Quantum Equations in Cosmological Spaces

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The Dirac equations for a free electron in a cosmological space are solved by means of separation of variables. It is shown that the wave functions depend on the angles θ and φ in the same manner as those of a free electron in flat space time. The radial functions are obtained and it is shown that they go over into the usual ones in the limit. The explicit form of the time dependence of the wave functions cannot be obtained until an arbitrary function $R(t)$ is specified. Three different cases are discussed. The energy of the free electron is then determined for each of these. Finally the connection between the equation used here and that proposed by Dirac for the DeSitter space is discussed. It is shown that they are similar and that the imaginary part of the complex mass that he was forced to introduce has a geometrical origin.

1. Introduction

VARIOUS extensions of the Dirac equation for an electron in arbitrary gravitational and electromagnetic fields have been proposed by Weyl, Fock, Schrödinger, Pauli, Schouten and Van Dantzig, and others. The relations between these equations and the class of equations of the Dirac type (linear, equations which are first order in the derivatives with respect to the coordinates, and which are invariant under arbitrary spin, gauge and coordinate transformations) were discussed in "The Dirac Equation in Projective Relativity."[1] There it is shown that one equation of the class reduces exactly to the Dirac equation for a charged particle in the special relativity case.

In this paper we obtain exact solutions of that equation for the free electron in any cosmological space, i.e., a space time whose metric is of the form[2]

$$ds^2=c^2dt^2-R^2(t)du^2, \tag{1.1}$$

where $R(t)$ is an arbitrary function of t and

$$du^2=h_{ij}dx^idx^j \tag{1.2}$$

defines a three dimensional space of constant Riemannian curvature, ρ^2, which may be positive, negative or zero. The angular functions are the same for the three cases. The radial functions are obtained explicitly for ρ^2 positive and those for the flat spaces are obtained by taking the limit as $\rho\rightarrow\infty$. The radial functions for spaces of negative curvature are obtained by making an imaginary transformation ($\rho=i\rho$, and $\alpha=i\alpha$).

The time dependence of the wave functions cannot be obtained until some assumptions are made regarding the arbitrary functions $R(t)$. In the Einstein universe R is a constant and $\rho^2>0$, it is shown that the wave functions contain a factor $e^{i\lambda t}$ where λ depends on the energy of the electron. The case where $\rho^2\gtrless 0$ and $R(t)=e^{ct/a}$ includes the DeSitter universe and the wave functions are obtained explicitly for this case. When $R(t)=ct$ and ρ^2 is negative,[3] the time dependence of the wave functions are obtained explicitly and the radial functions are obtained by an imaginary transformation.

Recently Dirac[4] proposed an equation for an electron in DeSitter space. He used the fact that the DeSitter space may be imbedded in a flat five space and used the coordinates of the latter space in his equation. The equation used here, when written in terms of the coordinates of the flat five space, gives an equation different from Dirac's but very similar to it.

The equation given in D.P.R. is invariant under abitrary spin, gauge and coordinate transformations, where the coordinate transformation

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[1] A. H. Taub, O. Veblen and J. v. Neumann, Proc. Nat. Acad. Sci. **20**, 383–385 (1934). Hereafter this paper will be referred to as D.P.R.

[2] H. P. Robertson, "Relativistic Cosmology," Rev. Mod. Phys. **5**, 62 (1933). Hereafter referred to as R.C.

[3] Robertson has shown that this case is equivalent to Milne's world structure: Zeits. f. Astrophys. **7**, 153–166 (1933). Professor Schrödinger discussed the Dirac equation in such a space in lectures at Princeton University during the spring of 1934.

[4] P.A.M. Dirac, Annals of Mathematics **36**, 657 (1935).

does not induce a spin transformation. This equation is[5]

$$\gamma^\alpha\left(\frac{\partial}{\partial x^\alpha}+S_\alpha-\frac{i}{h}\frac{e}{c}\varphi_\alpha\right)\psi=\mu\psi, \tag{1.3}$$

where φ_α is the electromagnetic four vector, $\mu=imc/2h$, m is the mass of the electron, e, its charge, h is Planck's constant divided by 2π, and in a proper choice of spin coordinate system[6]

$$S_\alpha=\gamma_\mu\left(\begin{Bmatrix}\mu\\ \beta\alpha\end{Bmatrix}\gamma^\beta+\frac{\partial\gamma^\mu}{\partial x^\alpha}\right). \tag{1.4}$$

The matrices γ_α are defined by means of the gravitational metric tensor $g_{\alpha\beta}$ as solutions of

$$\tfrac{1}{2}(\gamma_\alpha\gamma_\beta+\gamma_\beta\gamma_\alpha)=\tfrac{1}{4}g_{\alpha\beta}\cdot 1 \tag{1.5}$$

and the quantities $\begin{Bmatrix}\mu\\ \beta\alpha\end{Bmatrix}$ are the Christoffel symbols of the second kind formed from the $g_{\alpha\beta}$.

2. Computation of S_0 and γ^iS_i

The quantities S_0 and γ^iS_i will now be computed. The nonvanishing Christoffel symbols formed from the $g_{\alpha\beta}$ are[7]

$$\begin{Bmatrix}i\\ jk\end{Bmatrix}=\begin{Bmatrix}i\\ jk\end{Bmatrix}^*,\quad \begin{Bmatrix}0\\ ij\end{Bmatrix}=\frac{RR'}{c^2}h_{ij};\quad \begin{Bmatrix}i\\ 0j\end{Bmatrix}=\frac{R'}{R}\delta_j{}^i, \tag{2.1}$$

where the asterisk on the Christoffel symbol indicates that it is to be computed from the coefficients h_{ij} of the line element (1.5) and $R'=dR/dt$. Hence we have from (1.2)

$$S_0=\gamma_0\partial\gamma^0/\partial t+\gamma_i(R'\gamma^i/R+\partial\gamma^i/\partial t).$$

Since g_{00} is a constant, γ_0 and γ^0 may be taken independent of x^α. Therefore

$$S_0=\gamma_i\gamma^iR'/R+\gamma_i\partial\gamma^i/\partial t.$$

Let

$$\gamma_i=iR\sigma_i, \tag{2.2}$$

where

$$\tfrac{1}{2}(\sigma_i\sigma_j+\sigma_j\sigma_i)=\tfrac{1}{4}h_{ij}. \tag{2.3}$$

Then σ_i is independent of t. Also we have

$$\gamma^i=g^{ij}\gamma_j=-h^{ij}\gamma_j/R^2=-i\sigma^i/R. \tag{2.4}$$

Hence

$$S_0=\gamma_i\gamma^iR'/R+\gamma_i(\partial/\partial t)(-i\sigma^i/R)=0. \tag{2.5}$$

From Eq. (1.2) we have

$$S_i=\gamma_\mu\left(\begin{Bmatrix}\mu\\ \beta i\end{Bmatrix}\gamma^\beta+\frac{\partial\gamma^\mu}{\partial x^i}\right).$$

In virtue of Eqs. (2.1) and the fact that γ^0 is a constant matrix this reduces to

$$S_i=2\gamma_i\gamma_0\frac{R'}{c^2R}+\begin{Bmatrix}l\\ ji\end{Bmatrix}\gamma_l\gamma^j+\gamma_l\frac{\partial\gamma^l}{\partial x^i}. \tag{2.6}$$

From this we obtain

$$\gamma^iS_i=2\gamma^i\gamma_i\gamma_0\frac{R'}{c^2R}+\begin{Bmatrix}l\\ ji\end{Bmatrix}\gamma^i\gamma_l\gamma^j+\gamma^i\gamma_l\frac{\partial\gamma^l}{\partial x^i}.$$

[5] The summation convention is used throughout this paper. Also, we shall use the convention that Greek indices take on the four values: 0,1,2,3, whereas Latin indices take the values 1,2,3.

