

Resonance Effects in Nuclear Processes

H. A. BETHE

Cornell University, Ithaca, New York

AND

G. PLACZEK

Universitetets Institut for teoretisk Fysik, Copenhagen, Denmark

(Received November 9, 1936)

A general formula is developed for the probability of nuclear processes with particular consideration of resonance (§2). The dependence of the cross section on the energy of the incident particle can be divided into two parts: Firstly, the dependence over energy regions small compared to nuclear energies, and secondly that over large energy regions, of the order of a million volts or more. The first dependence is completely given by the resonance formula; it shows resonance maxima and besides a simple general trend with the particle energy such as the $1/v$ law. The dependence over large energy regions cannot be found without referring to a special nuclear model. (If the problem of nuclei were a one-body rather than a many-body problem, there would be only the dependence over large energy regions. Thus much more theoretical information of a general nature can be obtained for the many-body than for the one-body problem.) The nuclear processes may be divided into several classes according to whether light quanta or material particles are concerned. The selection rules for the various kinds of processes are given (§3). Another useful classification is according to the speed of the particles involved: Slow particles are such whose wave-length is long compared to nuclear dimensions. This means energies below about 300,000 volts for heavy, 1 MV for light nuclei. γ -rays are to be classed as fast particles. When a slow particle *produces* a nuclear reaction, the cross section contains a *factor* $1/v$ (v =velocity of the incident particle) besides the resonance factor; when a slow particle *is produced*, a *factor* v' (v' =velocity of the outgoing particle) appears in the cross section. If the reaction involves only fast particles, the resonance factor is the only significant one; the same is true for the scattering of slow particles. Explicit formulae for the various cases are given. The problem of the wave functions to be chosen for the incident particle is discussed in §4. Arguments are given for using

wave functions in a *repulsive* potential, corresponding to the assumption that the particle as a free particle cannot exist inside the nucleus. The "potential scattering" arising as a consequence of this assumption, is discussed and compared to the resonance scattering. In §§5 to 7 the capture of slow neutrons is discussed. The influence of the Doppler effect on the capture cross section is taken into account. Expressions are derived for the activation and for the absorption coefficient with self-indication, both for resonance and for thermal neutrons. These expressions allow for the influence of the line shape in the former case and for the $1/v$ law in the latter. Methods for the determination of the energy, radiation width and neutron width of the compound levels are discussed (§6) and applied (§7) to Ag, Rh, I and Cd. The importance of the interference of several resonance levels is emphasized, particularly for the capture of thermal neutrons. The properties of fast neutrons are briefly discussed (§8). In the case of charged particles (§10), the width of the resonance levels is reduced by the potential barrier. The width of resonance levels observed in the simple capture of protons is found in agreement with reasonable expectations. The widths of the levels in reactions produced by α -particles are probably smaller than has been observed. In the reaction of charged particles with heavy nuclei, no resonance effects can be observed because the energy of the incident particles cannot be defined accurately enough. The photodissociation of nuclei by γ -rays (§11) is not the inverse process of the radiative capture of particles. The cross section for the photodissociation of a heavy nucleus is about 10^{-28} cm² if the energy of the particle produced (neutron) is larger than about 1 MV. This should make the process just observable. The scattering of γ -rays by heavy nuclei has a cross section of the same order which makes it unobservably small compared to the Klein-Nishina scattering.

I. GENERAL THEORY

§1. Introduction

EVEN in the early experiments on the interaction of α -particles and protons with nuclei it was found that scattering and disintegration showed maxima at certain energies which were ascribed to a resonance phenomenon. With improved technique, it was shown that many of these resonances were much sharper

than was at first believed. Still more striking resonance effects have been revealed by the more recent experiments with slow neutrons.

The α -particle and proton resonances were formerly interpreted from a one-particle viewpoint, the incident particle being considered as moving in the potential field of the nucleus and as having certain virtual energy levels in that field. However, this picture has proved quite untenable in view of the neutron evidence. In

connection with this evidence, Bohr¹ has emphasized that the problem of nuclear dynamics is essentially a many-body problem and has shown how the experimental results in nuclear physics can be understood from this standpoint.

In particular, the theory of resonance effects is altered in the following way: In the one-particle picture, the distance between adjacent energy levels will be of the order of a few hundred thousand volts, i.e., smaller than, but still comparable to, nuclear binding energies. The width of these resonance levels would be of the same order as their distance apart.²

In the many-particle picture the distance between levels decreases extremely rapidly with increasing excitation energy and increasing number of particles in the nucleus.¹ At the same time, the levels are very much sharper than in the one-particle picture. Both spacing and width of the levels are negligibly small compared to nuclear binding energies except for very light nuclei.

This fact causes interesting variations of the probability of nuclear processes over energy regions small compared to nuclear binding energies about which general theoretical information may be obtained. In the one-particle picture, because of the large spacing of the resonance levels the dependence over *small* energy intervals is governed only by trivial factors such as the $1/v$ law for the capture of neutrons.³ Over *large* energy intervals, the probability depends on the particular model for the process considered.⁴

It is the purpose of this paper to develop the theory of the variations over small energy

regions. The formulae obtained are similar to those of Breit and Wigner.⁵

§2. The generalized dispersion formula

The problem of nuclear dynamics is only slightly more general than the ordinary problem of scattering of light. A nuclear process may generally be described as follows: A particle P (e.g., a light quantum, α -particle, proton, neutron, etc.) falls on an initial nucleus A in its ground state. A compound nucleus C is formed. This nucleus then emits a particle Q which may be either of the same or a different kind from the incident particle P . In this process, a residual nucleus B is left which may or may not be in an excited state.

The compound nucleus C possesses a large number of energy levels E_r . It may be considered in a Hohlraum which contains particles of various sorts Q' , all of which can interact with the nucleus C . The Hamiltonian $H_{Q'}$ of the interaction depends on the coordinates and possibly on the momenta of particle Q' and on the internal state of the nucleus. Because of this interaction, the nucleus C may emit particles or be formed by the absorption of a particle. The combination of absorption and emission leads to the transformation $A + P \rightarrow C \rightarrow B + Q$ described above. This transformation can occur for any energy of the incident particle P whether $E_A + E_P$ is equal to one of the levels E_r of the compound nucleus or not. This is the case because the levels E_r are broadened by the interaction with the particle-Hohlraum.

According to the foregoing, the problem can be treated by the well-known methods of the Dirac radiation theory. Following the customary procedure, we consider only the second-order perturbation, i.e., the lowest order in which the process $A + P \rightarrow C \rightarrow B + Q$ can occur, and neglect higher orders. This is justified if and only if the widths of all energy levels of nucleus C is small compared to the spacing of the levels.^{6, 6a} We

¹ Bohr, Nature **137**, 344 (1936). See also, Bohr and Kalckar, "On the Mechanism of Nuclear Reactions," Kgl. Dansk Vidensk. Selsk. Math. Phys. Medd. (1937) where a quantum-mechanical formulation of the general viewpoint developed in the former paper is given, as well as a detailed discussion of a number of typical nuclear reactions from this point of view.

² The width would be smaller for charged particles of "medium" energy, i.e., such energy that the penetrability of the Coulomb potential barrier is small compared to unity but still large enough to make the disintegration observable. We shall come back to this case in §10.

³ Amaldi, d'Agostino, Fermi, Pontecorvo, Rasetti and Segrè, Proc. Roy. Soc. **A149**, 522 (1935); Bethe, Phys. Rev. **47**, 747 (1935); Perrin and Elsassser, Comptes rendus **200**, 450 (1935).

⁴ Cf., eg., Ostrofsky, Breit and Johnson, Phys. Rev. **49**, 22 (1936).

⁵ Breit and Wigner, Phys. Rev. **49**, 519 (1936).

⁶ It might be thought that such a condition would lead to difficulties in the case of the ordinary theory of scattering of light by atoms when the atom has a continuous spectrum. That this is not the case, may be shown by considering, in the usual way, the continuous spectrum as the limiting case of a discrete spectrum, e.g., by considering the atom as enclosed in a Hohlraum of volume Ω . Then the spacing between the atomic energy levels will be inversely propor-

shall show in §4, that this condition is probably fulfilled for our problem but not very well fulfilled for highly excited states of the compound nucleus.

For the present, we assume the perturbation theory to be valid. Then the Dirac radiation theory yields for the probability of the process considered

$$w^{P_{Q'q}} = \frac{2\pi}{\hbar} \left| \sum_r \frac{H^P_{r,p} H^r_{Q'q}}{E_A + E_P - E_r + i\pi \sum_{Q'q'} |H^r_{Q'q'}|^2} \right|^2. \quad (1)$$

$H^P_{r,p}$ is the matrix element of the interaction between particle P and the nucleus referring to the state r of the nucleus C and to the given state p of the incident particle P . The sum over Q' extends over all kinds of particles which can be emitted by the compound nucleus. The index q' distinguishes various kinetic energies of the outgoing particles Q' which correspond to various excited states of the residual nucleus B . E_A is the energy of the initial nucleus A in the ground state, E_P the kinetic energy of the incident particle and E_r the energy of the state of the compound nucleus considered. In all the matrix elements $H^Q_{r,q}$, $H^{Q'}_{r,q'}$ the energy of the particle is determined by the requirement of conservation of energy, e.g., the kinetic energy of particle Q' must be taken as $E_{Q'}(q') = E_A + E_P - E_{B'}(q')$ where $E_{B'}(q')$ is the energy of the nucleus B' in the state in which it is left after the emission of particle Q' . All particle wave functions are normalized per unit energy. The sum over $Q'q'$ in the denominator of (1) represents the total probability of disintegration of the nuclear level r , multiplied by \hbar . We shall denote this expression by γ_r , and may consider it as composed of the contributions of the disintegrations with emission of the various sorts of particles Q' , viz.

tional to Ω . But the same will be true for the radiative width of the levels. This can be seen by using the fact that an electron will only radiate if it comes near the atom, the probability for which is inversely proportional to Ω . The ratio of width to spacing is independent of Ω , and very small compared to unity, viz. of the order $e^2\omega/mc^3$ for radiation of the frequency ω .

^{6a} This condition is also necessary in order to make the width of each level independent of the simultaneous excitation of other levels. For a more detailed discussion, cf. Bethe and Livingston, Rev. Mod. Phys., to appear shortly.

$$\gamma_r = \sum_{Q'} \gamma^{r_{Q'}}, \quad (2)$$

$$\gamma^{r_{Q'}} = \sum_{q'} \gamma^{r_{Q'q'}}, \quad (2a)$$

$$\gamma^{r_{Q'q'}} = 2\pi |H^r_{Q'q'}|^2. \quad (2b)$$

$\gamma^{r_{Q'q'}}$ is the probability, per unit time, that nucleus C in state r disintegrates into a particle of kind Q' and a residual nucleus B' in the state defined by q' .

Because of the normalization of the particle wave functions per unit energy, the matrix elements $H^r_{Q'q}$ have the dimension (energy)^{3/2}, and therefore the γ 's in (2) have the dimension of an energy as it should be. The total expression (1) has the dimension energy⁻¹ time⁻¹; it represents the probability that the process $A + P \rightarrow C \rightarrow B + Q$ is caused by a particle P in any of the Hohlraum states of energy around E_P , times the number of Hohlraum states per unit energy. The advantages of the normalization per unit energy are, firstly, that the volume of the Hohlraum does not appear in the formulae, only to drop out in the final result, and secondly, that the disintegration probability of a nuclear state is directly given by the squares of the matrix elements H , without any factors except 2π .

