We may rewrite (222) using the total momentum j rather than the "orbital" momentum L for labeling the terms:

$$e^{i\mathbf{k}\cdot\mathbf{r}}\frac{\partial}{\partial x}=\sum_{j=1}^{\infty}(Y_{j1}+Y_{j-1})\times\left\{[\alpha_j\chi_{j-1}(kr)+\alpha_{j+1}\chi_{j+1}(kr)]\frac{\partial}{\partial r}+\cdots\right\}. \quad (223)$$

For light, only small values of kr are important because the wave-length of the light is generally large compared to the dimensions of the radiating system. Now, according to (18), we have $\chi_{j-1}\propto(kr)^j$ and $\chi_{j+1}\propto(kr)^{j+2}$ so that the first term in the square bracket in (223) is always much more important than the second. The terms $j=1, 2, 3, \cdots$ (or $L=0, 1, 2, \cdots$) correspond to dipole, quadrupole, octopole . . . radiation.

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Relativistic Wave Functions in the Continuous Spectrum for the Coulomb Field

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For purposes of reference the continuous spectrum solutions of the Dirac wave equation for the Coulomb field are given. The solutions in the form of series and integral representations and the asymptotic behavior at large distances are included among the formulae.

THE solutions in the continuous spectrum of the Dirac wave equation for the Coulomb field have been known and used extensively for some time. However, the form in which they appear in the literature is not convenient for some purposes. It is with the intention of furnishing a reference from which one may obtain the wave functions with a minimum expenditure of time and labor that the following formulae are given.

For the sake of simplicity we adopt a system of units in which energy is measured in mc^2, length in $\hbar/mc$ and momentum in mc. The symbols which occur below have the following meanings:

$\psi_1\psi_2\psi_3\psi_4$, the four components of the wave function.
f_l and g_l, radial wave functions.
$Y_{l\mu}$, normalized spherical harmonics (see Eq. (2)).
j, total angular momentum quantum number.
m, magnetic quantum number.
l, auxiliary index characterizing the wave functions. (l is the orbital momentum for the electron *only*, in the nonrelativistic limit.)
W, *absolute* value of the energy.
p, *absolute* value of the momentum $=(W^2-1)^{\frac{1}{2}}$.
α, fine structure constant $=e^2/\hbar c$.
Z, nuclear charge.

The wave functions are of two types:
Type a, $j=l+\frac{1}{2}$,

$$\begin{aligned}
\psi_1&=i\left(\frac{l-m+\frac{3}{2}}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,\,m-\frac{1}{2}}f_l,\\
\psi_2&=i\left(\frac{l+m+\frac{3}{2}}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,\,m+\frac{1}{2}}f_l,\\
\psi_3&=\left(\frac{l+m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m-\frac{1}{2}}g_l,\\
\psi_4&=-\left(\frac{l-m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m+\frac{1}{2}}g_l,\\
&l\geqslant 0,\quad -(l+1)\leqslant m-\tfrac{1}{2}\leqslant l-1.
\end{aligned} \quad (1a)$$

Type b, $j=l-\frac{1}{2}$,

$$\begin{aligned}
\psi_1&=i\left(\frac{l+m-\frac{1}{2}}{2l-1}\right)^{\frac{1}{2}}Y_{l-1,\,m-\frac{1}{2}}f_{-l-1},\\
\psi_2&=-i\left(\frac{l-m-\frac{1}{2}}{2l-1}\right)^{\frac{1}{2}}Y_{l-1,\,m+\frac{1}{2}}f_{-l-1},\\
\psi_3&=\left(\frac{l-m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m-\frac{1}{2}}g_{-l-1},\\
\psi_4&=\left(\frac{l+m+\frac{1}{2}}{2l+1}\right)^{\frac{1}{2}}Y_{l,\,m+\frac{1}{2}}g_{-l-1},\\
&l\geqslant 1,\quad -l\leqslant m-\tfrac{1}{2}\leqslant l-1,
\end{aligned} \quad (1b)$$

$$Y_{l\mu}(\vartheta,\varphi)=\left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}}\left[\frac{(l-\mu)!}{(l+\mu)!}\right]^{\frac{1}{2}}\frac{\sin^{\mu}\vartheta e^{i\mu\varphi}}{2^l l!}$$
$$\times\left(\frac{d}{d\cos\vartheta}\right)^{l+\mu}(\cos^2\vartheta-1)^l. \quad (2)$$

It is convenient to treat the two types of solutions simultaneously by the introduction of the parameter κ.

$$\begin{aligned}\kappa&=-(j+\tfrac{1}{2})=-(l+1) &&\text{when } j=l+\tfrac{1}{2},\\ &=j+\tfrac{1}{2}=l &&\text{when } j=l-\tfrac{1}{2}.\end{aligned}$$

We give the solutions for the radial functions in the positive energy spectrum (electron). Suppressing the index κ,

$$f=(1-W)^{\frac{1}{2}}(G-G^*),\quad g=(1+W)^{\frac{1}{2}}(G+G^*),\quad (3)$$

so that f and g are real.[1] The solutions given below are normalized per unit energy interval.