[6] This spin coordinate system is characterized by the fact that the spinors γ_{AB} and γ_{4AB} are constants in it. See O. Veblen, "Spinors in Projective Relativity," Proc. Nat. Acad. Sci. **19**, 979–989 (1933) for a discussion of these spinors.

[7] R. C. p. 83 (Eq. A. 6).

But $\gamma^i\gamma_i=\frac{3}{4}\cdot 1$ and $\gamma^i\gamma_l=(\frac{1}{2}\delta_l{}^i\cdot 1-\gamma_l\gamma^i)$.

Hence
$$\gamma^i S_i=\frac{3}{2}\gamma_0\frac{R'}{c^2R}+\frac{1}{2}\gamma^i\left\{\begin{matrix} l\\ jl\end{matrix}\right\}-\frac{1}{4}\gamma_l g^{ij}\left\{\begin{matrix} l\\ ij\end{matrix}\right\}+\gamma^i\gamma_l\frac{\partial\gamma^l}{\partial x^i}$$

or
$$\gamma^i S_i=\frac{3}{2}\gamma_0\frac{R'}{c^2R}+\gamma^i\left(\frac{3}{4}\frac{\partial\log h^{\frac{1}{2}}}{\partial x^i}+\frac{1}{4}h_{li}\frac{\partial h^{lk}}{\partial x^k}+\gamma_l\frac{\partial\gamma^l}{\partial x^i}\right). \tag{2.7}$$

Since
$$h^{ij}\left\{\begin{matrix} l\\ ij\end{matrix}\right\}=-\left(\frac{\partial h^{lk}}{\partial x^k}+h^{lj}\left\{\begin{matrix} k\\ jk\end{matrix}\right\}\right).$$

3. No Field

In case $\varphi_\alpha=0$ Eq. (1.1) becomes
$$\left[\gamma^0\left(\frac{\partial}{\partial t}+\frac{3}{2}\frac{R'}{R}\right)-\frac{i\sigma^i}{R}\left(\frac{\partial}{\partial x^i}+\frac{3}{4}\frac{\partial\log h^{\frac{1}{2}}}{\partial x^i}+\frac{1}{4}h_{jl}\frac{\partial h^{lk}}{\partial x^k}+\sigma_l\frac{\partial\sigma^l}{\partial x^i}\right)\right]\psi=\mu\psi. \tag{3.1}$$

We can introduce a coordinate system such that[8]
$$du^2=\rho^2d\alpha^2+\rho^2\sin^2\alpha d\theta^2+\rho^2\sin^2\alpha\sin^2\theta d\varphi^2, \tag{3.2}$$
where ρ is a real constant, if the space defined by du^2 is a space of positive curvature.

In this coordinate system
$$\partial h^{lk}/\partial x^k=0$$
since $h^{lk}=0$ for $l\neq k$ and h^{kk} is independent of x^k. In this coordinate system a solution of the equations (2.3) in a spin coordinate system in which S_α is given by Eq. (1.4), is
$$\begin{aligned}\sigma^1&=(1/\rho)(\sin\theta\cos\varphi\delta_1+\sin\theta\sin\varphi\delta_2+\cos\theta\delta_3),\\ \sigma^2&=(1/\rho\sin\alpha)(\cos\theta\cos\varphi\delta_1+\cos\theta\sin\varphi\delta_2-\sin\theta\delta_3),\\ \sigma^3&=(1/\rho\sin\alpha\sin\theta)(-\sin\varphi\delta_1+\cos\varphi\delta_2),\end{aligned} \tag{3.3}$$
where
$$\tfrac{1}{2}(\delta_a\delta_b+\delta_b\delta_a)=\tfrac{1}{4}\delta_{ab}. \tag{3.4}$$
It is readily found that
$$\sigma_i\frac{\partial\sigma^i}{\partial x^j}=-\frac{1}{2}\frac{\cos\alpha}{\sin\alpha}\delta_j{}^1+\left[2\delta_3(\cos\varphi\delta_1+\sin\varphi\delta_2)-\frac{1}{4}\frac{\cos\theta}{\sin\theta}\right]\delta_j{}^2+2\delta_1\delta_2\delta_j{}^3; \tag{3.5}$$
hence
$$\sigma^i\left(\frac{3}{4}\frac{\partial\log h^{\frac{1}{2}}}{\partial x^i}+\sigma_l\frac{\partial\sigma^l}{\partial x^i}\right)=\sigma^1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma^1}{\rho\sin\alpha}. \tag{3.6}$$

Thus Eq. (3.1) may be written
$$\left[\gamma^0\left(\frac{\partial}{\partial t}+\frac{3}{2}\frac{R'}{R}\right)-\frac{i}{R}\left(\sigma^1\left(\frac{\partial}{\partial\alpha}+\frac{\cos\alpha}{\sin\alpha}-\frac{1}{\rho\sin\alpha}\right)+\sigma^2\frac{\partial}{\partial\theta}+\sigma^3\frac{\partial}{\partial\varphi}\right)\right]\psi=\mu\psi. \tag{3.7}$$

If $\psi'=R^{\frac{3}{2}}\rho\sin\alpha\psi$, Eq. (3.1) becomes
$$\left(\gamma^0\frac{\partial}{\partial t}-\frac{i}{R}\left[\sigma^1\left(\frac{\partial}{\partial\alpha}-\frac{1}{\rho\sin\alpha}\right)+\sigma^2\frac{\partial}{\partial\theta}+\sigma^3\frac{\partial}{\partial\varphi}\right]\right)\psi=\mu\psi. \tag{3.8}$$

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[8] R. C. p. 84 (Eq. B. 8).

The replacement of the wave function ψ by ψ' means that any normalization condition of the type

$$\int \sum_{A=1}^{4} |\psi^A|^2 g^{\frac{1}{2}} dx^1 dx^2 dx^3 = 1$$

is replaced by

$$\int \sum_{A=1}^{4} |\psi^A|^2 d\alpha \sin\theta d\theta d\varphi = 1.$$

Multiplying (3.8) by γ^0 and dropping primes we have

$$\left\{\frac{1}{4c^2}\frac{\partial}{\partial t} - \frac{1}{R}\left(i\gamma^0\sigma^1\left(\frac{\partial}{\partial\alpha} - \frac{1}{\rho\sin\alpha}\right) + i\gamma^0\sigma^2\frac{\partial}{\partial\theta} + i\gamma^0\sigma^3\frac{\partial}{\partial\varphi} - \mu\gamma^0\right)\right\}\psi = 0.$$

Now let $\gamma^0 = (1/2c)\alpha_4$;

$$\begin{aligned} i\gamma^0\sigma^1 &= \beta_\alpha/\rho = (1/\rho)(\sin\theta\cos\varphi\alpha_1 + \sin\theta\sin\varphi\alpha_2 + \cos\theta\alpha_3), \\ i\gamma^0\sigma^2 &= \beta_\theta/\rho\sin\alpha = (1/\rho\sin\alpha)(\cos\theta\cos\varphi\alpha_1 + \cos\theta\sin\varphi\alpha_2 - \sin\theta\alpha_3), \\ i\gamma^0\sigma^3 &= \beta_\varphi/\rho\sin\alpha\sin\theta = (1/\rho\sin\alpha\sin\theta)(-\sin\varphi\alpha_1 + \cos\varphi\alpha_2), \end{aligned} \tag{3.9}$$

where $\alpha_a = 4ci\gamma^0\delta_a$.