In order to obtain the *cross section* of our process from (1), we have to change from normalization per unit energy to normalization per unit incident current. A plane wave, normalized per unit incident current, has the form

$$\psi_{\text{curr}} = v^{-1/2} e^{ik \cdot r}, \quad (3)$$

where k is the "wave vector" ($k = \text{momentum}/\hbar$) and v the velocity. For normalization per unit energy we have

$$\psi_E = \left(\frac{4\pi k^2 dk}{(2\pi)^3 dE} \right)^{1/2} e^{ik \cdot r}. \quad (4)$$

Using the well-known formula

$$v = dE/\hbar dk \quad (5)$$

we find $\psi_{\text{curr}} = \psi_E (2\pi^2 \lambda^2 \hbar)^{1/2}$, (6)

where $\lambda = 1/k$ (6a)

is the wave-length of the incident particle, divided by 2π . Therefore the cross section is

$$\sigma^P Q^p q = 2\pi^2 \hbar \lambda^2 w^P Q^p q \quad (7)$$

$$= 4\pi^3 \lambda^2 \left| \sum_r \frac{H^P r^p H^r Q q}{E_A + E_P - E_r + \frac{1}{2} i \gamma_r} \right|^2. \quad (8)$$

In general, the levels of initial, compound and final nucleus will be degenerate. This fact is considered in detail in the appendix. The result is given in (217). If we transform from the probability to the cross section according to (8), we find

$$\sigma^P Q^p q = \frac{\pi \lambda^2}{(2s+1)(2i+1)} \sum_{li'j'} (2J+1) \times \left| \sum_r \frac{u^r_{A^J p i P l j} u^r_{B^J q i' Q l' j'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2, \quad (9)$$

where l , s and j are orbital momentum, spin⁷ and total angular momentum of the incident particle; l' , s' and j' the same quantities for the outgoing particle; i , J and i' the angular momenta of initial, compound and final nucleus; and $p r q$ denote all other quantum numbers of the three nuclei. The letters A , B , P , Q defining the various nuclei and particles precede the indices determining the state of the respective particle. $(u^r_{B^J q i' Q l' j'})^2$ is the sum of $2\pi |H^r_{B^J q^M i' m' Q l' j' \mu'}|^2$ over the various magnetic substates $m' \mu'$ of the degenerate final state q , the magnetic substate M of rJ being kept fixed. u is directly connected with the γ by

$$\gamma^r_{A^J p i P l j} = (u^r_{A^J p i P l j})^2. \quad (10)$$

γ_{rJ} is the total disintegration probability of the level rJ

$$\gamma_{rJ} = \sum_{Q'} \gamma^r_{Q'^J} = \sum_{Q' q' i' l' j'} \gamma^r_{B'^J q' i' Q' l' j'}. \quad (11)$$

Formula (9) simplifies considerably if only *one* level rJ of the compound nucleus contributes to the cross section, which will be the case if the energy of the incident particle is sufficiently near the resonance energy $E_{rJ} - E_A$. Then the sum over r reduces to a single term which may be written in the form

⁷ For light quanta, $2s+1=2$ because of the two directions of polarization, but $s=0$ for selection rules (cf. end of appendix). $l=1, 2, \dots$ for dipole, quadrupole \dots radiation.

$$\sigma^P Q^p q = \pi \lambda^2 \frac{2J+1}{(2s+1)(2i+1)} \times \frac{\gamma^r_{P^J p \gamma^r_{Q^J q}}}{(E_A + E_P - E_{rJ})^2 + \frac{1}{4} \gamma^2_{rJ}} \quad (12)$$

$$\text{with } \gamma^r_{P^J p} = \sum_{l_i} \gamma^r_{A^J p i P l i} = \sum_{l_i} (u^r_{A^J p i P l i})^2. \quad (13)$$

Formula (12) is known as the "one level formula"; it agrees with the formula of Breit and Wigner except for the slightly more complicated statistical weight factor in front, and it is the formula most used in applications.

As already mentioned, the matrix elements $H^r_{B^J q i' Q l' j'}$ and therefore $u^r_{B^J q i' Q l' j'}$ and $\gamma^r_{B^J q i' Q l' j'}$ have to be taken for *that* kinetic energy of the particle Q' for which energy is conserved, *viz.*,⁸

$$E_{Q'}(q') = E_A + E_P - E_{B'}(q'). \quad (14)$$

The γ 's will thus be functions of the energy of the incident particle, E_P . If we would insert, instead of the actual particle energy E_P , the energy corresponding to exact resonance with the state r of the compound nucleus, i.e., $E_{rJ} - E_A$, the probability γ_{rJ} would go over into the width Γ_{rJ} of the level rJ . The γ 's may therefore be called "effective widths."

The dependence of the γ 's and the u 's on the energy comes from the normalization of the particle wave function. A wave function of a free particle of orbital momentum l , normalized per unit energy, has the form⁹

$$\psi_P = \left(\frac{2}{\pi} \frac{dk}{dE} \right)^{\frac{1}{2}} \frac{1}{r} \chi_l(kr) Y_{lm}(\vartheta, \varphi). \quad (15)$$

Here k is the wave number of the particle,

$$E = (\hbar^2/2m)k^2, \quad (16)$$

its energy, Y_{lm} a normalized spherical harmonic and χ_l *that* solution of the radial wave equation

$$\frac{d^2 \chi_l}{dr^2} + \left(k^2 - \frac{l(l+1)}{r^2} \right) \chi_l = 0, \quad (17)$$

⁸ If the nucleus B' is left in an excited state, $E_{Q'}$ is defined only to an accuracy of the order of the width of the level $E_{B'}$. In this case, $u^r_{Q'^J q'}$ and the term $\gamma^r_{Q'^J q'}$ in γ_{rJ} must be averaged over this width. This is, however, of no practical importance.

⁹ *Handbuch der Physik* 24/1, p. 292. For our purpose the Coulomb field should be neglected.

which behaves asymptotically as $\sin(kr - \frac{1}{2}l\pi)$. At small r ,

$$\chi_l = \frac{1}{1^2 \cdot 3^2 \cdot \dots \cdot (2l-1)^2 (2l+1)} (kr)^{l+1}. \quad (18)$$

If the wave-length is large compared to nuclear dimensions, the matrix element $\int \psi_A \psi_P H \psi_C d\tau$ contains the wave function ψ_P only for small values of its argument kr . Therefore the matrix element depends on k as

$$(dk/dE)^{\frac{1}{2}} k^{l+1}. \quad (19)$$

This remains true even if one does not use free particle wave functions but rather the wave functions for an attractive or repulsive field of nuclear dimensions. As we shall show in §4, a repulsive field seems to be the most satisfactory assumption. For the present, we assume an arbitrary field, attractive or repulsive, to act on the particle for $r < R$ (R = nuclear radius) while the potential is zero for $r > R$. Then the logarithmic derivative of the wave function χ'/χ will have a certain value κ for $r=R$ which depends on the potential but not sensitively on the kinetic energy of the particle as long as this energy is small compared to the potential. Outside, i.e., for $r > R$, χ will again be a solution of the free particle wave Eq. (17) but in general not the "regular" solution (18) but a linear combination of the regular and the irregular solution. The latter behaves for large r 's as $\cos(kr - \frac{1}{2}l\pi)$ and for small values of r as

$$\varphi_l = 1^2 \cdot 3^2 \cdot \dots \cdot (2l-1)^2 (kr)^{-l}. \quad (20)$$

The correct solution which joins smoothly to the interior solution, will have the asymptotic form

$$\sin(kr - \frac{1}{2}l\pi + \delta) = \cos \delta \sin(kr - \frac{1}{2}l\pi) + \sin \delta \cos(kr - \frac{1}{2}l\pi)$$

and therefore the general form

$$\cos \delta \chi_l + \sin \delta \varphi_l = \cos \delta (\chi_l + \tan \delta \varphi_l). \quad (21)$$

Therefore we have for $r=R$

$$\chi'/\chi = (\chi'_l + \tan \delta \varphi'_l) / (\chi_l + \tan \delta \varphi_l) = \kappa \quad (22)$$

because κ is the value of χ'/χ for the interior wave function. With (18) and (20), we find from (22)

$$\tan \delta = \frac{\chi'_l - \kappa \chi_l}{\kappa \varphi_l - \varphi'_l} = \frac{1}{1^4 \cdot 3^4 \cdot \dots \cdot (2l-1)^4 (2l+1)} \times (kR)^{2l+1} \frac{l+1 - \kappa R}{l + \kappa R} = a(kR)^{2l+1}, \quad (23)$$

where a is a factor of order unity which depends on the special form of the potential acting on the particle but not on k . Since kR is supposed to be small compared to unity, δ will in general be very small. Then $\cos \delta \approx 1$ so that we have from (21), (18) and (20):

$$\chi(R) = \chi_l + a(kR)^{2l+1} \varphi_l = \frac{(kR)^{l+1}}{1^2 \cdot 3^2 \cdot \dots \cdot (2l-1)^2 (l + \kappa R)}. \quad (24)$$

This depends on the energy as k^{l+1} just as without any potential acting on the particle.

Since dE/dk is proportional to k (cf. 16), the matrix element is, according to (19), proportional to $k^{l+\frac{1}{2}}$. Thus we may write

$$u^r_{q^J q} = b^r_{q^J q} k^{l+\frac{1}{2}} = b^r_{q^J q} \lambda^{-(l+\frac{1}{2})} \quad (25)$$

and, because of (13)

$$\gamma^r_{q^J q} = (b^r_{q^J q})^2 \lambda^{-(2l+1)}. \quad (25a)$$

However, if the wave-length is long, only the partial wave $l=0$ is of any importance. Therefore, for long particle wave-length, the important u 's behave as $k^{\frac{1}{2}}$ which is proportional to $v^{\frac{1}{2}}$ so that

$$u^r_{q^J q} = b^r_{q^J q} \lambda^{-\frac{1}{2}} = a^r_{q^J q} v^{\frac{1}{2}} = c^r_{q^J q} E^{\frac{1}{4}}, \quad (26)$$

where abc are constants independent of the energy of the particle.

For fast particles, the dependence of u on the energy is not quite so simple because formula (18) for χ_l can no longer be used throughout the nucleus. Moreover, orbital momenta larger than zero will become important. Formula (26) can, however, be safely applied for neutrons and protons if the energy does not exceed about 300,000 volts for interaction with heavy nuclei, and about 1 MV for light nuclei. For α -particles, the limits are one-quarter of these figures.

The nonapplicability of (26) for fast particles does not cause any difficulties for our purposes. If the energy of the incident particle is as large as a few hundred thousand volts or more, we restrict ourselves to variations of the energy small compared to the energy of the incident particle itself (cf. the program outlined in §1). Then the variation of $k^{l+\frac{1}{2}}$ in (25) is negligible, and although u is no longer given by (25), the order of magnitude of its variation with energy will be the same and will therefore also be negligible. We may then replace the u by its value for exact resonance, $U^r_{p^J p}$. This is equivalent to the replacement of the effective width γ by the "true width" Γ discussed above, since by definition $u^2 = \gamma$, $U^2 = \Gamma$.

Alternatively,¹⁰ we may still retain formally the relations (25) (25a) with the only difference that b is now no longer exactly constant but varies slowly with the energy. This variation is again outside the scope of our considerations

¹⁰ This alternative is useful in order to write the same formulae for fast and slow particles.

because it is only appreciable over energy regions of the order of nuclear energies.

The restriction to variations of the energy small compared to a few hundred thousand volts, is quite irrelevant for heavy nuclei, since in this case the compound nucleus possesses very narrowly spaced energy levels in the energy region considered, i.e., about 10 MV above its ground state.¹ For light nuclei, cases may arise where the spacing between the resonance levels is more than a few hundred thousand volts, in this case no simple dependence of the cross section on the energy for fast particles can be deduced from our considerations.

The true widths Γ and the U 's may also be introduced for slow particles. In this case, since $l=0$, we have from (26)

$$u^r_{Q^J_q} = U^r_{Q^J_q} (\lambda^r_{Q^J_q} / \lambda)^{\frac{1}{2}} = U^r_{Q^J_q} (E / E^r_{Q^J_q})^{\frac{1}{2}}, \quad (27a)$$

$$\gamma^r_{Q^J_q} = \Gamma^r_{Q^J_q} (\lambda^r_{Q^J_q} / \lambda) = \Gamma^r_{Q^J_q} (E / E^r_{Q^J_q})^{\frac{1}{2}}, \quad (27b)$$

where $\lambda^r_{Q^J_q}$ and $E^r_{Q^J_q}$ are the wave-length and energy corresponding to exact resonance.

These formulae are directly applicable only if the energy of particle Q corresponding to exact resonance, $E^r_{Q^J_q} = E_{rJ} - E_B(q)$ is positive and not too large. If the resonance energy is negative, the true width is zero.¹¹ However, a quantity $\Gamma^r_{Q^J_q}$ may be defined such that (27b) holds, and this Γ has *cet. par.* the same magnitude as if the resonance energy were $|E_Q|$ instead of $-|E_Q|$. For $\lambda^r_{Q^J_q}$ we have also to insert the wave-length for the energy $|E_Q|$. The same argument holds for U .

We shall now consider the case of light quanta. For dipole radiation, the γ 's have the familiar form

$$\gamma^r_{Q^J_q} = \lambda^{-3} (b^r_{Q^J_q})^2, \quad (28)$$

where the magnitude of the b 's is related to the matrix elements of the electric moment vector X by

$$(b^r_{Q^J_q})^2 = \frac{2}{3} |X^r_{Q^J_q}|^2, \quad (29)$$

while the sign of b may be found from the

¹¹ This is not strictly correct. It is true that a negative energy level of nucleus C would have exactly zero width if only particles of kind Q could be emitted. However, in reality there is always a finite probability of emission of particles other than Q , at least the emission of γ -rays is possible in any case. This possibility broadens the negative energy level of the nucleus, and the broadened level may extend into the region of positive energies. For this reason, also the emission of a particle of kind Q by the negative level becomes possible so that the emission of Q may give some contribution to the true width of the negative level. However, this contribution has no relation to the $\Gamma^r_{Q^J_q}$ used in (27b.)

explicit evaluation of the matrix elements of the electric moment.¹² The u 's may be expressed by the b 's

$$u^r_{Q^J_q} = b^r_{Q^J_q} \lambda^{-\frac{3}{2}}. \quad (30)$$

As in (27), we may express u and γ by the quantities U and Γ for exact resonance, *viz.*

$$\begin{aligned} u^r_{Q^J_q} &= U^r_{Q^J_q} (\lambda^r_{Q^J_q} / \lambda)^{\frac{3}{2}}, \\ \gamma^r_{Q^J_q} &= \Gamma^r_{Q^J_q} (\lambda^r_{Q^J_q} / \lambda)^3. \end{aligned} \quad (31)$$

For quadrupole radiation, similar formulae hold; only the exponent of λ is $5/2$ instead of $3/2$.

