If instead of Eq. (9.7) we had used the equation

$$A\sigma^\alpha\left(\frac{\partial}{\partial x^\alpha}-\frac{3}{2}\frac{\partial \log A}{\partial x^\alpha}\right)\psi=i\sigma^4\mu\psi, \tag{9.25}$$

which is obtained from (9.7) by a constant spin transformation with $T=1/\sqrt{2}(1+i\sigma^4)$, we would have obtained instead of Eq. (9.23) the equation

$$\left[\sigma^\rho p_\rho-\frac{\sigma^\rho u^\rho}{a}u^\sigma p\sigma+\frac{2ih}{a^2}u^\rho\sigma^\rho\right]\psi=i\left(\frac{\sigma^\rho u^\rho}{a}\right)mc\psi. \tag{9.26}$$

This is Eq. (9.24) except that m is replaced by im.

In conclusion I wish to thank Professor H. P. Robertson for his valuable aid and inspiring advice.

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Stable Isobars

D. R. Inglis
University of Pittsburgh, Pittsburgh, Pennsylvania
and
L. A. Young
Carnegie Institute of Technology, Pittsburgh, Pennsylvania

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RECENTLY Wigner[1] has derived some interesting results relating the lowest values of the mass number A having a certain isotopic number A-2Z. We have found it possible to derive all of those results in a much simpler manner, and have extended the calculations to include the more general interaction

$$V=\Sigma J(r_{ij})O_{ij}$$
$$O_{ij}=g_qP_{ij}{}^q+gP_{ij}+g_11_{ij}+g_\sigma P_{ij}{}^\sigma, \tag{1}$$
$$g_q+g+g_1+g_\sigma=1.$$

In Eq. (1) the sum is to be taken over all pairs of particles in the nucleus, P^q is the space-interchange operator, P is the space-spin operator and P^σ the spin operator. Wigner carried through calculations only for the special case $g=g_1=g_\sigma=0$.

We approximate the wave function ψ of a nucleus containing Z protons, π, and A-Z neutrons, ν, by a sum, antisymmetric in like particles, of products of single-particle wave functions:

$$\psi=\langle\Sigma_a u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\pi \times\langle\Sigma_a u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\nu.$$

[1] E. Wigner, Phys. Rev. **51**, 106 (1937).

Single particle states $u_1, u_2 \cdots$ are each filled with four particles (two protons and two neutrons) so long as there are enough particles to fill them. Such a filled state may be called an α group. In evaluating $(0|U|0)=\int\psi^*U\psi d\tau$ we may omit the antisymmetry in ψ^* (making the normalization factor unity). Since the interaction involves pairs of particles only, we need retain only those terms in ψ arising from single interchanges P of like particles. We follow Wigner and approximate $(0|U|0)$ by its high density limit $(0|U_0|0)$ obtained from $(0|U|0)$ by replacing $J(r)$ by $J(o)$:

$$\begin{aligned}(0|U_0|0)&=J(o)(0|\Sigma O_{ij}|0)\\&=J(o)\{g_q(0|\Sigma P_{ij}{}^q|0)+g(0|\Sigma P_{ij}|0)\\&\quad+g_1(0|\Sigma 1_{ij}|0)+g_\sigma(0|P_{ij}{}^\sigma|0)\}.\end{aligned}$$

We therefore have to compute expressions of the form

$$\begin{aligned}(0|\Sigma P^q|0)=\int &\langle u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\pi{}^*\\&\langle u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\nu{}^*(\Sigma P^q)\\&\langle(1-\Sigma P)u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\pi\\&\langle(1-\Sigma P)u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots\rangle_\nu d\tau.\end{aligned}$$

These are easy to evaluate:

$$\begin{aligned}(0|\Sigma P^q|0) &= \text{(number of bonds)—(number of like-particle, like-spin pairs)},\\ (0|\Sigma P|0) &= \text{(number of like-spin bonds)—(number of like-particle pairs)},\\ (0|\Sigma 1|0) &= \text{(number of pairs)},\\ (0|\Sigma P^\sigma|0) &= \text{(number of like-spin pairs)—(number of like-particle bonds)}.\end{aligned} \quad (2)$$

By "bond" we mean pair of particles in the same space state.