Since γ^0 anticommutes with δ_a and since the δ_a anticommute among themselves, we have

$$\tfrac{1}{2}(\alpha_a\alpha_b + \alpha_b\alpha_a) = \delta_{ab} \quad (a, b = 1, 2, 3) \tag{3.10}$$

and

$$\alpha_4\alpha_a + \alpha_a\alpha_4 = 0, \quad (\alpha_4)^2 = 1. \tag{3.11}$$

The matrices α_a and α_4 are the usual Dirac matrices.[9]

Eq. (3.8) may now be written as

$$(h/i)(1/c)(\partial/\partial t)\psi = H\psi, \tag{3.12}$$

where

$$H\psi = \frac{1}{R}\left[\frac{\beta_\alpha}{\rho}\frac{h}{i}\left(\frac{\partial}{\partial\alpha} - \frac{1}{\rho\sin\alpha}\right) + \frac{1}{\rho\sin\alpha}\frac{h}{i}\left(\beta_\theta\frac{\partial}{\partial\theta} + \frac{\beta_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) + mc\alpha_4\right]\psi. \tag{3.13}$$

4. Angular Momentum Integral

A first integral of Eqs. (3.12) may be obtained by noting that the operator

$$K = -\alpha_4\left[\beta_\alpha\left(\beta_\theta + \frac{\beta_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) - 1\right] \tag{4.1}$$

commutes with H. To prove this we first write Eq. (3.13) as

$$H = \frac{1}{R\rho}\left[\beta_\alpha\frac{h}{i}\frac{\partial}{\partial\alpha} - \frac{h}{i}\frac{\beta_\alpha\alpha_4}{\sin\alpha}K + mc\alpha_4\right]. \tag{4.2}$$

Since α_4 is a constant and anticommutes with the β's, it commutes with K. We must now prove that β_α commutes with K.

From Eqs. (3.9) we see that

$$\partial\beta_\alpha/\partial\theta = \beta_\theta \quad \text{and} \quad \partial\beta_\alpha/\partial\varphi = \sin\theta\beta_\varphi. \tag{4.3}$$

[9] E. U. Condon: G. H. Shortley, *Theory of Atomic Spectra* (Cambridge Press, 1935), p. 126, Eq. (3).

Hence $$\alpha_4\left[\beta_\alpha\left(\beta_\theta\frac{\partial}{\partial\theta}+\frac{\beta_\varphi}{\sin\varphi}\frac{\partial}{\partial\varphi}\right)-1\right]\beta_\alpha=2\alpha_4\beta_\alpha+\beta_\alpha\alpha_4\beta_\alpha\left(\beta_\theta\frac{\partial}{\partial\theta}-\frac{\beta_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)-\alpha_4\beta_\alpha$$
$$=\beta_\alpha\alpha_4\left[\beta_\alpha\left(\beta_\theta\frac{\partial}{\partial\theta}+\frac{\beta_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)-1\right].$$

The operator defined by Eqs. (4.1) is the operator

$$\alpha_4((2/h^2)L\cdot S+1), \tag{4.4}$$

which occurs in the usual treatment of the Dirac equation as may be seen from Eqs. (3.9). Thus we see that the angular momentum integral of the Dirac equation in flat space holds in the cosmological spaces. That is, the dependence of the wave functions in these spaces on the variables θ and φ is given as in the flat case by solutions of the equations

$$K\psi=k\psi, \tag{4.5}$$

where k is a constant. It follows that k is an integer and equal to $l+1$ where l is the orbital angular momentum quantum number whose z component is labeled by the integer m.

The solutions of Eqs. (4.5) are[10]

$$\begin{aligned}\psi_1&=A_1(\alpha,t)\left(\frac{k+m}{2k-1}\right)^{\frac{1}{2}}\varphi(k-1,m), & \psi_3&=A_3(\alpha,t)\left(\frac{k-m}{2k+1}\right)^{\frac{1}{2}}\varphi(k,m),\\ \psi_2&=A_1(\alpha,t)\left(\frac{k-m-1}{2k-1}\right)^{\frac{1}{2}}\varphi(k-1,m+1), & \psi_4&=A_3(\alpha,t)\left(\frac{k+m+1}{2k+1}\right)^{\frac{1}{2}}\varphi(k,m+1),\end{aligned} \tag{4.6}$$

where $\varphi(l, m)$ is the normalized spherical harmonic and A_1 and A_3 are arbitrary functions.

It is readily verified that

$$\beta_\alpha\varphi=\varphi, \tag{4.7}$$

where φ is the spinor whose components are φ_1, φ_2, φ_3 and φ_4.

The angular part of the wave functions in an arbitrary spherically symmetric electromagnetic field (i.e., one that depends on α and t) are the same as for the free electron, since the operator K will still commute with H if such a field is present.

5. Radial Functions

We now proceed with the determination of the functions A_1 and A_3. Eqs. (3.12) may now be written as

$$\frac{h}{i}\frac{1}{c}\frac{\partial}{\partial t}\psi=\frac{1}{R\rho}\left(\beta_\alpha\frac{h}{i}\frac{\partial}{\partial\alpha}-\frac{h}{i}\frac{\beta_\alpha\alpha_4 k}{\sin\alpha}+mc\alpha_4\right)\psi. \tag{5.1}$$

In virtue of Eqs. (4.6) and (4.7) these equations reduce to

$$R(t)\left(\frac{h}{i}\frac{1}{c}\frac{\partial}{\partial t}-mc\right)A_1=-\frac{1}{\rho}\frac{h}{i}\left(\frac{\partial}{\partial\alpha}+\frac{k}{\sin\alpha}\right)A_3,\quad R(t)\left(\frac{h}{i}\frac{1}{c}\frac{\partial}{\partial t}+mc\right)A_3=-\frac{1}{\rho}\frac{h}{i}\left(\frac{\partial}{\partial\alpha}-\frac{k}{\sin\alpha}\right)A_1. \tag{5.2}$$

Let[11] $$\frac{u}{c}=\int_{t_0}^{t}\frac{dt}{R(t)},\quad\text{that is}\quad \frac{R}{c}\frac{\partial}{\partial t}=\frac{\partial}{\partial u}.$$

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[10] Reference 9, pp. 127–128.

[11] Then u as the spatial distance light has traveled during the time $t-t_0$. See R.C. p. 68.

Eqs. (5.2) become

$$\left(\frac{h}{i}\frac{\partial}{\partial u}-mcR(u)\right)A_1=\frac{h}{i\rho}\left(\frac{\partial}{\partial\alpha}+\frac{k}{\sin\alpha}\right)A_3,\quad \left(\frac{h}{i}\frac{\partial}{\partial u}+mcR(u)\right)A_3=\frac{h}{i\rho}\left(\frac{\partial}{\partial\alpha}-\frac{k}{\sin\alpha}\right)A_1. \tag{5.3}$$

We may now obtain a second-order differential equation by eliminating A_3 from the first of these. Thus

$$\left(\frac{h}{i}\frac{\partial}{\partial u}+mcR(u)\right)\left(\frac{h}{i}\frac{\partial}{\partial u}-mcR(u)\right)A_1=\frac{-h^2}{\rho^2}\left(\frac{\partial}{\partial\alpha}+\frac{k}{\sin\alpha}\right)\left(\frac{\partial}{\partial\alpha}-\frac{k}{\sin\alpha}\right)A_1.$$

Let $A_1(\alpha, u)=A_1(\alpha)T_1(u)$, then by separating variables we have

$$\frac{-h^2}{\rho^2}\left[\frac{d^2}{d\alpha_2}+\frac{k^2-k\cos\alpha}{\sin^2\alpha}\right]A_1=W_1{}^2A_1, \tag{5.4}$$

$$\left[\left(\frac{h}{i}\right)^2\frac{d^2}{du^2}-mc\frac{h}{i}\frac{dR}{du}-R^2m^2c^2\right]T_1=W_1{}^2T_1, \tag{5.5}$$

where W_1 is a constant.