To derive these formulae, we may again use (19) but we must consider that for light dE/dk is independent of k , *viz.*, equal to hc . Moreover, the interaction with particles is not proportional to the density of the light quanta but to the electric field strength \mathbf{E} . For unit density of light quanta we have $\mathbf{E}^2/8\pi = h\nu$; therefore $\mathbf{E} \propto \nu^{\frac{1}{2}} \propto k^{\frac{1}{2}}$, which introduces another factor $k^{\frac{1}{2}}$ in the matrix element. The matrix element is therefore proportional to $k^{l+\frac{1}{2}} = \lambda^{-(l+\frac{1}{2})}$, which is $\lambda^{-\frac{3}{2}}$ for dipole radiation ($l=0$) and $\lambda^{-5/2}$ for quadrupole radiation. We may *formally* retain the same formula for light as for particles if we arbitrarily determine that, for light quanta, l shall denote the total momentum j rather than the orbital momentum; since the total momentum is one unit larger than the orbital momentum (cf. appendix, end; j is 1, 2, \dots for dipole, quadrupole \dots radiation), we thus come back to our previous formulae. This is, however, entirely accidental. The dependence of u and γ on the wave-length of the γ -rays is of no practical importance, for the same reasons as discussed for fast particles.

If we insert (25) and (30) into (9) we find for the cross section corresponding to the transition from a definite l to a definite l' :

$$\begin{aligned} \sigma^P_{Q^J_q}{}^{l'l'} &= \frac{\pi}{(2s+1)(2i+1)} \lambda^{1-2l} \lambda'^{-1-2l'} \sum_{JjJ'} (2J+1) \\ &\times \left| \sum_{\tau} \frac{b^r_{A^J_p i P l j} b^r_{B^J_q i' Q l' j'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2, \quad (32) \\ b^r_{A^J_p \dots} &= U^r_{A^J_p \dots} (\lambda^r_{A^J_p})^{l+\frac{1}{2}}, \\ b^r_{B^J_q \dots} &= U^r_{B^J_q \dots} (\lambda^r_{B^J_q})^{l'+\frac{1}{2}}, \end{aligned} \quad (32a)$$

¹² Cf. Placzek, *Handbuch der Radiologie*, second edition, Vol. 6/2, p. 245, formula (27), or appendix of this article.

where λ and λ' are the wave-lengths of incident and outgoing particle, l and l' the corresponding orbital momenta and s, s' the spins. i, J and i' are the angular momenta of initial, compound and final nucleus. b is defined in (25) and (30) and γ in (11). The b 's are exact constants if the wave-length of the particle is long compared to nuclear dimensions, while for short wave-length they are slowly varying functions of the energy, the variation being irrelevant for our considerations (see above). In the case of long wave-length, only $l=0$ is important for particles and $l=1$ for light. If only one level of the compound nucleus is important, (32) reduces similarly to (9), to the "one-level formula"

$$\sigma^P Q^p q^{l'l'} = \frac{\pi}{(2s+1)(2i+1)} \frac{(\lambda^r P^J)^{2i+1}}{\lambda^{2i-1}} \times \left(\frac{\lambda^r Q^J q^J}{\lambda'} \right)^{2i'+1} \frac{\Gamma^r P^J_{pl} \Gamma^r Q^J_{q'l'}}{(E_A + E_P - E_{rJ})^2 + \frac{1}{4} \gamma^2_{rJ}}, \quad (33)$$

$$\text{where } \Gamma^r P^J_{pl} = \sum_i (U^r_{A^J_{piPl_i}})^2 \quad (33a)$$

is that part of the true width of the level rJ which corresponds to the emission of particles of sort P with orbital momentum l such that the residual nucleus A is left in state p .

Formulae (32), (33) solve the problem stated at the beginning of this section. They express the cross section of nuclear processes in terms of constants (or, in the case of fast particles, quantities varying only over energy regions of the order of nuclear energies) and of simple functions of the energy of the incident particle.

§3. The processes of nuclear dynamics

For the discussion of (32), (33), we may systematize the nuclear processes according to two schemes. Firstly, we may distinguish according to the particles involved, and secondly, between fast and slow particles.

According to the particles involved, we have four groups of processes:

- (1) Incident: light quantum, outgoing: light quantum. Scattering of γ -rays.
- (2) Incident: light quantum, outgoing: corpuscle.¹³ Photodissociation.

¹³ We denote by "corpuscle" a material particle, while the term "particle" shall, in this section, include light quanta.

- (3) Incident: corpuscle, outgoing: light quantum. Radiative capture.
- (4) Incident: corpuscle, outgoing: corpuscle.
 - (a) Outgoing corpuscle same kind as incident. Elastic and inelastic scattering.
 - (b) Outgoing corpuscle different from incident. Corpuscle transmutation.

The distinction between the types 1 to 4 of processes is of importance primarily for the selection rules. *The selection rules* for the compound nucleus and either initial or final nucleus are

$$|i-J| \leq j \leq l+s, \quad i+J \geq j \geq |l-s|. \quad (34)$$

The parity (even or odd character) of the states of the two nuclei is the same if l is even, and is different if l is odd.

Between initial and final nucleus, we have the selection rule

$$|i'-i| \leq l+l'+s+s', \quad (35)$$

$i'+i$ must be larger than or equal to the smallest difference between any of the numbers $|l-s| \cdots l+s$ and any of the numbers $|l'-s'| \cdots l'+s'$. The parity remains unchanged if $l-l'$ is even and changes if $l-l'$ is odd.

(35) represents the selection rules between initial and final nucleus, if states of the compound nucleus with *every* J value possible according to (34) contribute to the cross section. If this is not the case, a more stringent rule than (35) applies which may be found directly from (34) considering the contributing states of the compound nucleus.

For radiation, $s=0$ and $l=1$ for dipole, $l=2$ for quadrupole radiation (cf. appendix). For corpuscles, the selection rules are simplified if the corpuscles are slow (λ large compared to nuclear dimension) because then $l=0$.

The more important selection rule is the one determining the J of the compound nucleus. In the case of incident radiation or slow corpuscles, the explicit selection rules are given in Table I.

According to *velocities*, we distinguish between slow and fast particles. We call a particle "slow" if its wave-length is large compared to nuclear dimensions. For corpuscles this means that the energy must be small compared to a few hundred

thousand volts. γ -rays are always to be considered as fast.

A. Incident particle fast, outgoing particle fast.—In this case, the variation of the λ -factors in (32) with energy is negligible, and the λ 's may be replaced by the corresponding resonance λ 's. Introducing the quantity U (cf. (25), (27a)) we have then

$$\sigma^P Q^p q = \frac{\pi}{(2s+1)(2i+1)} \lambda^2 \sum_{i'j'J} (2J+1) \times \left| \sum_r \frac{U^r_{A^J} U^r_{P l s} U^r_{B^J} U^r_{q Q l' s'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (36)$$

This formula applies to the scattering of γ -rays,¹⁴ to the capture and the elastic scattering of fast corpuscles. To the photodissociation, the inelastic scattering of fast corpuscles and the particle transmutation produced by fast corpuscles it applies if the outgoing corpuscle is fast.

TABLE I. Selection rule between initial and compound nucleus.

PARTICLE	J	SPECIAL RULES	PARITY
Light, dipole radiation	$i, i \pm 1$	0→0 forbidden 0→0, 1	changes
Light, quadrupole rad.	$i, i \pm 1, i \pm 2$	0→1, 1, 2 forbidden $\frac{1}{2} \rightarrow \frac{1}{2}$	unchanged
Slow protons or neutrons	$i \pm \frac{1}{2}$	—	unchanged
Slow α -particles	i	—	unchanged
Slow deuterons	$i, i \pm 1$	0→0 forbidden	unchanged

¹⁴ Formula (36) differs from the Kramers-Heisenberg formula in three respects: Firstly, the apparent dependence on λ is different; secondly, it contains for each rJ only a term with the difference of the incident energy $h\nu$ and the resonance energies $E_r - E_A$ while the Kramers-Heisenberg formula contains also another term with denominators of the form $E_r - E_A + h\nu$. Thirdly, in (36) the scattering amplitudes due to states with different J do not interfere. The correct dependence on wave-length is given in the original formula (32). However, as we have mentioned above, this dependence is negligible for our purpose. The terms with $E_r - E_A + h\nu$ have purposely been neglected in (1) because they are of no practical importance in all cases we are interested in. For particles, $E_r - E_A$ and E_P are each of the order Mc^2 where M is the mass of particle P . Therefore the terms with the sum in the denominator are entirely negligible (Besides, even the occurrence of these terms for particles is theoretically doubtful). In the case of γ -rays, the circumstances are not quite so extreme; but we may still neglect the sum terms because in this paper we deal only with processes leading to highly excited states of the compound nucleus. Then $h\nu$ and $E_r - E_A$ are of the order 10 MV. On the other hand, the terms which give the main contributions to (36) contain energy differences of the order of the distance between neighboring energy levels of the compound nucleus which is very small compared to 10 MV. The third difference mentioned above is connected to the second because interference between the scattering amplitudes due to different J 's occurs in the Kramers-Heisenberg formula only because of the presence of the "sum terms."

The cross section (36) shows resonance maxima, which, for heavy nuclei, occur at frequent intervals. Apart from these maxima, there is no general trend of the cross section as a function of energy over regions small compared to nuclear energies, since the variation of λ over such regions is negligible.

The total actual width γ_{rJ} is given by (11). In this formula, we may replace the contributions $\gamma^r_{Q^J q'}$ by their values at exact resonance for all those emitted particles Q' which have high energy. It is, however, possible that the compound nucleus may also be capable of emitting slow particles. This may, e.g., happen if γ -rays are scattered by a nucleus which at the same time may be dissociated by these γ -rays with the emission of a slow corpuscle. The contributions $\gamma^r_{Q^J q'}$ due to these slow particles depend sensitively on the energy. Formula (11) may thus be rewritten

$$\gamma_{rJ}(E_P) = \sum_{\text{fast particles}} \Gamma^r_{Q^J q'} + \sum_{\text{slow particles}} (c^r_{Q^J q'})^2 E_{Q^J q'}, \quad (37)$$

where E_{Q^J} depends on the energy of the incident particle according to (14). If no slow particles are emitted, or if, as is true in most cases, the second sum is negligible compared to the first by virtue of the small factors E_{Q^J} , the total actual width γ_{rJ} may be replaced by the total true width Γ_{rJ} which is independent of E_P . However, it may occur that the coefficients $c^r_{Q^J q'}$ are very large, so that the second sum becomes comparable to the first.

B. Incident particle slow, outgoing particle fast.

—The variation of λ' in (32) with energy is negligible. For the incident particle, $l=0$ and $j=s$. We have then

$$\sigma^P Q^p q = \frac{\pi}{(2s+1)(2i+1)} \lambda \sum_{J l' j'} (2J+1) \times \left| \sum_r \frac{b^r_{P^J} U^r_{B^J} U^r_{q Q l' j'}}{E_A + E_P - E_r + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (38)$$

This formula applies to the radiative capture of slow particles, and to transmutations produced by slow particles. It is important mainly for slow neutron phenomena, and also for the disintegration of very light nuclei by slow protons.

Apart from the resonance maxima, the cross section is proportional to the wave-length λ of the incident particle. For very small energies, there will always be a region in which the factor λ is predominant ("1/ ν law"). How small these energies must be, depends on the spacing and width of the resonance levels. These questions and their application on slow neutrons will be discussed in detail in §5. γ_{rJ} is again given by

(37) and may be replaced by Γ_{rJ} if the contribution of the incident particle (and possibly other slow particles which may be emitted by the compound nucleus) to γ is small compared to the total γ_{rJ} .

C. *Incident particle fast, outgoing particle slow.*
—The cross section is

$$\sigma^P Q^p q = \frac{\pi}{(2s+1)(2i+1)} \frac{\lambda^2}{\lambda' \sum_{Jl_j} (2J+1)} \times \left| \sum_r \frac{U_{rA}^J \rho_{Plj} b^r Q^J q}{E_A + E_P - E_r + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (39)$$

(39) applies to photodissociation with the emission of a slow particle, and to particle transmutations and inelastic scattering if the incident particle is fast and the outgoing one slow. Apart from resonance, the cross section decreases as the velocity of the outgoing particle, i.e., as $(E_P + E_A - E_B)^{\frac{1}{2}}$. About the width γ , see case A.