Series representation:

$$r\begin{Bmatrix}f\\g\end{Bmatrix}=\frac{(1\mp W)^{\frac{1}{2}}(2pr)^{\gamma}e^{\pi\alpha ZW/2p}|\Gamma(\gamma+i\alpha ZW/p)|}{2(\pi p)^{\frac{1}{2}}\Gamma(2\gamma+1)}$$
$$\times\{e^{-ipr+i\eta}(\gamma+i\alpha ZW/p)F(\gamma+1+i\alpha ZW/p,$$
$$2\gamma+1;2ipr)\mp\text{c.c.}\},\quad (4)$$

where c.c. denotes the complex conjugate.

$$\gamma=(\kappa^2-\alpha^2Z^2)^{\frac{1}{2}},$$
$$e^{2i\eta}=-(\kappa-i\alpha Z/p)/(\gamma+i\alpha ZW/p),$$

and F is the confluent hypergeometric function defined by

$$F(a,b;x)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)}{\Gamma(b+n)}\frac{x^n}{n!}.$$

Integral representation:

$$r\begin{Bmatrix}f\\g\end{Bmatrix}=\frac{(1\mp W)^{\frac{1}{2}}e^{\pi\alpha ZW/2p}(pr/2)^{\gamma}}{2(\pi p)^{\frac{1}{2}}|\Gamma(\gamma+i\alpha ZW/p)|}$$
$$\times\left\{e^{i\eta}\int_{-1}^{1}e^{ipru}(1-u)^{\gamma-1-i\alpha ZW/p}\right.$$
$$\left.\times(1+u)^{\gamma+i\alpha ZW/p}du\mp\text{c.c.}\right\}.\quad (5)$$

Asymptotic behavior:

$$\begin{aligned}rf&=-[(W-1)/\pi p]^{\frac{1}{2}}\sin(pr+\delta),\\ rg&=[(W+1)/\pi p]^{\frac{1}{2}}\cos(pr+\delta),\\ \delta&=(\alpha ZW/p)\log 2pr-\arg\Gamma(\gamma+i\alpha ZW/p)\\ &\quad+\eta-\pi\gamma/2.\end{aligned}\quad (6)$$

It is, of course, simple to obtain the solutions with any other normalization. For example, to obtain the solutions normalized to one particle in a large sphere of radius R the above solutions need only be multiplied by a factor $(\pi p/WR)^{\frac{1}{2}}$.

The dimensionality of rf (and rg) is (energy length)$^{-\frac{1}{2}}$ for normalization per unit energy interval and (length)$^{-\frac{1}{2}}$ for normalization to one particle in a large sphere. Hence to write the solution in ordinary units, besides replacing W by W/mc^2, p by p/mc and pr by $pr/\hbar$, rf (and rg) should be multiplied by $(\hbar c)^{-\frac{1}{2}}$ and $(mc/\hbar)^{\frac{1}{2}}$ for the two methods of normalization, respectively. The transition to any other system of units is easy. For example, to write the above formulae in atomic (Hartree) units W should be replaced by α^2W, p by αp, pr remains unchanged and rf is to be multiplied by $\alpha^{\frac{1}{2}}$ and $\alpha^{-\frac{1}{2}}$ for the two cases of normalization, respectively.

To obtain the solutions in the negative energy spectrum (positron) we have[1]

$$f=(1+W)^{\frac{1}{2}}(G+G^*),\quad g=(1-W)^{\frac{1}{2}}(G-G^*),\quad (7)$$

in which G and G^* are obtained from the formulae given above by changing the sign of W and $e^{2i\eta}$. Thus, for example, in the asymptotic expressions (6) the forms of f and g are interchanged.

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[1] The relative sign of f and g as determined from the differential equations which they must satisfy is obtained correctly if we take $(1-W)^{1/2}=+i(W-1)^{1/2}$.