If we are given the function $R(t)$, $R(u)$ is determined and we can solve Eq. (5.5) for T_1 and thus determine the time dependence of the wave functions. Later we shall assume $R(t)$ has a definite form and solve Eqs. (5.5). The function $A_1(\alpha)$ may be determined from Eq. (5.4). Thus if we set $r=\cos\alpha$, Eq. (5.4) becomes

$$(r^2-1)\frac{d^2A_1}{dr^2}+r\frac{dA_1}{dr}-\left(\frac{\rho^2W_1{}^2}{h^2}+\frac{k^2-kr}{r^2-1}\right)A_1=0. \tag{5.6}$$

A nonsingular solution of this Riemann P equation is

$$A_1=C_1(1-r)^{k/2}(1+r)^{k+1/2}F\left(\frac{\rho W_1}{h}+\frac{2k+1}{2},\frac{-\rho W_1}{h}+\frac{2k+1}{2};\frac{2k+1}{2};\frac{1-r}{2}\right), \tag{5.7}$$

where C_1 is a constant and F denotes the hypergeometric series and k is assumed to be positive.

Since A_1 must be finite for $\frac{1}{2}(1-r)=\pm1$, the series must contain only a finite number of terms and we must have either

$$-\rho W_1/h+(2k+1)/2=-n \quad\text{or}\quad \rho W_1/h+(2k+1)/2=-n, \tag{5.8}$$

where n is a positive integer. That is,

$$(\rho W_1/h)^2=(n+(2k+1)/2)^2. \tag{5.9}$$

Thus the allowed values of W are determined in terms of k. Since k must be an integer, we see that $(\rho W_1/h)^2$ is the square of a half-integer. We shall show later that the kinetic energy of a free electron is expressible in terms of W_1.

Similarly we may eliminate A_1 from Eqs. (5.3) and we obtain

$$\left(\frac{h}{i}\frac{\partial}{\partial u}-mcR(u)\right)\left(\frac{h}{i}\frac{\partial}{\partial u}+mcR(u)\right)A_3=\frac{-h^2}{\rho^2}\left(\frac{\partial}{\partial\alpha}-\frac{k}{\sin\alpha}\right)\left(\frac{\partial}{\partial\alpha}+\frac{k}{\sin\alpha}\right)A_3.$$

Now let $A_3(\alpha, u)=A_3(\alpha)T_3(u)$; then by separating variables we have

$$-\frac{h^2}{\rho^2}\left[\frac{d^2}{d\alpha^2}-\frac{k^2+k\cos\alpha}{\sin^2\alpha}\right]A_3=W_3{}^2A_3, \tag{5.10}$$

$$\left[\left(\frac{h}{i}\right)^2\frac{d^2}{du^2}+mc\frac{h}{i}\frac{dR}{du}-R^2n^2c^2\right]T_3=W_3{}^2T_3. \tag{5.11}$$

Setting $r=\cos\alpha$, Eq. (5.10) becomes

$$(r^2-1)\frac{d^2A_3}{dr^2}+r\frac{dA_3}{dr}-\left(\frac{k^2+kr}{r^2-1}+\frac{W_3{}^2\rho^2}{h^2}\right)A_3=0. \tag{5.12}$$

A nonsingular solution of this equation is

$$A_3=C_3(1-r)^{k+1/2}(1+r)^{k/2}F\left(\frac{\rho W_3}{h}+\frac{2k+1}{2},\frac{-\rho W_3}{h}+\frac{2k+1}{2},\frac{2k+3}{2},\frac{1-r}{2}\right). \tag{5.13}$$

The condition that A_3 be finite at $\frac{1}{2}(1-r)=\pm 1$ implies

$$(\rho W_3/h)^2=(n+(2k+1)/2)^2. \tag{5.14}$$

The functions $A_1(\alpha)$ and $A_3(\alpha)$ given by Eqs. (5.7) and (5.13) will be solutions of Eqs. (5.3) if $W_1=W_3$. We shall henceforth set $W=W_1=W_3$.

In the above we have assumed that k is positive. However, from Eqs. (5.3) we see that if (A_1, A_3) is a solution for positive k then (A_3, A_1) is the solution for negative k.

The constants C_1 and C_3 may be chosen so that

$$\int_0^\pi |A_1|^2 d\alpha=\int_0^\pi |A_2|^2 d\alpha=1. \tag{5.15}$$

This is readily done by expressing A_1 and A_3 as Jacobi polynomials, and using their orthogonality relations.[12] We readily see that Eqs. (5.15) are satisfied if

$$C_1=\frac{1}{2^k\Gamma((2k+1)/2)}\left(\frac{\Gamma(\rho W/h+(2k+1)/2)}{\Gamma(\rho W/h-(2k-1)/2)}\right)^{\frac{1}{2}},\quad C_3=\frac{\rho W/h}{2^k\Gamma((2k+3)/2)}\left(\frac{\Gamma(\rho W/h+(2k+1)/2)}{\Gamma(\rho W/h-(2k-1)/2)}\right)^{\frac{1}{2}}. \tag{5.16}$$

The neighborhood of the point $\alpha=0$ on the three dimensional sphere of radius ρ goes into a flat three dimensional space as becomes infinite. We shall now show that the functions A_1 and A_3 go over into the usual radial functions as ρ becomes infinite (i.e., Bessel functions).

Let
$$\sigma=\lim_{\substack{\alpha\to 0\\ \rho\to\infty}}\rho\sin\alpha=R\alpha.$$

Then
$$\frac{1-\cos\alpha}{2}=\sin^2\frac{\alpha}{2}\to\frac{\alpha^2}{4}=\frac{\sigma^2}{4\rho^2}$$

and $A_3=C_3(\sigma^2/4\rho^2)^{k+1/2}F(n+2k+1, -n, (2k+3)/2; \sigma^2/4\rho^2)$, where $n=\rho W/h+(2k+1)/2$.

In virtue of Eq. (5.16) we have

$$\lim_{\rho\to\infty}A_3=\frac{1}{2^k}\left(\frac{W}{h}\frac{\sigma}{2}\right)^{\frac{1}{2}}J_{k+\frac{1}{2}}\left(\frac{W}{h}\sigma\right),$$

where J_n is the Bessel function of order n.

[12] H. Bateman: *Partial Differential Equations of Mathematical Physics* (Cambridge Press, 1932), p. 392.

Similarly we can show that

$$\lim_{\rho\to\infty} A_1 = \frac{1}{2^k}\left(\frac{W}{h}\frac{\sigma}{2}\right)^{\frac{1}{2}} J_{k-\frac{1}{2}}\left(\frac{W}{h}\sigma\right).$$

But σ is the distance from the origin to the point whose coordinates are x, y, z. Hence the radial functions we have obtained go over into the usual radial functions as $\rho\to\infty$.