D. *Incident particle slow, outgoing particle slow.*
—We have

$$\sigma^P Q^p q = \frac{\pi}{(2s+1)(2i+1)} \frac{\lambda}{\lambda' \sum_J (2J+1)} \times \left| \sum_r \frac{b^r P^J p b^r Q^J q}{E_A + E_P - E_r + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (40)$$

This cross section is, apart from resonance, proportional to the ratio of the velocities of outgoing and incident particle. For γ_{rJ} the arguments given in A apply again, because there is, besides the possibility of emission of slow particles, always the possibility of emission of γ -radiation, and in some cases of fast corpuscles.

(40) applies to transmutations in which both the incident and the outgoing particle are slow. By far the most important case is the elastic scattering of slow particles. In this case, the cross section does not depend on λ except for the resonance factor.

There is also some simplification of the formula (40) for the case of elastic scattering of slow particles,¹⁵ because in this case $b^r P^J p = b^r Q^J q$.

¹⁵ A similar simplification of (36) for the elastic scattering of fast particles is not possible because (36) contains b 's referring to different lj and $l'j'$ so that the numerator of (36) does not reduce simply to a square, even for elastic scattering. The only exception is the scattering of γ -rays for which only the term $l=1, j=1$ is important so that a formula similar to (41) holds.

Only the square of b occurs in the cross section while the sign of b does not matter. b^2 may be expressed by Γ according to (25a), (27b), and we have

$$\sigma^P P^p p = \frac{\pi}{(2s+1)(2i+1)} \sum_J (2J+1) \times \left| \sum_r \frac{\Gamma_{rP}^J \lambda^r P^J p}{E_A + E_P - E_r + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (41)$$

It must be noted that all formulae for the cross section refer to a coordinate system in which the *center of gravity* of incident particle and nucleus is *at rest*. To obtain the cross section in the system of the observer we have to multiply by the ratio of the incident currents in the two systems, i.e., by $v_{\text{center of gravity}}/v_{\text{observer}}$. In the case B discussed above, we have therefore to put the factor λ in (38) equal to $\hbar/M_{\text{red}} v_{\text{obs}}$ where M_{red} is the reduced mass and v_{obs} the velocity relative to the observer, while all the other quantities in (38) are to be taken in the center of gravity system. In the particular case of the $1/v$ law, the cross section is therefore independent of the velocity of the initial nucleus. If the $1/v$ law does not hold, the motion of the initial nucleus has an effect on the phenomena with slow incident particles.¹⁶

In concluding this section, we may say a few words about *inverse processes*. For a given process $A+P \rightarrow B+Q$ and its inverse we have the statistical relation

$$g_{BQq} \sigma^B A^Q p^q \lambda_{Qq}^{-2} = g_{APp} \sigma^A B^P Q^p \lambda_{Pp}^{-2}. \quad (42)$$

However, it must be kept in mind that the concept of inverse processes has only limited applications. If we take a given incident particle with given velocity, the outgoing particle Q may, in general, have any of a great number of different energies, corresponding to different states of the resultant nucleus B . The only inverse process which can be produced experimentally, is that starting from nucleus B in the ground state, and the particle Q with the corresponding kinetic energy. This process will, again, lead to many different energies of particle P and different excited states of nucleus A . Therefore only a

¹⁶ A discussion of these effects will be given in a paper by Placzek and Teller, to appear shortly in the Phys. Rev.

small, very often a negligible, part of the experimentally observed cross section in the process $A + P \rightarrow B + Q$, is related by (42) to a part of the cross section in process $B + Q \rightarrow A + P$. Only for light nuclei, where the number of levels is small, inverse processes can be observed. Even here, it is necessary to make sure in every case that the processes compared lead to the ground states of the respective final nuclei.

§4. Particle wave functions and potential scattering¹⁷

We have shown in the preceding sections how the probability of nuclear processes may be reduced to the knowledge of certain matrix elements H or U . We shall now discuss how these matrix elements are to be calculated from the wave functions of the nuclei $A B C$ and of the particle (P or Q).

The matrix element $H^{P,r,p}$ is

$$H^{P,r,p} = \int \psi_{A,p} \psi_{P,p} H \psi_{C,r}^* d\tau, \quad (43)$$

where $\psi_{A,p}$ is the eigenfunction of the initial nucleus A in state p , $\psi_{C,r}$ that of the compound nucleus in state r , $\psi_{P,p}$ that of the incident particle and H the Hamiltonian of the interaction between incident particle and initial nucleus. If the nuclear forces are known, H is given; and by solving the Schrödinger equation the nuclear wave functions $\psi_{A,p}$ and $\psi_{C,r}$ may be obtained without ambiguity. Thus the only factor in (43) about which doubt may arise, is the particle wave function $\psi_{P,p}$.

We know about $\psi_{P,p}$ its asymptotic behavior at infinity: it is a plane or spherical wave, normalized per unit energy. However, in (43) we need the wave function $\psi_{P,p}$ *inside* the nucleus. To obtain $\psi_{P,p}$ inside, we may suppose that it is the solution of a Schrödinger equation with a certain potential $V(r_P)$. This assumption is convenient in order to make the functions $\psi_{P,p}$, corresponding to different states p of the particle, orthogonal to each other. At large distances r_P from the nucleus, the potential $V(r_P)$ will of course be zero if the particle is a neutron, and it will be the Coulomb potential for charged particles.

About the behavior of the potential V *inside*

the nucleus, three different assumptions may be made:

- (a) V is negative (attractive potential),
- (b) V is zero (free particles),
- (c) V is positive (repulsive potential).

Assumption (c) expresses the idea that the nucleus is (practically) impenetrable for incident particles: This is quite plausible, because a particle falling on the surface of a nucleus will give part of its energy to the nuclear particles and will thus become amalgamated with the initial nucleus, forming the compound nucleus. This will prevent the particle from traversing the nucleus unperturbed. This assumption (c) seems most in accord with the general ideas of the compound nucleus.

Previously it has been argued (cf. references 3 and 5) that the most obvious choice of the particle potential $V(r)$ is to take it equal to the average potential energy of the particle in the field of the initial nucleus A , i.e.,

$$V(r_P) = \int H(r_A, r_P) |\psi_{A,p}(r_A)|^2 d\tau_A \quad (44)$$

where r_A denotes all the coordinates of the particles inside nucleus A . This average potential is certainly attractive. However, we shall show in the following that assumptions (a) and (b) are to be rejected in favor of (c).

The three assumptions (a), (b) and (c) will give widely different results for the matrix elements H , or the widths γ . Obviously, (a) will in general give the largest result because with assumption (a) we may have a resonance phenomenon already for the incident particle.³ If such a one-body-resonance occurs, $\psi_{P,p}$ will evidently be very large inside the nucleus and therefore the matrix element H will be large. On the other hand, assumption (c) will give the smallest matrix elements: In case (c), the wave function $\psi_{P,p}$ will decrease exponentially as we go from the surface of the nucleus inside. The rate of decay of $\psi_{P,p}$ is determined by the magnitude of $V(r_P)$ inside the nucleus. If we assume V of the order of nuclear binding energies, and the energy of the particle E_P small compared to V , $\psi_{P,p}$ will decrease to $1/e$ of its value in a distance of the order of the range of the nuclear forces (say, $2 \cdot 10^{-13}$ cm). Therefore the whole contribution

¹⁷ Our thanks are due to Professor E. Teller, to whom we owe most of the ideas presented in this chapter.

to the matrix element (43) will come from a surface layer whose thickness is of the order of the range of the nuclear forces, while in case (a) and (b) the whole volume of the nucleus contributes. Moreover, the wave function ψ_{Pp} will be smaller than for a free particle even right at the surface (cf. 50). The matrix elements in case (c) will, therefore, in general be smaller than in case (b).

Evidently, only one of the three values obtained for the matrix element H by the three assumptions (a), (b) and (c) can give the correct width of the level r of the compound nucleus. On the other hand, if we continue the perturbation calculation to sufficiently high approximations, we shall obtain the correct answer from whatever wave functions for the incident particle we have started. Therefore, if we start from the "wrong" wave functions, the first approximation is not sufficient to calculate the width of a nuclear level, and the second approximation will not be sufficient to give the correct dispersion formula.

The condition for a good wave function is the smallness of the higher approximations of the perturbation theory. This condition will be best fulfilled if the matrix elements H are as small as possible. Therefore, *assumption (c) will come nearest to the truth.*

After having decided for assumption (c), the question arises how to choose height and radius of the repulsive potential V . On one hand, a high repulsive potential makes the wave function ψ_{Pp} fall off more steeply and therefore makes the matrix elements in the average smaller. On the other hand, the perturbing potential H must, of course, include the auxiliary potential V with opposite sign: If V is chosen too large, this will again increase the average matrix element. This increase will (again in the average) be small as long as V is small compared to the average nuclear forces, a good measure of which are the nuclear binding energies. Therefore it seems an appropriate choice to make V of the same order as a nuclear binding energy, i.e., about 10 MV.

For the radius of the potential V , the obvious choice is the nuclear radius itself. A smaller radius would again make the matrix elements larger. A larger radius of the potential is even more to be avoided: The perturbation theory as

used in this paper, is based on an expansion of a certain function, *viz.*, $F = H\psi_A\psi_P$, in terms of the wave functions of the compound nucleus. This is only possible if in the region in which F is appreciable, the wave functions ψ_C form a complete (or practically complete) system. But this is, obviously, only the case if F is only large inside the nuclear radius. Therefore ψ_P must only fall off inside that radius.

An indication for the importance of higher approximations and therefore of the accuracy of the wave functions chosen may be obtained from the elastic scattering. The scattering of slow particles, e.g., slow neutrons, consists in our scheme of three parts:

(1) The scattering of zero order which is already contained in the "unperturbed" wave functions ψ_P of the incident particle. It is due to the existence of the potential V and amounts, for slow neutrons, to a scattering cross section¹⁸

$$\sigma_1 = 4\pi R^2, \quad (45)$$

where R is the radius of the sphere inside which the repulsive potential exists. This part may be called potential scattering in the narrower sense.

(2) The contribution of the low energy levels of the compound nucleus. This contribution shows resonance maxima; in between the maxima it is very small compared to σ_1 , because the neutron width of the levels of the compound nucleus is small compared to the spacing of the levels.

(3) The contribution of the high levels of the compound nucleus. For a given neutron wave function, the matrix elements $H^{P,r,p}$ will, in the average, be of the same order of magnitude for high levels r of the compound nucleus as for low ones, because there is no essential difference in character between the levels. Only when the energy E_r becomes very high, the matrix elements will fall off with increasing energy. Let us say that this will be the case for $E_r > E_c$. The critical energy E_c will be of the order of 20 or 30 MV, as may be estimated without great difficulty.¹⁹

The *density* of nuclear levels, i.e., the number of levels per unit energy, increases rapidly with

¹⁸ Cf. Mott and Massey, *Atomic Collisions*, p. 30.

¹⁹ Bethe and Livingston, "Nuclear Physics," Part B, to appear shortly in *Rev. Mod. Phys.*

increasing energy E_r . For a special model, this increase has been calculated.²⁰ Now the contributions of the various levels to the elastic scattering (41) all have the same sign. Therefore the most important contributions will come from very high levels where the density of levels is already large and the matrix elements have not yet decreased, i.e., from the levels in the neighborhood of E_c . *These high levels of the compound nucleus are responsible for all the difficulties in the convergence of the perturbation calculation mentioned above.* Therefore the condition for convergence is that the contribution of the high levels should not be too large.

The contribution of the high levels to the scattering can easily be estimated. The resonance denominators in (1) may all be replaced by $-E_c$ where E_c is the critical energy at which the matrix elements begin to fall off.²¹ Then we have

$$w^{P_r p} = \frac{2\pi}{\hbar E_c^2} \left| \sum_r |H^{P_r p}|^2 \right| = \frac{2\pi}{\hbar E_c^2} |(H^2)^{P_r p}|^2, \quad (46)$$

using the fact that the wave functions ψ_{Cr} of the compound nucleus form a complete system. The cross section is obtained by multiplying the probability (46) by $2\pi^2 \hbar \lambda^2$ (cf. (7)) so that

$$\sigma_3 = \frac{4\pi^3 \lambda^2}{E_c^2} \left| \int H^2 |\psi_A|^2 |\psi_{P_r p}|^2 d\tau \right|^2. \quad (47)$$

We denote by $d\tau_A$ the volume element in the configuration space of all particles contained in the nucleus A and by $d\tau_P$ that of the incident particle, so that $d\tau_A d\tau_P = d\tau$. Then, because of normalization,

$$\int |\psi_A|^2 d\tau_A = 1 \quad (48)$$

and
$$\int H^2 |\psi_A|^2 d\tau_A = U^2(r_P), \quad (49)$$

where U is an energy of the order of magnitude of a nuclear interaction (about 10 MV), which is an irregularly varying function of the position r_P of the particle. Of the wave function of the particle, only the part $l=0$ is important because

²⁰ Bethe, Phys. Rev. 50, 332 (1936).

