\right]+\left[\frac{1}{3!}\left(\frac{2y}{a}\right)^3+\frac{1}{4!}\left(\frac{2y}{a}\right)^4\right]\left[\tan^{-1}\frac{ap}{\hbar}-\frac{ap}{\hbar}+\frac{1}{3}\left(\frac{ap}{\hbar}\right)^3\right]+\cdots\right\}.$$

If $\sin(\mathbf{y},\mathbf{p})=1$, the integration is more difficult but the resulting expression reduces to a similar series of which the first two terms are identical with the first two of the above terms.

For $ap/\hbar$ small, say 0.1 (this would require a relative velocity of about $c/30$)

$$V(\mathbf{y},\mathbf{p})\cong-8Be^{-2y/a}.$$

This approximation to the interaction has $1/e$-width of $a/2$ or 0.585×10^{-13} cm and a depth 8 times the "effective" depth B or 386.4 Mev. Bethe and Bacher[10] show, however, that for a strictly exponential hole of $1/e$-width 0.5×10^{-13} cm, the depth would have to be 310 Mev to fit the binding energy of the deuteron. Wider holes require smaller depths. The above approximation is therefore not justified. The velocity dependent interaction of Wheeler can be thought of as a deep and narrow exponential type of interaction modified in a complicated way by the relative momentum of the particles.

The writer is very grateful to Dr. John A. Wheeler for much helpful advice during the whole course of the work as well as for the suggestions about the area under the cross section curves.

[10] Bethe and Bacher, Rev. Mod. Phys. **8**, 111 (1936).