6. Einstein Universe

To obtain the time dependence of the wave functions we must make some assumptions regarding the arbitrary function $R(t)$. The simplest assumption, namely $R(t)$ is constant, corresponds to the Einstein universe.[13] When the radial functions given by Eqs. (5.7) and (5.13) are substituted into Eqs. (5.2) they become

$$\left(\frac{h}{i}\frac{1}{c}\frac{d}{dt} - mc\right)T_1 = -i\frac{W}{R}T_3, \qquad \left(\frac{h}{i}\frac{1}{c}\frac{d}{dt} + mc\right)T_3 = i\frac{W}{R}T_1. \tag{6.1}$$

Two linearly independent solutions of these equations are

$$T_1 = c_1 e^{i/h\lambda t}, \quad T_3 = ie_3 e^{i/h\lambda t}, \tag{6.2}$$

and

$$T_1 = d_1 e^{i/h\lambda t}, \quad T_3 = id_3 e^{-i/h\lambda t}, \tag{6.3}$$

where

$$\lambda = \left(m^2c^4 + c^2\frac{W^2}{R^2}\right)^{\frac{1}{2}} = \left(m^2c^4 + \frac{c^2h^2}{R^2\rho^2}\left(n + \frac{2k+1}{2}\right)^2\right)^{\frac{1}{2}}$$

and c_1, c_3, d_1 and d_3 are constants.

The constant λ is the energy of a free electron in an Einstein space. The existence of the two solutions (6.2) and (6.3) means that both positive and negative energy states are allowable as in flat space. Since n and k are integers we see that the square of the kinetic energy is proportional to the square of half an odd integer.

The existence of two linearly independent solutions for each value of λ corresponding to the two orientations of spin for given energy is obtained as in the usual theory.

Thus there exist four linearly independent solutions of the Dirac equations in an Einstein space. They are of the form

$$\begin{aligned}
\psi_1 &= a_1 e^{\pm i/h\lambda t} A_1(\alpha)\varphi_1(\theta,\varphi) = a_1 e^{\pm i/h\lambda t}\Phi_1(\alpha,\theta,\varphi),\\
\psi_2 &= a_1 e^{\pm i/h\lambda t} A_1(\alpha)\varphi_2(\theta,\varphi) = a_1 e^{\pm i/h\lambda t}\Phi_2(\alpha,\theta,\varphi),\\
\psi_3 &= a_3 e^{\pm i/h\lambda t} A_3(\alpha)\varphi_3(\theta,\varphi) = a_3 e^{\pm i/h\lambda t}\Phi_3(\alpha,\theta,\varphi),\\
\psi_4 &= a_3 e^{\pm i/h\lambda t} A_3(\alpha)\varphi_3(\theta,\varphi) = a_3 e^{\pm i/h\lambda t}\Phi_4(\alpha,\theta,\varphi),
\end{aligned} \tag{6.4}$$

where φ's are given by Eqs. (4.6), the A's are given by Eqs. (5.7) and (5.13) and a_1 and a_3 are constants.

The current vector whose divergence vanishes as a consequence of the Dirac equations is[14]

$$J^\alpha = \psi^+\gamma^\alpha\psi, \tag{6.5}$$

where $\psi^+ = \psi^*\gamma^0$ (ψ^* is the complex conjugate of ψ) and ψ is a solution of Eqs. (1.3). Hence

$$\int J^0 dV = \int \sum_{A=1}^{4} |\psi_A|^2 dV, \tag{6.6}$$

where dV is an arbitrary three dimensional volume.

In virtue of the normalization conditions we imposed we have

$$\int J^0 dV = 2(|a_1|^2 + |a_3|^2). \tag{6.7}$$

It is evident that we may normalize our constants so that the left member of Eqs. (6.7) is unity.

The form of the wave functions given by Eqs. (6.4) was to be expected since the space with the metric

$$du^2 = h_{ij}dx^idx^j$$

is a three dimensional sphere. The wave functions must therefore be a representation of the group

[13] R.C. p. 69.

[14] D.P.R. p. 384. The matrix $\gamma_{\dot{A}B}$ may be taken equal to the matrix γ^0.

which leaves this space invariant, namely, the rotations in a flat four space. However, the rotation group in four dimensions is a direct product of two three dimensional rotation groups. Hence the representations of this group are obtained by taking the direct product of representations of the three dimensional groups. The Jacobi polynomials and the spherical harmonics are such representations. Hence we expect that the wave functions for the Einstein space should have as factors Jacobi polynomials in cos α and spherical harmonics in θ and φ.

7. DeSitter Universe

The DeSitter universe is a cosmological space in which $R(t)=e^{ct/a}$ and the three dimensional space whose metric is $du^2=h_{ij}dx^idx^j$ is a flat space.[15] Hence we may obtain the solutions of the Dirac equations in a DeSitter universe by solving Eqs. (5.2) where $R(t)=e^{ct/a}$ and passing to the limit as $\rho\rightarrow\infty$. We have already seen that the radial functions go over into those of the flat space as we pass to the limit. Hence the only difference between the wave functions in this space and in the flat space-time is in the time dependence of the wave functions given by the functions $T_1(t)$ and $T_3(t)$.

Eqs. (5.5) and (5.11) which determine $T_1(t)$ and $T_3(t)$ become

$$\left[\frac{d^2}{du^2}-\frac{\mu^2a^2+\mu a}{u^2}+\frac{W^2}{h^2}\right]T_1=0 \qquad (7.1)$$

and
$$\left[\frac{d^2}{du^2}-\frac{\mu^2a^2-\mu a}{u^2}+\frac{W^2}{h^2}\right]T_3=0, \qquad (7.2)$$

where $\mu=\frac{mc}{h}i,\quad W=\pm\frac{h}{\rho}\left(n+\frac{2k+1}{2}\right),$

and

$$\frac{u}{c}=\int_{t_0}^{t}e^{-ct/a}dt=-\int_{t}^{\infty}e^{-ct/a}dt=-\frac{a}{c}e^{-ct/a}. \qquad (7.3)$$

We must choose such solutions of Eqs. (7.1) and (7.2) which are solutions of Eqs. (6.1).

Eqs. (7.1) and (7.2) are the differential equations for the Bessel functions. Hence for each value of W we have as the two linearly independent solutions of Eqs. (6.1) the functions

$$T_1=c\sqrt{u}J_{\mu a+\frac{1}{2}}(Wu/h)\,;\quad T_3=c\sqrt{u}J_{\mu a-\frac{1}{2}}(Wu/h) \qquad (7.4)$$

and
$$T_1=d\sqrt{u}J_{-\mu a-\frac{1}{2}}(Wu/h)\,;\quad T_3=d\sqrt{u}J_{-\mu a+\frac{1}{2}}(Wu/h), \qquad (7.5)$$

where c and d are constants and J_p is a Bessel function of complex order. Thus we again find four linearly independent solutions to the Dirac equations. This corresponds to the fact that there are two orientations of the spin for each of two possible values of the energy.

The constants c and d may be chosen so that the time component of the current vector is one. From Eqs. (6.1) it is evident that

$$\int J^0dV=2(|T_1|^2+|T_3|^2) \qquad (7.6)$$

is a constant. This constant is one for the solutions (7.4) and (7.5) if

$$c=d=\tfrac{1}{2}(|W|/h)^{\frac{1}{2}}\Gamma(\mu a+\tfrac{1}{2}). \qquad (7.7)$$

The magnitude of the number μa is the ratio of the radius of the universe to the Compton wave-length. This number is of the order of magnitude of 10^{37}. From the asymptotic expansions of T_1 and T_3 we see that as $a\rightarrow\infty$ T_1 and T_3 tend to the exponential functions. Since the functions T_1 and T_3 can be normalized for any value of μa there is no reason for assigning any particular value to this number. However, such might not be the case when there is a field present.