²¹ More accurately, E_c is the energy for which the product of the average square of the matrix element and the density of the levels reaches its maximum.

we assume the particle to be slow. Considering the normalization per unit energy, we have

$$\psi_{P_r p} = (M^2 v / 2\pi^2 \hbar^3)^{\frac{1}{2}} \sin k(r-r_0)/kr \text{ for } r > R, \quad (50)$$

$$\psi_{P_r p} = (M^2 v / 2\pi^2 \hbar^3)^{\frac{1}{2}} (1/\kappa r) e^{-\kappa(R-r)} \text{ for } r < R, \quad (50a)$$

$$\text{with } \kappa = (2MV)^{\frac{1}{2}}/\hbar, \quad (50b)$$

$$r_0 = R - 1/\kappa. \quad (50c)$$

Quantities of the relative order $1/\kappa R$ have been neglected. Then we have

$$\begin{aligned} & \int H^2 |\psi_A|^2 |\psi_{P_r p}|^2 d\tau \\ &= \frac{4\pi}{2\pi^2} \frac{M^2 v}{\hbar^3} \int^R \frac{U^2 r^2 dr}{\kappa^2 r^2} e^{-2\kappa(R-r)} \\ &= \frac{1}{\pi} \frac{M^2 v}{\hbar^3 \kappa^3} \frac{1}{U^2} = \frac{1}{\pi} \frac{1}{2V\kappa\lambda} \overline{U^2} \quad (51) \end{aligned}$$

where $\lambda = \hbar/Mv$ is the wave-length of the incident particle. Inserting into (47), we have

$$\sigma_3 = \frac{\pi}{\kappa^2} \left(\frac{\overline{U^2}}{E_c V} \right)^2. \quad (52)$$

U , E_c and V are all of the same order of magnitude; probably E_c is a little larger than the other quantities. To stay on the safe side, we replace the ratio $\overline{U^2}/E_c V$ by unity. $1/\kappa$ is about $2 \cdot 10^{-13}$ cm, i.e., one-fifth of the nuclear radius. Thus σ_3 is about a hundred times smaller than σ_1 , (cf. 45). This would justify our choice of the wave function. It must, of course, be admitted that some of the estimates leading to (52) are very crude so that we prefer to regard our wave function still as provisional.

On the other hand, we can see immediately from our estimate that plane waves (possibility (b) above) and *a fortiori* an attractive potential (assumption (a)) are out of the question. For plane waves, κ^{-3} in (51) would be replaced by $\frac{2}{3}R^3$ so that (52) would be multiplied by a factor $4/9(\kappa R)^6$, i.e., about $5^6 \cdot 4/9 \approx 7000$. The contribution of the high nuclear levels to the scattering would then be enormous, and by analogy we must conclude that the contribution of these levels to the higher approximations of the perturbation theory would also be extremely large

so that the dispersion formula based on the second-order perturbation would be quite insufficient. The assumption of an impenetrable nucleus is, therefore, at present the most satisfactory for the calculation of particle wave functions.

Our estimates show also that with the use of the correct wave functions, the higher approximations of the perturbation theory may be neglected, which justifies formula (1).

As regards the elastic scattering itself, the contribution of the high levels is, of course, not so small as it might seem from the comparison of σ_3 and σ_1 . Actually, the "direct scattering" σ_1 and the contribution σ_3 of the high levels interfere with each other. Now the scattering due to the high levels is certainly in phase with the incident wave just as is the light scattered by an atom if the frequency of the light is below the resonance frequency. On the other hand, the scattered wave due to the impenetrability of the nucleus, has certainly a phase opposite to that of the incident wave. Therefore the amplitudes of the two scattered waves must be subtracted from each other, so that the total potential scattering is

$$\sigma_{\text{pot}} = (\sigma_1^{\frac{1}{2}} - \sigma_3^{\frac{1}{2}})^2. \quad (53)$$

Since σ_3 is about one percent of σ_1 , the total potential scattering will be about 20 percent less than the scattering from a hard sphere of radius R .

Near resonance, the resonance scattering σ_2 must be added to the potential scattering. Again there will be interference between the two kinds of scattering which, in the general case, is complicated by the fact that there is only interference between waves of the same total angular momentum J . The potential scattering must therefore be analyzed according to J . The phases are equal if the energy is above, opposite below the resonance level. Neglecting the contribution σ_3 , the total elastic scattering cross section becomes for slow particles (cf. 41)

$$\sigma_{Pp}^P = \frac{\pi}{(2s+1)(2i+1)} \sum_J (2J+1) \times \left| 2R + \sum_r \frac{\Gamma_{rP}^J \lambda_{rP}^J}{E_A + E_P - E_{rJ} + \frac{1}{2}i\gamma_{rJ}} \right|^2. \quad (54)$$

If only one level is important, this reduces to

$$\sigma_{Pp}^P = 4\pi R^2 + \frac{\pi(2J+1)\Gamma_{rP}^J \lambda_{rP}^J}{(2s+1)(2i+1)} \times \frac{4R(E_A + E_P - E_{rJ}) + \Gamma_{rP}^J \lambda_{rP}^J}{(E_A + E_P - E_{rJ})^2 + \frac{1}{4}\gamma_{rJ}^2}. \quad (55)$$

This cross section has its maximum at an energy E_P slightly higher than the resonance energy $E_{rJ} - E_A$, and a minimum at some energy below the resonance energy. If $\Gamma_{rP}^J \lambda_{rP}^J / \gamma_{rJ} R \gg 1$ (large resonance scattering), the maximum cross section is approximately

$$\sigma_{\text{max}} = \frac{4\pi(2J+1)}{(2s+1)(2i+1)} \left(\frac{\Gamma_{rP}^J}{\Gamma_{rJ}} \right)^2 \lambda^2, \quad (56)$$

the minimum

$$\sigma_{\text{min}} = 4\pi R^2 \left(1 - \frac{2J+1}{(2s+1)(2i+1)} \right). \quad (57)$$

The elastic cross section will always be at least of the order of σ_1 unless i or s is zero.

II. APPLICATIONS

§5. Capture of slow neutrons: theory

A. General formula.—For the capture cross section of a slow neutron with emission of fast particles or light quanta we have (cf. (38), $s = \frac{1}{2}$, (26), (27a))

$$\sigma_{Pq}^P = \frac{\pi \lambda}{2(2i+1)} \sum_{J'q'} (2J+1) \times \left| \sum_r \frac{|\lambda_{rP}^J|^{\frac{1}{2}} U_{rP}^J U_{rQ}^J}{E_A + E_P - E_{rJ} + \frac{1}{2}i\gamma_{rJ}} \right|^2. \quad (58)$$

This gives the capture cross section for a process which leads to a definite final level q of the remaining nucleus. Especially for radiative capture a great number of final levels will be possible. Hence for obtaining the observed cross section (58) has to be summed over all these final states. This makes the application of the formula rather cumbersome, and the effects of the interference become complicated, even if only a small number of compound states contribute to (58). We shall therefore, in the following, only consider two

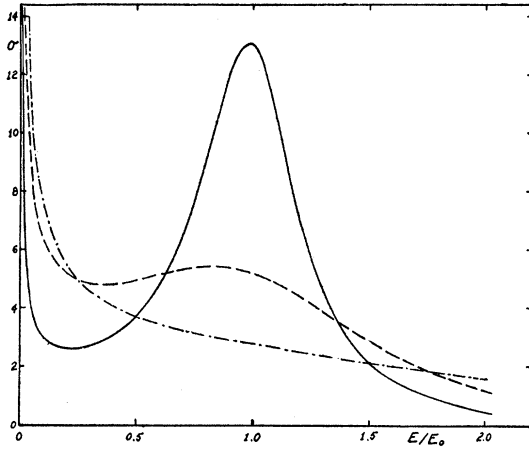


FIG. 1. The cross section as a function of energy for various ratios of width to resonance energy. Abscissa: energy as a fraction of the resonance energy. Ordinate: cross section. The curves are adjusted so that the area is unity in each case. — Width $\Gamma = \frac{1}{2}E_0$ (pronounced resonance); - - - $\Gamma = \sqrt{2}E_0$ (case of Cd); - · - $\Gamma = 4E_0$ (almost $1/v$ law).

important limiting cases which permit the discussion of some but by far not all applications.

First, we shall discuss the case that only a single compound state is of importance in the sum in (58), and hence (58) can be simplified according to (12). (12) will be generally applicable for light nuclei because of the small density of energy levels; for heavier nuclei it will often be true in the immediate neighborhood of the resonance.

Secondly, we shall consider the case of constant capture probability which leads to the $1/v$ law. As already mentioned in §3B, it is quite generally true that for low energies σ varies as $1/v$. The extension of the energy region in which this is the case is, however, very different according to the circumstances and will be discussed below.

B. Single resonance level.—(58) reduces to (cf. (12))

$$\sigma = \sigma_{0N} \frac{(\Gamma/\gamma)^2}{1 + ((E - E_0)/(\gamma/2))^2} \left(\frac{|E_0|}{E} \right) \quad (59)$$

with

$$\begin{aligned} \sigma_{0N} &= 2\pi\lambda_g^2 \frac{\Gamma_n \Gamma_r}{\Gamma^2} \left(1 \pm \frac{1}{2i+1} \right) \\ &= \frac{1.30 \cdot 10^{-18} \Gamma_r \Gamma_n}{|E_0| \Gamma^2} \left(1 \pm \frac{1}{2i+1} \right). \end{aligned} \quad (60)$$

Here, E_0 and E are the energy of the level and

the neutron, respectively, both measured from zero kinetic energy of the neutron. Hence E_0 can be positive as well as negative. Γ_n is the neutron width at the resonance energy E_0 , Γ_r the radiation width or width for emission of a fast particle, γ is the total width ($\gamma = \Gamma_r + \Gamma_n(E/|E_0|)^{\frac{1}{2}}$) and $\Gamma = \gamma(E_0)$ the total width at the resonance energy. All energies are measured in volts, σ in cm^2 . The factor $1 \pm 1/2i+1$ corresponds to the two possibilities $J = i \pm \frac{1}{2}$; for $i=0$, it has to be replaced by 2. If, as is true in most of the cases to be considered afterwards, the contribution of the neutron width to the total width is negligible, γ can be considered as independent of the energy E and replaced by $\Gamma = \Gamma_r$. We shall in the following, if not otherwise stated, always make this assumption. Then we have

$$\sigma = \sigma_{0N} \frac{1}{1 + ((E - E_0)/(\Gamma/2))^2} \left(\frac{|E_0|}{E} \right)^{\frac{1}{2}}; \quad (61)$$

$$\sigma_{0N} = \frac{1.30 \cdot 10^{-18} \Gamma_n}{|E_0| \Gamma_r} \left(1 \pm \frac{1}{2i+1} \right). \quad (61a)$$

If $E_0 > 0$, and $\Gamma/E_0 \ll 1$, the cross section has, according to (61), a maximum for $E = E_0$ and a minimum for $E = E_0/5$. For larger values of the ratio $\Gamma/|E_0|$ the extrema get less pronounced, their positions come nearer together and finally coalesce at $\frac{3}{5}E_0$ for $\Gamma/E_0 = 4/\sqrt{5} = 1.79$. Generally we have

$$E_{\min} = \frac{E_0}{5} \left\{ 3 \pm 2 \left(1 - \frac{5}{16} \left(\frac{\Gamma}{E_0} \right)^2 \right)^{\frac{1}{2}} \right\}. \quad (62)$$

For $\Gamma/E_0 \geq 1.79$ there is no longer a maximum and minimum, and the cross section increases continuously with decreasing E up to the $1/v$ -region.

If $E_0 < 0$, the cross section increases monotonously with decreasing energy for all values of Γ/E_0 . Fig. 1 illustrates the energy dependence of the cross section for a number of values of the ratio $\Gamma/E_0 = c$.

The form (61) of the cross section is modified by the motion of the capturing nuclei. For free nuclei,²² this leads to a Doppler broadening of the amount:

$$\Delta = 4(mE_0 k T / M)^{\frac{1}{2}} \quad (63)$$

²² For the discussion of the validity of the assumption of free nuclei in the cases treated in the present paper cf. G. Placzek and E. Teller, reference 16.