In using these formulas to determine the change in binding energy when a proton in state i becomes a neutron in state j, its interaction with an α group $k(k\neq i,j)$ is unchanged. This conforms with Wigner's division of all nuclei into types $A=4n$, $4n+1$, $4n+2$, $4n+3$.

Still adhering to Wigner's assumptions we write for the change in binding energy in an isobaric transition

$$\Delta W=\Delta(0|U_0|0)+\Delta C = \{J(o)\Delta(0|\Sigma O_{ij}|0)/\Delta Z+\Delta C/\Delta Z\}\Delta Z,$$

where C is the Coulomb energy. Any such transition may be described by indicating in parentheses the types of particles in the same space state before and after the transition (omitting an arbitrary number of α groups, if desired). If there are m paired-neutron groups these are indicated by $(\nu\nu)^m$. All transitions leading to stable isobars may be divided into the following three types

$$(\pi\nu\nu)(\nu\nu)^m \rightarrow (\nu\nu)^{m+1}\nu, \quad (3.1)$$

$$(\pi\pi\nu\nu)(\nu\nu)^m \rightarrow (\nu\nu)^{m+2}, \quad (3.2)$$

$$(\pi\pi\nu\nu)(\nu\nu)^m\nu \rightarrow (\pi\nu\nu)(\nu\nu)^{m+1}. \quad (3.3)$$

For all such transitions we find

$$\Delta(0|\Sigma O_{ij}|0)/\Delta Z=(m+3)g_q+(2m+3)g. \quad (4)$$

The quantity $\Delta C/\Delta Z$ increases with Z (approximately as $Z^{\frac{2}{3}}$), so the transitions (3.1), (3.2), (3.3) should, for $m=0$ begin to yield stable isobars at about the same value of Z. Indeed they do, Ca^{43}, A^{40}, and Cl^{37} being the lightest products of these transitions. Similarly for transitions with $m=1$, whose lightest products are V^{51}, Ti^{50} and Ti^{49}. For $g>-\frac{1}{2}g_q$, Eq. (4) says that $\Delta(0|\Sigma O_{ij}|0)/\Delta Z$ increases with increasing m, i.e., the isobaric number is an increasing function of Z. This result is, however, to be expected from any sensible theory. To facilitate comparison with Wigner's paper, one may note that the ratio 3/4 of the critical slopes which he discusses on p. 118 is just the ratio of the coefficients of g_q in (4) for $m=0$ and $m=1$.

Further, it is possible to derive the general empirical rule that *odd* Z does not occur with *even* A; a rule which has no exceptions for $A>2Z$. We simply compare the value (4) of $\Delta(0|\Sigma O_{ij}|0)/\Delta Z$ for transition (3.2) with the value

$$\Delta(0|\Sigma O_{ij}|0)/\Delta Z=(m+4)g_q+(2m+2)g$$

for the competing transition

$$(\pi\pi\nu\nu)(\nu\nu)^m \rightarrow (\pi\nu\nu)(\nu\nu)^m\nu$$

to see that the latter does not lead to stable nuclei (if only $g_q>g$).

Thus far the results seem very satisfactory but, as Wigner has remarked, this may be partially fortuitous. Indeed, if we consider the transitions of Eqs. (3) for general m we find that they first occur at

$$Z_m\sim[(m+3)g_q+(2m+3)g]^{\frac{3}{2}}J(o)^{\frac{3}{2}}\sim m^{\frac{3}{2}},$$

whereas empirically $Z_m\sim m^{3/5}$. This disagreement may perhaps be ascribed to the high density approximation, rather than to the type of interaction used. Surely we should, however, consider as remarkable any success of a theory in which differences of binding energies of isobars are computed from theoretical binding energies which have even the wrong sign, as is the case here for any simple choice of the coefficients g. It seems that the successes which have followed the use of interactions of the type (1) are very insensitive to the precise form of the interaction, and could be achieved in other ways.