The energy of the electron in this type of a universe is a function of the time. The exact expression for it may be obtained by computing the time-time component of the stress energy tensor given by Fock.[16] However, since a is of the order of 10^{27} cm, we may consider $R=e^{ct/a}$ as a constant and obtain as a first approximation that

$$T_3=cT_1=e^{i\lambda t/h}, \qquad (7.8)$$

where c is a constant and λ is the energy and is given as a function of time by the relation

[15] R.C. p. 70.

[16] V. Fock, Zeits. f. Physik **57**, 274 (1929).

$$\frac{\lambda^2}{c^2}-m^2c^2=e^{-2ct/a}\left(\frac{Wh}{\rho}\right)^2\left(n+\frac{2k+1}{2}\right)^2. \quad (7.9)$$

Also we see that λ may take on positive and negative values.

The wave functions for a universe in which $R(t)=e^{ct/a}$ and $\rho^2>0$ are therefore

$$\begin{aligned}\varphi_1&=T_1\Phi_1(\alpha,\theta,\varphi), & \varphi_3&=T_3\Phi_3(\alpha,\theta,\varphi),\\ \varphi_2&=T_1\Phi_2(\alpha,\theta,\varphi), & \varphi_4&=T_3\Phi_4(\alpha,\theta,\varphi),\end{aligned} \quad (7.10)$$

where T_1 and T_3 are given by Eqs. (7.6) and (7.7), respectively, and the functions $\Phi(\alpha, \theta, \varphi)$ are given by Eqs. (6.5).

The wave functions for the DeSitter universe are obtained by going to the limit $\rho\rightarrow\infty$. They are of the same type as Eqs. (7.10) but we must replace the Jacobi polynomials of $\cos\alpha$ that occur in the functions $\Phi(\alpha, \theta, \varphi)$ by the Bessel functions of $\sigma=\rho\sin\alpha$. The energy in excess of the rest energy need not be an odd integer since $n\rightarrow\infty$ as $\rho\rightarrow\infty$. Hence we have that the energy of a free electron in DeSitter space is approximately

$$\lambda^2/c^2-m^2c^2=e^{-2ct/a}W, \quad (7.11)$$

where W is a constant.

8. Milne Case

If the three spaces $t=$constant are spaces of negative Riemannian curvature, then we can introduce a coordinate system such that the metric takes on the form

$$du^2=\rho^2(d\alpha^2+\sinh^2\alpha(d\beta^2+\sin^2\beta d\varphi^2)) \quad (8.1)$$

instead of that given in Eqs. (3.2). If we make the imaginary transformation $\rho=i\rho$ and $\alpha=i\alpha$, Eq. (8.1) goes over into Eq. (3.2). Thus the solutions of the Dirac equation for the case of 3 space of negative curvature can be obtained from that of positive curvature by replacing ρ by $i\rho$ and α by $i\alpha$ in the solutions we have obtained. Since this transformation leaves β and φ unaltered, we see that the angular part of the wave functions is the same for both cases. The radial functions differ from those given in Section 5 in that we have Jacobi polynomials in $\cosh\alpha$ instead of $\cos\alpha$.

The time dependence of the wave functions depends again on the nature of $R(t)$. Robertson has shown[2] that Milne's theory of world structure is equivalent to the theory of a cosmological space in which the three spaces t constant are spaces of negative Riemannian curvature and $R(t)=ct$. We shall now obtain the time dependence of the wave functions for this case. The equations determining T_1 and T_3 are

$$\begin{aligned}ct\left(\frac{h}{i}\frac{1}{c}\frac{d}{dt}-mc\right)T_1&=-iWT_3,\\ ct\left(\frac{h}{i}\frac{1}{c}\frac{d}{dt}+mc\right)T_3&=iWT_1,\end{aligned} \quad (8.2)$$

where $$W=\pm\frac{h}{\rho}\left(n+\frac{2k+1}{2}\right).$$

The two second-order equations obtained from Eq. (8.2) by elimination are

$$\left(\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\mu^2c^2-\frac{\mu c}{t}+\frac{W^2}{h^2t^2}\right)T_1=0 \quad (8.3)$$

and $$\left[\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\mu^2c^2+\frac{\mu c}{t}+\frac{W^2}{h^2t^2}\right]T_3=0, \quad (8.4)$$

where $\mu=imc/h$. The two linearly independent solutions of Eq. (8.3) for each value of W are

$$T_1^{(1)}=c_1e^{\mu ct}e^{iW\log t/h}\times{}_1F_1\left(i\frac{W}{h},\frac{2iW}{h}+1\,;\,-2\mu ct\right), \quad (8.5)$$

$$T_1^{(2)}=d_1e^{\mu ct}e^{-iW\log t/h}\times{}_1F_1\left(-i\frac{W}{h},-2\frac{iW}{h}+1,-1\mu ct\right), \quad (8.6)$$

where ${}_1F_1(a, b, x)$ is the confluent hypergeometric function. The two linearly independent solutions of Eq. (8.4) are

$$T_3^{(1)}=c_3e^{-\mu ct}e^{iW\log t/h}\times{}_1F_1\left(i\frac{W}{h},2i\frac{W}{h}+1,2\mu ct\right), \quad (8.7)$$

$$T_3^{(2)}=d_3e^{-\mu ct}e^{-iW\log t/h}\times{}_1F_1\left(-i\frac{W}{h},\frac{-2iW}{h}+1,2\mu ct\right). \quad (8.8)$$

If we take $c_1 \neq 0$ and $d_1=0$, then in order to satisfy Eqs. (8.2) we must take $c_3 \neq 0$ and $d_3=0$. Similarly, if we take $c_1=0$, we must take $c_3=0$. Again the two possible solutions for T_1 and T_3 correspond to the two possible orientations of the spin of the electron. We shall restrict ourselves to the case $d_1=d_3=0$, and show how c_1 and c_3 must be taken so that

$$\int J^0 dV = 2(|T_1|^2+|T_3|^2)=1. \tag{8.9}$$

Eq. (8.9) will be satisfied if

$$c_1=\frac{h}{W}\frac{1}{\sqrt{2}}\frac{\Gamma(iW/h)}{\Gamma(2iW/H+1)}e^{-\pi W/2h}, \tag{8.10}$$

$$c_3=\frac{h}{W}\frac{1}{\sqrt{2}}\frac{\Gamma(iW/h)}{\Gamma(2iW/h+1)}e^{\pi W/2h}. \tag{8.11}$$

Hence we have as solutions of the Dirac equation for this type of space

$$\begin{aligned} \varphi_1&=T_1\Phi_1(i\alpha,\beta,\varphi), & \varphi_3&=T_3\Phi_3(i\alpha,\beta,\varphi),\\ \varphi_2&=T_1\Phi_2(i\alpha,\beta,\varphi), & \varphi_4&=T_3\Phi_4(i\alpha,\beta,\varphi), \end{aligned} \tag{8.12}$$

where the functions T_1 and T_3 are given by Eqs. (8.5) to (8.8), and the functions Φ are defined in Eqs. (6.5). Since there are two sets of solutions of Eqs. (8.2) for every value of W and since W may either be positive or negative, we see that for any set of quantum numbers n, k, m, there are four possible solutions of the Dirac equation.