(m , M mass of the neutron and capturing nucleus).²³

C. *Immediate neighborhood of the resonance.*—
For

$$\frac{|E_g - E|}{E_g} \ll 1, \quad \frac{\Delta}{E_g} \ll 1,^{24}$$

the combination of Doppler and natural width leads to the result familiar from optics:

$$\sigma = \sigma_{0N} \psi(\xi, x), \quad (64a)$$

$$\psi(\xi, x) = \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\xi^2(x-y)^2}}{1+y^2} dy, \quad (64)$$

$$x = \frac{E_g - E}{\Gamma/2}, \quad \xi = \Gamma/\Delta. \quad (64b)$$

The maximum cross section σ_0 is

$$\begin{aligned} \sigma_0 &= \sigma_{0N} \psi(\xi, 0) = \sigma_{0N} \frac{\xi}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\xi^2 y^2}}{1+y^2} dy \\ &= \sigma_{0N} \sqrt{\pi} \xi e^{\xi^2} \{1 - \Phi(\xi)\}, \end{aligned} \quad (65)$$

²³ If the resonance energy is $E_g = \frac{1}{2} m v_g^2$ in the system where the capturing particle is at rest, we have for the energy in the system of the observer

$$E = \frac{1}{2} m (v_g - u_x)^2, \quad (63a)$$

where u_x is the velocity component of the capturing particle in the direction of the incident particle. Supposing $|u_x| \ll |v_g|$ i.e., $E_g \gg k T m/M$ we may neglect the term with u_x^2 and obtain from (63a)

$$E_g - E = u_x (2mE_g)^{\frac{1}{2}}. \quad (63b)$$

Now the probability that u_x lies between u_x and $u_x + du_x$ is

$$w(u_x) du_x = (M/2\pi kT)^{\frac{1}{2}} e^{-Mu_x^2/2kT} du_x. \quad (63c)$$

Hence from (63b) and (63c)

$$w(E) dE = 2\pi^{-\frac{1}{2}} e^{-4(E-E_g)^2/\Delta^2} dE/\Delta, \quad (63d)$$

where

$$\Delta = 4(mE_g kT/M)^{\frac{1}{2}}. \quad (63)$$

In the relativistic case we have instead of (63b)

$$E_g - E = u_x p.$$

(p momentum of the incident particle) and hence

$$\Delta = 2p_g (2kT/M)^{\frac{1}{2}}. \quad (63e)$$

(63e) is valid both for light quanta and for particles; it reduces for light to the familiar expression

$$\Delta = 2(\hbar\omega/c)(2kT/M)^{\frac{1}{2}}. \quad (63f)$$

The Doppler width for particles is equal to the Doppler width for light quanta with the same momentum.

²⁴ This latter assumption was already used in the derivation of (63), because it is equivalent with $u_x \ll v$. It is fulfilled for all practical cases.

where $\Phi(\xi)$ is the Gaussian error function

$$\Phi(\xi) = 2\pi^{-\frac{1}{2}} \int_0^{\xi} e^{-t^2} dt.$$

For $\xi \gg 1$ (natural width) (64), (65) go over into²⁵

$$\sigma = \sigma_{0N} (1/(1+x^2)), \quad (66)$$

$$\sigma_0 = \sigma_{0N}. \quad (66a)$$

For $\xi \ll 1$, $\xi x < 1$ (Doppler width)

$$\sigma = \sigma_{0D} e^{-(E-E_g)/(\Delta/2)^2} \quad (67)$$

with

$$\begin{aligned} \sigma_{0D} &= \sqrt{\pi} \xi \sigma_{0N} \\ &= 2.31 \cdot 10^{-18} \cdot \frac{\Gamma_n}{\Delta E_g} \left(1 \pm \frac{1}{2i+1}\right). \end{aligned} \quad (67a)$$

With the help of (64) and (65) we can calculate the cross section for the important case, that the same substance is used as absorber and indicator (self-indication)^{26, 27}

$$\sigma_g = \int \sigma^2 dE / \int \sigma dE. \quad (68)$$

If $\Gamma/E_g \ll 1$, the integration can be extended from $x = -\infty$ to $x = \infty$ without appreciable error, as in this case only those regions contribute to the integral, in which the condition $|E_g - E| \ll E_g$ is fulfilled and hence (64) is valid. The result of the integration²⁸ is:

²⁵ The factor $(|E_g|/E)^{\frac{1}{2}}$ in (59) does not appear in (64a) as $E_g - E$ was supposed small compared with E_g .

²⁶ We denote throughout with the indices 0 and g the maximum cross section for monochromatic incident particles and the effective cross section for self-indication while the letters N and D refer to natural and Doppler width, respectively.

²⁷ (68) holds, of course, only for thin absorbers.

²⁸ Instead of the integral $\int \psi^2(\xi, x) dx$ which occurs in (68) (cf. (64)), we calculate immediately the somewhat more general integral $\int \psi(\xi, x) \psi(\eta, x) dx$. Using the definition (64), we have

$$\begin{aligned} I &= \int \psi(\xi, x) \psi(\eta, x) dx \\ &= \frac{\xi \eta}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy dz \frac{e^{-\xi^2(x-y)^2}}{1+y^2} \frac{e^{-\eta^2(x-z)^2}}{1+z^2}. \end{aligned} \quad (69a)$$

We integrate first over x and obtain

$$I = \frac{\xi \eta}{\pi^{\frac{1}{2}} (\xi^2 + \eta^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dy dz \frac{e^{-\xi^2 \eta^2 (y-z)^2 / (\xi^2 + \eta^2)}}{(1+y^2)(1+z^2)}. \quad (69b)$$

Here we introduce the new coordinates $u = \frac{1}{2}(y+z)$ and $v = \frac{1}{2}(y-z)$. The integration over u is straightforward and yields

$$I = \frac{1}{2} \pi^{\frac{1}{2}} \xi \int_{-\infty}^{\infty} dv e^{-\xi^2 v^2} / (1+v^2) \quad (69c)$$

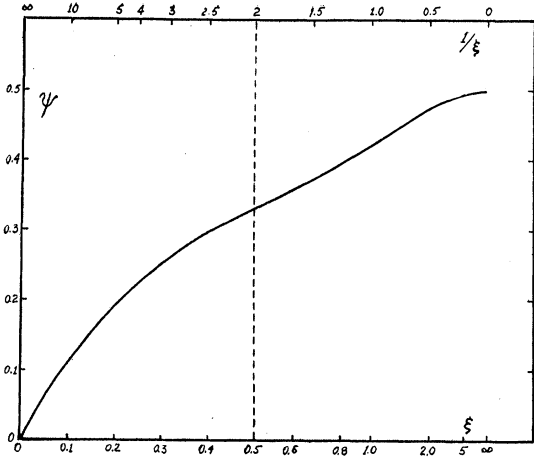


FIG. 2. The ratio χ of the average absorption of the resonance group to the absorption coefficient at exact resonance, as a function of the ratio ξ of natural to Doppler width. Note the change of scale at $\xi=0.5$. (By mistake the ordinate in the figure is marked " ψ " instead of " χ ").

$$\sigma_g = \sigma_{0N} \frac{\int_{-\infty}^{\infty} \psi^2 dx}{\int_{-\infty}^{\infty} \psi dx} = \frac{1}{2} \sigma_{0N} \psi(\xi\sqrt{2}, 0). \quad (69)$$

The cross section with self-indication is thus equal to half the maximal cross section, measured at $\frac{1}{2}$ of the original temperature. Hence, for $\xi \gg 1$ (natural width), as there is no effect of the temperature, $\sigma_{gN} = \frac{1}{2} \sigma_{0N}$ while for $\xi \ll 1$ (Doppler width) $\sigma_{gD} = \sigma_{0D}/\sqrt{2}$ since the Doppler width (*viz.* (63)) is proportional to the square root of the temperature and the maximal cross section in this case (*viz.* (67a)) proportional to ξ .

From (65) and (69) we obtain:

$$\sigma_g = \sigma_{0N} \chi(\xi), \quad (70)$$

$$\chi(\xi) = \frac{1}{2} \psi(\xi\sqrt{2}, 0) = (\pi/2)^{1/2} \xi e^{2\xi^2} \{1 - \Phi(\xi\sqrt{2})\}. \quad (70a)$$

For $\xi \gg 1$

$$\sigma_g = \sigma_{0N} \left\{ 1 - \frac{1}{4\xi^2} + \frac{1 \cdot 3}{(4\xi^2)^2} - \frac{1 \cdot 3 \cdot 5}{(4\xi^2)^3} + \dots \right\}, \quad (71)$$

with
$$\zeta = 2\xi\eta(\xi^2 + \eta^2)^{-1/2}. \quad (69d)$$

But the integral in (69c) is exactly the same as in (65), so that we find

$$I = \frac{1}{2} \pi \psi(\zeta, 0). \quad (69e)$$

In the special case $\xi = \eta$ (69d) reduces to

$$\zeta = \xi\sqrt{2}. \quad (69f)$$

The integral in the denominator of (68) gives simply $\pi\sigma_{0N}$ which, together with (69e) (69f), leads immediately to (69).

$$\sigma_{gN} = \frac{1}{2} \sigma_{0N}. \quad (71a)$$

For $\xi \ll 1$

$$\sigma_g = \sigma_{gD} \left\{ 1 - 2(2/\pi)^{1/2} \xi + 2\xi^2 - (8/3)(2/\pi)^{1/2} \xi^3 + 2\xi^4 - \dots \right\}, \quad (72)$$

$$\sigma_{gD} = \sigma_{0N} (\pi/2)^{1/2} \xi = 1.63 \cdot 10^{-18} (\Gamma_n/\Delta E_g). \quad (72a)$$

A plot of the function $\chi(\xi)$ is given in Fig. 2. If the Doppler width is not negligible with respect to the natural width, the cross section for self-indication σ_g will depend on ξ and hence on the temperature of the absorber and indicator. Measurements of the temperature-dependence of σ_g in a region where both widths are comparable, may therefore serve to determine ξ and, if the energy E_g is known, also Γ_n .²⁹ If indicator and absorber are held at different temperatures, the temperature to be introduced in the ξ in (70), (71), (72) is (*cf.* (69d) and (63)), the arithmetic mean $\frac{1}{2}(T_1 + T_2)$.

The capture cross section is not only important for absorption experiments but also for the measurement of the activation. Amaldi and Fermi³⁰ have given a relation between absorption coefficient, activation and width. The absorption coefficient was supposed to be constant within the line. Instead of this assumption we shall use (64). For the activation (number of disintegrations per second after infinite time of neutron irradiation = number of neutrons captured per second) of a thin layer we have

$$A = \delta F \int \rho(E) v \kappa(E) dE, \quad (73)$$

δ and F thickness (in g/cm²) and surface of indicator, $\rho(E)dE$ density of neutrons of energy E per cm³; v velocity; κ absorption coefficient in cm²/g ($\kappa = L\sigma/A$; L , Avogadros number; A , atomic weight).

According to Amaldi and Fermi, the density of neutrons above 1 volt is

$$\rho(E)dE = (1/v) q l_g dE/E, \quad (74)$$

q number of neutrons emitted by the source per second and cm³, l_g mean free path of neutrons

²⁹ This was first pointed out by O. R. Frisch, who also reported about preliminary experiments of this kind at the Copenhagen conference (June, 1936).

³⁰ Amaldi and Fermi, *Ricerca Sci.* 7/1, no. 11-12 (1936) and *Phys. Rev.* 50, 899 (1936).

of energy E_0 in paraffin. Introducing (74), (64) and (70) into (73) we have

$$A_0 = \delta Fq \frac{l_0}{E_0} \int \kappa(E) dE = \frac{\delta Fql_0}{E_0} \kappa_{0N} \int_{-\infty}^{\infty} \psi dE$$

$$= \frac{\delta Fql_0}{E_0} \kappa_{0N} \frac{\Gamma}{2} \pi = \frac{\delta Fql_0}{E_0} \frac{\Gamma}{\chi} \frac{\pi}{2}. \quad (75)$$

Γ/χ can be considered as a measure of the effective width, we have

$$\text{for } \xi \gg 1, \quad \Gamma/\chi = 2\Gamma$$

$$\xi \ll 1, \quad \Gamma/\chi = (2/\pi)^{1/2} \Delta. \quad (76)$$

We may also write (75) (cf. (61a))

$$A_0 = \frac{\pi}{2} \frac{\delta Fql_0}{M} \cdot 1.30 \cdot 10^{-18} \frac{\Gamma_n}{E_0^2} \left(1 \pm \frac{1}{2i+1} \right)$$

$$= \frac{\pi}{2} \frac{\delta Fql_0}{M} \cdot 1.30 \cdot 10^{-18} \frac{\Gamma_n'}{E_0^3} \left(1 \pm \frac{1}{2i+1} \right), \quad (77)$$

$$\text{where } \Gamma_n' = \Gamma_n E_0^{-3/2} \quad (77a)$$

is the neutron width at 1 volt and M the atomic mass in grams. Thus, the activation is proportional to the neutron width and varies with the position of a level with given properties (symbolized by Γ_n') as $E_0^{-3/2}$.

D. The 1/v region.—The condition for the validity of the 1/v law is that the variation of the 1/v factor with the energy is large compared to the variation of the capture probability. Thus, according to (61)

$$\left| \left(\frac{|E_0|}{E} \right)^{1/2} \frac{d}{dE} \frac{1}{1 + ((E_0 - E)/(\Gamma/2))^2} \right|$$

$$\ll \left| \frac{1}{1 + ((E_0 - E)/(\Gamma/2))^2} \frac{d}{dE} \left(\frac{|E_0|}{E} \right)^{1/2} \right|, \quad (78a)$$

$$\text{which gives } \frac{4E|E_0 - E|}{(\Gamma/2)^2 + (E_0 - E)^2} \ll 1. \quad (78)$$

We see that (78) is always satisfied if *either*³¹

$$E \ll \frac{1}{4} |E_0| \quad (79)$$

$$\text{or } E \ll \frac{1}{4} \Gamma. \quad (80)$$

³¹ We may write (78) in the form

$$E \ll \frac{1}{4} \Gamma \left\{ \frac{1}{2} \Gamma / |E_0 - E| + |E_0 - E| / \frac{1}{2} \Gamma \right\}$$

from which (79) and (80) are obvious.