As before

$$W^2=\frac{h^2}{\rho^2}\left(n+\frac{2k+1}{2}\right)^2. \tag{8.13}$$

The relation between the constant W and the energy of the electron may be obtained by computing the time-time component of the stress energy tensor given by Fock.[16] It is

$$T_0{}^0=\frac{h}{2i}\int\left(\varphi^+\gamma^0\frac{\partial\varphi}{\partial t}-\frac{\partial\varphi^+}{\partial t}\gamma^0\varphi\right)dV, \tag{8.14}$$

where φ are the solutions of Eq. (1.3) and $\varphi^+=\varphi^*\gamma^0$. Therefore

$$T_0{}^0=mc^2(|T_1|^2-|T_3|^2)+\frac{iW}{t}(T_1{}^*T_3-T_3{}^*T_1) \tag{8.15}$$

as a consequence of Eqs. (8.2) and the normalization of Φ. We shall evaluate (8.15) for large values of W/h. In this case

$${}_1F_1\left(i\frac{W}{h},\ \frac{2iW}{h}+1,\ -2\mu ct\right)\sim e^{-\mu ct};$$

therefore $T_1{}^{(1)}\sim c_1 e^{-iW\log t/h}$

and similarly $T_3{}^{(1)}\sim c_3 e^{-iW\log t/h}$.

Therefore $T_0{}^0=WC/t=W/t$,

where C is twice the imaginary part of c_1c_3, and equals one from Eqs. (8.10) and (8.11).

9. Comparison with Dirac's Equation for DeSitter Space[3]

In order to compare the equation used here with the one recently proposed by Professor Dirac for an electron in DeSitter space, we will evaluate our equation for the DeSitter space in a different coordinate system. Since the DeSitter space is a space of constant curvature $1/a^2$, there exists a coordinate system in which the line element takes the form

$$ds^2=A^{-2}(-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2), \tag{9.1}$$

where $x^0=ct$, and

$$A=1+[-(x_0)^2+(x^1)^2+(x^2)^2+(x^3)^2/4a^2]=(1+s^2/4a^2). \tag{9.2}$$

A solution of the equations

$$\tfrac{1}{2}(\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha)=\tfrac{1}{4}g^{\alpha\beta}\cdot 1$$

in a spin frame in which S_α takes on the form given by Eq. (1.3) is

$$\gamma^0=\tfrac{1}{2}iA\sigma^0 \quad \gamma^j=\tfrac{1}{2}A\sigma^j, \tag{9.3}$$

where the matrices σ^α are constant matrices following the relation

$$\tfrac{1}{2}(\sigma^\alpha\sigma^\beta+\sigma^\beta\sigma^\alpha)=\delta^{\alpha\beta}. \tag{9.4}$$

By a calculation similar to that given in section one we find

$$\gamma^\alpha S_\alpha=\gamma^\alpha\left(\frac{1}{2}\frac{\partial\log\sqrt{-g}}{\partial x^\alpha}+\frac{1}{4}g_{\alpha\mu}\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}\,g^{\mu\tau}}{\partial x^\tau}+\gamma_\mu\frac{\partial\gamma^\mu}{\partial x^\alpha}\right). \tag{9.5}$$

From the values of the $g_{\mu\nu}$ and the γ_μ given by Eqs. (9.1) and (9.3) we find

$$\gamma^\alpha S_\alpha=-\frac{3}{2}\frac{\partial\log A}{\partial x^\alpha}. \tag{9.6}$$

Thus Eq. (1.1) for a free electron becomes

$$\gamma^\alpha\left(\frac{\partial}{\partial x^\alpha}-\frac{3}{2}\frac{\partial\log A}{\partial x^\alpha}\right)=\mu\psi. \tag{9.7}$$

In order to compare this equation with that proposed by Professor Dirac, we must find a transformation to a flat five space which has the space given by (9.1) as a four-dimensional hypersurface given by the equation

$$(u^1)^2+(u^2)^2+(u^3)^2-(u^0)^2+(u^4)^2=a^2. \tag{9.8}$$

Let us replace x^0 by ix^0 and u^0 by iu^0; then we must find a transformation that carries the space whose metric is given by

$$ds^2=(1+s^2/4a^2)^{-2}((dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2),\quad s^2=(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2 \tag{9.9}$$

into the hypersurface

$$(u^0)^2+(u^1)^2+(u^2)^2+(u^3)^2+(u^4)^2=a^2.$$

The transformation needed is

$$u^\beta=x^\beta/A,\quad u^4=a(1-s^2/4a^2)/A,\quad A(u)=2a/u^4+a. \tag{9.10}$$

Therefore
$$\frac{\partial}{\partial x^\alpha}=\frac{\partial u^\beta}{\partial x^\alpha}\frac{\partial}{\partial u^\beta}+\frac{\partial u^4}{\partial x^\alpha}\frac{\partial}{\partial u^4}=\frac{1}{A}\left(\delta_\alpha{}^\beta-\frac{u^\beta u^\alpha}{2a^2}A\right)\frac{\partial}{\partial u^\beta}-\frac{1}{A}\frac{u^\alpha}{a}\frac{\partial}{\partial u^4},$$

Hence
$$A\sigma^\alpha\frac{\partial}{\partial x^\alpha}=\left(\sigma^\beta-\frac{u^\beta}{a}\left(\frac{\sigma^\alpha u^\alpha}{u^4+a}\right)\right)\frac{\partial}{\partial u^\beta}-\frac{\sigma^\alpha u^\alpha}{a}\frac{\partial}{\partial u^4}=\rho^\beta\frac{\partial}{\partial u^\beta}+\rho^4\frac{\partial}{\partial u^4}, \tag{9.11}$$

where
$$\rho^\beta=\sigma^\beta-\frac{u^\beta}{a}\left(\frac{\sigma^\alpha u^\alpha}{u^4+a}\right),\quad \rho^4=-\frac{\sigma^\alpha u^\alpha}{a}. \tag{9.12}$$

Then we will have
$$\rho^\beta\rho^\alpha+\rho^\alpha\rho^\beta=2\delta^{\alpha\beta}-2u^\alpha u^\beta/a^2,\quad \rho^4\rho^\alpha+\rho^\alpha\rho^4=-2u^\alpha u^4/a. \tag{9.13}$$

Hence five matrices defined by the relations

$$\alpha^\beta=\rho^\beta-\mu^\beta\sigma^4/a,\quad \alpha^4=\rho^4-u^4\sigma^4/a, \tag{9.14}$$

where σ^4 is a matrix which anticommutes with the four matrices σ^α and hence with ρ^β and ρ^4, satisfy

$$\alpha^\sigma\alpha^\rho+\alpha^\rho\alpha^\sigma=2\delta^{\rho\sigma}. \tag{9.15}$$

Eq. (9.11) may now be written as

$$A\sigma^{\alpha}\frac{\partial}{\partial x^{\alpha}}=\alpha^{\rho}\frac{\partial}{\partial u^{\rho}}+\sigma^{4}\frac{u^{\rho}}{a}\frac{\partial}{\partial u^{\rho}}. \tag{9.16}$$

The matrices α^{ρ} are a set of five anticommuting matrices which we will want to identify with those used by Dirac. However, since they are not constant matrices we will first perform a spin-transformation on Eq. (9.7). Setting $\psi=T\varphi$, we have

$$A\sigma^{\alpha*}\frac{\partial\varphi}{\partial x^{\alpha}}+A\sigma^{\alpha*}T^{-1}\frac{\partial T}{\partial x^{\alpha}}-\tfrac{3}{2}\sigma^{\alpha*}\frac{\partial A}{\partial x^{\alpha}}\varphi=2u\varphi, \tag{9.17}$$

where

$$\sigma^{\alpha*}=T^{-1}\sigma^{\alpha}T,\quad \sigma^{4*}=T^{-1}\sigma^{4}T. \tag{9.18}$$

Let

$$T=\frac{1}{A^{\frac{1}{2}}}\left(\sigma^{4}+\frac{x^{\beta}\sigma^{\beta}}{2a}\right)=\left(\frac{u^{4}+a}{2a}\right)^{\frac{1}{2}}\left(\sigma^{4}+\frac{u^{\beta}\sigma^{\beta}}{u^{4}+a}\right). \tag{9.19}$$

Then since $T^2=1$, $T=T^{-1}$ and

$$\sigma^{\alpha*}=-\sigma^{\alpha}+(u^{\alpha}/a)(\sigma^{4}+u^{\beta}\sigma^{\beta}/(u^{4}+a)),\quad \sigma^{4*}=(1/a)(u^{4}\sigma^{4}+u^{\beta}\sigma^{\beta}). \tag{9.20}$$

It is readily verified that

$$\alpha^{\alpha}=\sigma^{\alpha*}-(\sigma^{4*}+u^{\beta}\sigma^{\beta*}/(u^{4}+a))u^{\alpha}/a=-\sigma^{\alpha},\quad \alpha^{4}=-(1/a)(\sigma^{\alpha*}u^{\alpha}+\sigma^{4*}u^{4})=-\sigma^{4}, \tag{9.21}$$

and hence are constants.

From Eq. (9.9) it follows that

$$A\sigma^{*\alpha}T^{-1}\frac{\partial T}{\partial x^{\alpha}}-\tfrac{3}{2}\sigma^{*\alpha}\frac{\partial A}{\partial x^{\alpha}}=\frac{2}{a}\frac{\sigma^{4}}{A}\left(1-\frac{s^{2}}{4a^{2}}\right)+\frac{x^{\beta}\sigma^{\beta}}{a^{2}}\left(\frac{1-s^{2}/4a^{2}}{A}+1\right)=\frac{2}{a^{2}}(\sigma^{4}u^{4}+u^{\alpha}\sigma^{\alpha}). \tag{9.22}$$

Thus as a consequence of Eqs. (9.16), (9.21) and (9.22) Eq. (9.17) becomes

$$\left[-\sigma^{\rho}\frac{\partial}{\partial u^{\rho}}+\frac{\sigma^{\rho}u^{\rho}}{a^{2}}u^{\sigma}\frac{\partial}{\partial u^{\sigma}}+\frac{2}{a^{2}}\sigma^{\rho}u^{\rho}\right]\psi=2\mu\psi.$$

Multiplying by $-h/i$, we have

$$\left[\sigma^{\rho}p_{\rho}-\left(\frac{\sigma^{\rho}u^{\rho}}{a^{2}}\right)\left(u^{\sigma}p_{\sigma}+\frac{2ih}{a^{2}}\right)\right]\psi=-mc\psi. \tag{9.23}$$

The equation proposed by Dirac,[17] when evaluated on the DeSitter space, may be written in the form

$$(\sigma^{\rho}p_{\rho}-\sigma^{\rho}u^{\rho}u^{\sigma}p\sigma/a^{2})\psi=(\sigma^{\rho}u^{\rho}/a)(mc-2ih/a), \tag{9.24}$$

where m is real and the term $-2ih/a$ is introduced in order that Eq. (9.24) be Hermitian. It is interpreted as the imaginary part of a complex mass.

From Eq. (9.23) we see that this term arises from the choice of spin coordinate system used and hence is purely a geometrical term and has no physical significance. Eq. (9.23) is exactly the same as Eq. (9.24) if in the former we replace $-m$ by $m\sigma^{\rho}u^{\rho}/a$.

[17] Dirac, reference 4, p. 664.

If instead of Eq. (9.7) we had used the equation

$$A\sigma^\alpha\left(\frac{\partial}{\partial x^\alpha}-\frac{3}{2}\frac{\partial \log A}{\partial x^\alpha}\right)\psi=i\sigma^4\mu\psi, \tag{9.25}$$

which is obtained from (9.7) by a constant spin transformation with $T=1/\sqrt{2}(1+i\sigma^4)$, we would have obtained instead of Eq. (9.23) the equation

$$\left[\sigma^\rho p_\rho-\frac{\sigma^\rho u^\rho}{a}u^\sigma p\sigma+\frac{2ih}{a^2}u^\rho\sigma^\rho\right]\psi=i\left(\frac{\sigma^\rho u^\rho}{a}\right)mc\psi. \tag{9.26}$$

This is Eq. (9.24) except that m is replaced by im.

In conclusion I wish to thank Professor H. P. Robertson for his valuable aid and inspiring advice.

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Stable Isobars

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RECENTLY Wigner[1] has derived some interesting results relating the lowest values of the mass number A having a certain isotopic number A-$2Z$. We have found it possible to derive all of those results in a much simpler manner, and have extended the calculations to include the more general interaction

$$V=\Sigma J(r_{ij})O_{ij}$$

$$O_{ij}=g_qP_{ij}{}^q+gP_{ij}+g_11_{ij}+g_\sigma P_{ij}{}^\sigma, \tag{1}$$

$$g_q+g+g_1+g_\sigma=1.$$

In Eq. (1) the sum is to be taken over all pairs of particles in the nucleus, P^q is the space-interchange operator, P is the space-spin operator and P^σ the spin operator. Wigner carried through calculations only for the special case $g=g_1=g_\sigma=0$.

We approximate the wave function ψ of a nucleus containing Z protons, π, and A-Z neutrons, ν, by a sum, antisymmetric in like particles, of products of single-particle wave functions:

$$\psi=<\Sigma_a u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\pi \times<\Sigma_a u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\nu.$$

Single particle states $u_1, u_2\cdots$ are each filled with four particles (two protons and two neutrons) so long as there are enough particles to fill them. Such a filled state may be called an α group. In evaluating $(0|U|0)=\int\psi^*U\psi d\tau$ we may omit the antisymmetry in ψ^* (making the normalization factor unity). Since the interaction involves pairs of particles only, we need retain only those terms in ψ arising from single interchanges P of like particles. We follow Wigner and approximate $(0|U|0)$ by its high density limit $(0|U_0|0)$ obtained from $(0|U|0)$ by replacing $J(r)$ by $J(o)$:

$$\begin{aligned}(0|U_0|0)&=J(o)(0|\Sigma O_{ij}|0)\\&=J(o)\{g_q(0|\Sigma P_{ij}{}^q|0)+g(0|\Sigma P_{ij}|0)\\&\quad+g_1(0|\Sigma 1_{ij}|0)+g_\sigma(0|P_{ij}{}^\sigma|0)\}.\end{aligned}$$

We therefore have to compute expressions of the form

$$\begin{aligned}(0|\Sigma P^q|0)=\int &<u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\pi{}^*\\&<u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\nu{}^*(\Sigma P^q)\\&<(1-\Sigma P)u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\pi\\&<(1-\Sigma P)u_1{}^+u_1{}^-u_2{}^+u_2{}^-\cdots>_\nu d\tau.\end{aligned}$$

[1] E. Wigner, Phys. Rev. **51**, 106 (1937).