A more complete expression for the extension of the 1/v range as a function of E_0 and Γ is obtained by putting the left side of (78) equal to a small quantity ϵ and solving this as an equation for E . In this way it is seen that the 1/v range may even extend to much higher energies than indicated by (79) and (80).

Thus the 1/v law certainly holds if the incident energy is small compared to the *larger* of the two quantities E_0 and Γ .³² No serious error will be committed by applying this condition also in the cases, where the one-level formula is to be replaced by the general formula (58), provided it holds for all the compound levels contributing to the sum.

Actually, the extension of the 1/v range is very different according to the position and width of the resonance levels. It is a fraction of a volt for nuclei like Ag and Rh (see below), and considerably higher for light elements, because of the smaller density of nuclear levels. It is true that accidentally the first resonance level may lie at a low energy even for light nuclei. Even then, the 1/v law will hold up to fairly high energies if the capture of the neutron is followed by the emission of a charged particle. For in this case the width of the level will be of the order of several 10,000 volts (cf. §6) so that the validity of the 1/v law is ensured at least up to several thousand volts, irrespective of the position of the resonance level. Considerations of this kind have led to the use of the absorption in boron, caused by the reaction



as an energy gauge for slow neutrons.³³ This method will be discussed in §6A1.

For the radiative capture of neutrons by light nuclei two cases may occur. In the usual case of a high resonance level we shall have a wide 1/v range but the cross section will *et. par.* be smaller than for the emission of particles in the ratio of the γ -ray to the particle width. The γ -ray width may be of the order of 1 volt (cf. §10) which

³² If, however, the contribution of the neutron width to the total width is not negligible, this condition is not strictly sufficient because of the energy dependence of γ . In this case (80) is invalidated, but (79) remains sufficient with the supplementary condition $\gamma_n \ll E_0$ as can be derived from (59).

³³ Frisch and Placzek, *Nature* **137**, 357 (1936); Weekes, Livingston and Bethe, *Phys. Rev.* **49**, 471 (1936).

makes the radiative capture by a factor of the order of 10,000 less probable than the emission of particles. A large cross section for radiative capture by light nuclei can only occur if the first resonance level lies exceptionally low; in this case, however, the $1/v$ law will break down at quite a low energy.

For a great number of nuclei the $1/v$ law holds in the region of thermal energies. We shall therefore in the following give expressions for the cross section of and the activation by thermal neutrons, assuming this law for absorber and indicator. The question, how far the so-called C neutrons (i.e., neutrons stopped by cadmium) are identical with thermal neutrons, shall not be discussed at this stage.

For the total density of thermal neutrons, we have (cf. Amaldi and Fermi,³⁰ or (83d) for large x)

$$\int_{\text{th}} \rho(E) dE = q' \tau, \quad (81)$$

where q' , the total number of neutrons produced per cm^3 and second, is different from the q used in (74) because of the different density distribution in paraffin of resonance and thermal neutrons. Integrating over the whole volume, we get the total number Q of neutrons produced per second:

$$Q = \int q dV = \int q' dV. \quad (82)$$

τ is the lifetime of a thermal neutron in paraffin. We may put

$$\tau = Nl/\bar{v} = Nl(8kT/\pi m)^{-1/2}. \quad (83)$$

N : number of collisions of a thermal neutron in paraffin before its capture, l : mean free path of a thermal neutron.³⁴

³⁴ That this particular average of v has to be taken (rather than, say, $\tau = Nl\bar{v}^{-1}$) can be seen by going back to the theory of diffusion of neutrons (cf. Amaldi and Fermi, reference 30).

The stationarity equation for the neutron diffusion is

$$D\Delta F - F/\tau + q' = 0, \quad (83a)$$

where $F(x, y, z)$ is the total neutron density per cm^3 , τ the mean life of a neutron in paraffin, q' the number of neutrons produced by the source per cm^3 and sec., and

$$D = \frac{1}{3}\bar{v}, \quad (83b)$$

the diffusion coefficient of thermal neutrons. τ is independent of the neutron velocity. We solve (83a) for the case

Assuming complete thermal equilibrium, we get for the energy distribution of the thermal neutrons

$$\rho(E) dE = \frac{q' \tau E^{3/2} e^{-E/kT}}{(\pi/4)^{1/2} (kT)^{3/2}} dE. \quad (84)$$

If the indicator is covered on one side by Cd which absorbs the thermal neutrons, the density

that a Cd absorber, sufficient to absorb all thermal neutrons, is placed at $x=0$. This corresponds in first approximation to the boundary condition

$$F=0 \quad \text{at} \quad x=0. \quad (83c)$$

(We shall find below that F is not exactly zero; in fact, it is just the value of F for $x=0$ we wish to calculate. But it is sufficient for our arguments that F at $x=0$ is small compared to its value in the paraffin.) The solution of (83a) with the boundary condition (83c) is for $x>0$

$$F = q' \tau (1 - e^{-x(D\tau)^{-1/2}}). \quad (83d)$$

The current of neutrons near $x=0$ is then

$$S = D(dF/dx)_0 = q' L, \quad (83e)$$

where

$$L = (D\tau)^{1/2} \quad (83f)$$

is the "diffusion length" measured by Amaldi and Fermi. The neutron current determines the activity produced in a detector. If the angular distribution of the neutrons is

$$f(\vartheta) = a \cos \vartheta + b \cos^2 \vartheta, \quad (83g)$$

the activity produced in a thin detector of thickness δ and absorption coefficient κ will be

$$A = S\kappa\delta \frac{\int_0^{\pi/2} (a+b \cos \vartheta) \sin \vartheta d\vartheta}{\int_0^{\pi/2} (a \cos \vartheta + b \cos^2 \vartheta) \sin \vartheta d\vartheta} = S\kappa\delta \frac{a + \frac{1}{2}b}{\frac{1}{2}a + \frac{1}{3}b}. \quad (83h)$$

For the neutrons emerging from paraffin, we have (Amaldi and Fermi, (5))

$$b = a\sqrt{3}, \quad A = S\kappa\delta\sqrt{3}. \quad (83i)$$

On the other hand, *inside* a uniform paraffin block the activity would be

$$A = F_0 \bar{v} \kappa \delta, \quad (83j)$$

where \bar{v} is the average velocity and F_0 the density of the neutrons. From (83i) and (83j) we obtain the "effective density" at the surface

$$F' = 3^{1/2} S/\bar{v} = 3^{1/2} q' L/\bar{v}. \quad (83k)$$

Now the number of collisions before capture N is *defined* by Amaldi and Fermi in terms of the observed diffusion length L and the observed mean free path l by the relation

$$L = l(N/3)^{1/2}. \quad (83l)$$

Inserting this in (83k), we find

$$F' = q' l N^{1/2} / \bar{v}, \quad (83m)$$

which is identical with (81) (83) (85). It might be objected against (83h) and (83j) that the absorption coefficient κ depends on the velocity. This objection is not valid because the energy distribution of the thermal neutrons emerging from the paraffin near the Cd absorber is the same as that in the interior of the paraffin: In both cases, the number of neutrons of energy E striking the detector per sec., is proportional to the number of such neutrons in the Maxwell distribution, times the neutron velocity.

is hereby reduced by the factor (cf. Amaldi and Fermi,³⁰ and reference 34)

$$\rho'/\rho = 1/N^{1/2}. \quad (85)$$

Thus we have

$$\rho'(E)dE = \frac{q'l(\frac{1}{2}Nm)^{1/2}}{(kT)^2} E^{1/2} e^{-E/kT} dE. \quad (86)$$

With the help of (86) we can calculate the cross section and activation.

Putting $\sigma = XE^{-1/2}$, where X is a constant, and considering that also the sensitivity of the indicator varies with $1/v$ which compensates the v factor coming in by the change from density to current, we have

$$\sigma_{th} = \frac{\int \sigma \rho' dE}{\int \rho' dE} = \frac{X}{((\pi/4)kT)^{1/2}} \quad (87)$$

and hence $\sigma(E) = \sigma_{th}(\frac{1}{4}\pi kT/E)^{1/2}. \quad (88)$

Thus, the effective energy for the absorption of thermal neutrons in a thin layer is $(\pi/4)kT$.

For the activation by thermal neutrons B_{th} we get from (86) and (88):

$$B_{th} = \delta F \int \kappa(E)v\rho'(E)dE = \frac{1}{4}\pi\delta Fq'N^{1/2}l\kappa_{th} \quad (89)$$

(κ_{th} absorption coefficient in cm^2/g , $\kappa_{th} = \sigma_{th}/M$).

§6. Slow neutrons: Methods for the determination of energy, neutron and radiation width of the compound levels

In the following we discuss the application of the formulae derived in §5 to the determination of the constants E_p , Γ_n and Γ_r from absorption and activation experiments. We shall limit ourselves throughout to a consideration of the case of thin absorbers and indicators. Corrections for finite thickness, which become important in nearly all practical cases, will be discussed in a subsequent paper by one of the authors (H.A.B.). When referring to experimental data, we shall not recorrect the values for cross section and activation for finite thickness, but use the values as given and corrected by the respective authors. The constants thus obtained must therefore be considered as provisional.

The application of the preceding formulae is based on the possibility of isolating, among the

neutrons emerging from paraffin, (a) thermal neutrons, (b) neutrons, the capture of which leads to a *definite* level of the compound nucleus formed in the capture process (resonance neutrons), The only means of isolating the thermal neutrons is at present the absorption in cadmium. As is shown by various experiments, practically all thermal neutrons are stopped by 0.45 g/cm^2 Cd. Under the assumption, that the numerosity of the thermal neutrons is very large compared to the numerosity of the neutrons of higher energies absorbed by Cd, the thermal neutrons have therefore been identified with the so-called *C* neutrons, i.e., neutrons stopped by 0.45 g/cm^2 Cd. It is, however, probable—see below—that the Cd level responsible for this absorption lies at about 0.1 volt and has a width of 0.2 volt, so that the strong absorption in 0.45 g/cm^2 Cd extends to about 0.3 volt.

Hence, a certain number of neutrons of energies considerably higher than thermal energies will be among the *C* neutrons. The fact, that the slowing down process in paraffin in the region between the energy of the chemical hydrogen bond ($\frac{1}{3} - \frac{1}{8}$ volt) and the thermal region is rather inefficient, increases the number of neutrons in this region. Furthermore, in the thermal region itself, the energy distribution of the neutrons will not be accurately represented by the Maxwell curve, as the neutrons slowed down in paraffin are captured by the hydrogen nuclei before complete thermal equilibrium is reached.

No accurate estimates of the magnitude of these effects are at present available, but it seems from a theoretical discussion that they are not very large,³⁵ so that at least the order of magnitude of the results will not be changed. This is also shown by the fact that the energy distribution curve of the neutrons, as found with the mechanical velocity selector of Cd sheets, has a maximum in the thermal region.³⁵ Therefore we shall in this paper retain throughout the identification of *C* neutrons and thermal neutrons, substituting thus σ_{th} , κ_{th} , and B_{th} by σ_c , κ_c , and B_c and using for the effective energy of the *C* neutrons with $1/v$ indication the value $E_c = (\pi/4)kT$ (cf. (88)).

³⁵ Dunning, Pegram, Fink, Mitchell and Segrè, Phys. Rev. 48, 704 (1935).

For the isolation of the resonance neutrons the absorption or indication in the element to be investigated is used. A source of uncertainty lies in the fact, that the resonance neutrons thus defined may belong not to a single resonance level but to several of them. The total resonance activation A_{res} observed will then be the sum of the activations of the single compound levels, so that we have (cf. (77)):

$$A_{\text{res}} = \sum_g A_g = c \sum_g (\Gamma'_{ng}/E_g^{\frac{1}{2}}), \quad (90)$$

where c is a constant, and Γ'_{ng} the reduced neutron width of the level E_g (i.e., neutron width at 1 volt; cf. (77), (77a)). Except for the irregular variation of Γ'_{ng} , the contributions of the higher levels to the activation will decrease as $E_g^{-\frac{1}{2}}$. For a quite schematic picture of the situation, we may assume the levels to be equally spaced with a distance D corresponding to the actual mean distance and Γ'_{ng} to be the same for all levels. Then we see that the importance of the contributions of the higher levels to the activation will be determined by the ratio of the energy of the first positive level E_1 to the mean distance D . Indeed we have

$$A_{\text{res}} = A_1 \cdot \sum_{g=1}^{\infty} \{1 + (g-1)(D/E_1)\}^{-\frac{1}{2}} \quad (91)$$

and may now distinguish three cases:

(a) $E_1 \ll D$. The activation is practically determined by the first level. This case will be realized, if resonance capture is observed in a light element, because here the resonance activation will in general only be observable if E_1 by accident lies exceptionally low.³⁶ Occasionally, this may also occur for a heavy element.

(b) $E_1 \sim D$. The total contribution of the higher levels to the activation is of the same order of magnitude as the contribution of the first level and is caused by a great number of levels; e.g., for $E_1 = \frac{1}{2}D$ we have $A_{\text{res}} = 2.6A_1$, and the activation due to the levels with $g > 50$ is still 30 percent of the activation of the first level.

³⁶ It is true that with decreasing atomic number, not only the mean distance D of the levels, but also the neutron width $\bar{\Gamma}_n$ increases rapidly, so that for a light nucleus a noticeable activation might still be produced by resonance neutrons of much higher energy than for a heavy nucleus. It is, however, probable, that the neutron width increases less rapidly than $D^{\frac{1}{2}}$, so that here the possibility for the detection of resonance-capture is in general limited to the case $E_1 \ll D$.

We shall later give arguments to the effect that this case is more or less realized for elements like Ag, Rh, In; D being of the order of some volts.

(c) $E_1 \gg D$. Each of a great number of higher levels produces an activation of the order of the effect of the first one. This may occasionally happen for a heavier nucleus. It seems (cf. below) that iodine may come near to this case.

In a similar way, the influence of the higher levels on the absorption coefficient with self-indication can be discussed.

A certain—though not very sharp—control of the effect of higher levels is supplied by the analysis of the absorption curves with self-indication, supplemented with boron absorption curves. Such an analysis would permit one to detect a pronounced heterogeneity of the neutrons.³⁷ (Cf. Fermi's groups *A* and *B* in Ag.) In the following, we shall give formulae under the assumption, that only a single level is effective, and discuss for every method how the results are modified by the presence of several levels.

A. Determination of energy

1. Absorption in boron³⁸

Case a. Resonance energy larger than the cadmium absorption limit.—The absorption coefficient in boron for the neutrons penetrating cadmium and indicated by the element under investigation is compared to the absorption coefficient of the *C* neutrons in boron, measured with a boron chamber or another indicator obeying the $1/v$ law in the thermal region. Because of the $1/v$ law for the capture in boron, the reasons for which have been given in §5D, and because of (88), we have for the resonance energy³⁹

³⁷ Also if only a single resonance level is of importance, the heterogeneity caused by the variation of the absorption coefficient within the line will manifest itself in the absorption curves. It will often be possible, however, to decide with the help of the constants determined according to Section C whether this reason is sufficient to explain the shape of the absorption curves or whether the presence of several levels must be assumed.

³⁸ Frisch and Placzek, reference 33. Weekes, Livingston and Bethe, reference 33. Collie, Nature 137, 619 (1936). Amaldi and Fermi, reference 30. Halban and Preiswerk, Helv. Phys. Acta 9, 318 (1936). Goldsmith and Rasetti, Phys. Rev. 50, 328 (1936). Goldsmith and Manley, Phys. Rev., in press.

³⁹ Goldsmith-Rasetti take kT instead of $\pi/4 kT$ as a correction for the hardening of the thermal neutrons in the thick layers used. The actual value of E_0 is probably still a little higher because of the reasons exposed at the beginning of this section.

$$E_g = E_c \left(\frac{k^{\text{boron } c}}{k^{\text{boron } g}} \right)^2 = \frac{\pi}{4} k T \left(\frac{k^{\text{boron } c}}{k^{\text{boron } g}} \right)^2. \quad (92)$$

If several resonance levels are effective, the energy determined according to (92) will be a rather ill defined mean value, namely (cf. 90)

$$E_{\text{res}} = \left(\frac{\sum_g \Gamma'_{ng} E_g^{-\frac{1}{2}}}{\sum_g \Gamma'_{ng} E_g^{-2}} \right)^2. \quad (93)$$

If all Γ'_{ng} are equal, and $E_1 = D$ (cf. above), this gives $E_{\text{res}} = 2.5E_1$.

(93) holds for a thin boron layer; for thick layers, the influence of the higher levels will be still more pronounced. It is therefore essential to base the energy determination on complete absorption curves and not on single points. Furthermore the influence of the higher levels may be reduced by defining the resonance neutrons not by the indication alone, but also by the absorption, which can be done by adding to the cadmium filter a filter consisting of the same element as the indicator.

At present, the energy values found by different observers with the boron method seem to differ to a certain extent in some cases. The reason for this appears to be not so much lack of precision of the measurements, but different—and perhaps insufficient—correction for geometrical factors.

Case b. E_g smaller than the Cd absorption limit.

If the resonance energy is smaller than the energy at which Cd becomes transparent (about 0.3 volt, *viz.* below), the boron method in the form described will not be applicable. Here, instead, some information about the position of the level may be obtained by the study of the boron absorption with boron indication of the neutrons stopped by and penetrating through layers of various thickness of the element to be investigated, as was done originally in the case of cadmium.⁴⁰

Important supplementary information about the position of low energy levels may be obtained from the rotating wheel experiment,⁴¹ as will be discussed for the case of cadmium (§7B).

⁴⁰ Frisch and Placzek, reference 33.

⁴¹ Rasetti, Segrè, Fink, Dunning and Pegram, Phys. Rev. 49, 104 (1936); Rasetti, Mitchell, Fink and Pegram, Phys. Rev. 49, 777 (1936).

The extension of both methods to other elements, where the presence of a low level is suspected (Dy, Sm, Gd) is made difficult by the rarity of most of these elements, which forbids the use of absorbers of sufficient dimensions.

2. Other methods

The energy of resonance neutrons may also be estimated by the study of the diffusion of the resonance neutrons in paraffin.⁴² It seems, however, that these methods are hardly accurate enough to give quantitative results; they may chiefly serve to check the relative order of the energies as given by the boron method.

B. Neutron width

The neutron width may be found directly from the resonance activation with the help of (77), if the absolute number of the neutrons q is known. To determine q , we will use the procedure of Amaldi and Fermi, who express q by the thermal activation and cross section. Integrating resonance and thermal activations over the whole paraffin volume, we get from (77), (89) and (82):

$$\Gamma_n = \frac{CE_g^2 \sigma_c}{1.30 \cdot 10^{-18}}, \quad (94)^{43}$$

$$\text{where } C = \frac{1}{2} \frac{\int A_g dV N^{\frac{1}{2}}}{\int B_c dV l_g}. \quad (94a)$$

For purposes of comparison it is better to use the reduced neutron width Γ'_n (neutron width at 1 volt, $\Gamma'_n = \Gamma_n E_g^{-\frac{1}{2}}$)

$$\Gamma'_n = \frac{CE_g^{\frac{3}{2}} \sigma_c}{1.30 \cdot 10^{-18}}. \quad (95)$$

(95) is very useful as it gives an expression for the neutron width which, being independent of ξ , holds irrespective of the form of the line.

In the case of the presence of several resonance levels, the Γ'_n determined from (95) will be considerably too large. Indeed, we have from (95), (90) and (93)

$$(\Gamma'_n)_a = \frac{(\sum \Gamma'_{ng} E_g^{-\frac{1}{2}})^4}{(\sum \Gamma'_{ng} E_g^{-2})^2}, \quad (96)$$

⁴² Halban-Preiswerk, Comptes rendus 202, 849 (1936); Amaldi-Fermi, reference 30.

⁴³ Here and in the following, we shall always incorporate the unknown factor $(1 \pm 1/(2i+1))$ in the neutron width, i.e., write Γ_n instead of $\Gamma_n(1 \pm 1/(2i+1))$.

which is a weighted sum rather than a mean value. This is also clear from the fact that Γ'_n is proportional to the activation. In the special case (cf. above, after (91) and after (93)) that all Γ'_{ng} are equal, and $E_1=D$, we have $(\Gamma'_n)_a = 10.4 \Gamma'_{ng}$.

A way of estimating Γ'_n , if E_g lies near the thermal region, will be discussed for the example of cadmium.

C. *Effective width Γ_r/χ and radiation width Γ_r*

The effective width can be found—as shown by Amaldi and Fermi—from the resonance activation and absorption coefficient for self-indication. For the relative effective width $(\Gamma_r/\chi)/E_g$ we find from (75), (89), (82) and (94a)

$$(\Gamma_r/\chi(\xi))/E_g = C\kappa_c/\kappa_g \quad (97)$$

For $\xi \gg 1$ and $\xi \ll 1$ we may express the effective width by the natural and Doppler width, respectively. Thus (cf. 70 to 72a)

for $\xi \gg 1$:

$$\Gamma_r/E_g = \frac{1}{2}C\kappa_c/\kappa_g \cdot (1 - 1/4\xi^2 + \dots) \quad (97a)$$

for $\xi \ll 1$:

$$\Delta/E_g = (\pi/2)^{\frac{1}{2}} \cdot C\kappa_c/\kappa_g (1 - (8/\pi)^{\frac{1}{2}}\xi + \dots). \quad (97b)$$

The relative width found by Amaldi and Fermi under the assumption of constant cross section both for resonance and thermal neutrons is $2C(\kappa_c/\kappa_g)$, their values have hence to be divided by 4 in the case of natural and by $(8/\pi)^{\frac{1}{2}}$ in the case of Doppler width.⁴⁴ (97) also provides a mean for checking the assumptions $\Gamma_r \ll E_g$, $\Delta \ll E_g$, made for the derivation of the formulae in §5C. (Δ/E_g can, of course, also be directly calculated, *viz.* (63), if E_g is known from the boron method.)

Instead of finding from (97) the effective width, we may use the relation also as an equation for ξ and hence for Γ_r . Dividing both sides of (97) by Δ , we get

$$\frac{\chi(\xi)}{\xi} = 4 \left(\frac{kTm/M}{E_g} \right)^{\frac{1}{2}} \cdot \frac{\kappa_g}{\kappa_c} \cdot \frac{1}{C}, \quad (98)$$

from which ξ may be determined. If, however, ξ is small, (98) is very insensitive, since for $\xi \ll 1$,

$\chi(\xi)$ is proportional to ξ . Thus small natural widths cannot be determined in the presence of large Doppler widths.

The presence of several levels has a similar effect upon the radiation width Γ_r determined from (98) as upon the neutron width Γ_n . If all levels have natural width, the Γ_r found from (98) is

$$(\Gamma_r)_a = \frac{(\sum \Gamma'_{ng} E_g^{-\frac{1}{2}})^4}{(\sum (\Gamma'_{ng}{}^2/\Gamma_{rg}) E_g^{-2}) (\sum \Gamma'_{ng} E_g^{-2})^2}, \quad (99)$$

which, as in the case of $(\Gamma_n)_a$, is a sum rather than a mean value. In the special case that all Γ'_{ng} and all Γ_{rg} are equal, that the levels are equally spaced and $E_1=D$, we have $\Gamma_r = 10.4 \Gamma_{rg}$.

In the formulae derived up to now, the one level formula (61) was only made use of in the immediate neighborhood of the resonance but no relation was assumed between the cross section for thermal neutrons σ_c and the resonance cross section σ_g . If we suppose that the thermal cross section σ_c is determined by the influence of a single resonance level, the comparison between thermal and resonance cross section gives us another method for finding the radiation width.⁴⁵ It is obvious, that this assumption, which involves the extrapolation of the one level formula up to the thermal region, is a much more special one than the assumption discussed before, *viz.* that the measured resonance absorption is mainly due to one compound level only.

Firstly, for the thermal cross section the levels of negative energy E_g , which do not contribute to the absorption, are of as much importance as the levels of positive energy. Secondly, the interference of the capturing amplitudes in (58) may considerably increase the effect of the higher levels. The importance of formulae based on this assumption will therefore consist not so much in that they supply an independent method to determine Γ_r , but rather in that they give a possibility of finding out in actual cases, to what extent the thermal cross section is connected to the measured resonance absorption.

If $E_g \gg kT$, we have from (61), (63) and (70)⁴⁶

⁴⁵ Bethe, Phys. Rev. **49**, 888 (1936).

⁴⁶ The formulae which are based on the extrapolation of the one level formula in the thermal region are marked by an asterisk.

⁴⁴ Cf. however, reference 49.

$$\sigma_c = \frac{\sigma_{0N}}{2} \frac{\Gamma_r^2}{E_g^{\frac{3}{2}} (\pi kT)^{\frac{1}{2}}} = \sigma_g \cdot \frac{8}{\pi^{\frac{1}{2}}} \frac{m}{M} \frac{\xi^2}{\chi(\xi)} \left(\frac{kT}{E_g} \right)^{\frac{1}{2}}. \quad (100)^*$$

This gives for the relative radiation width Γ_r/E_g for $\xi \gg 1$ (cf. 71):

$$\Gamma_r/E_g = (\sigma_c/\sigma_g)^{\frac{1}{2}} (\pi kT/E_g)^{\frac{1}{2}} (1 - 1/8\xi^2 + \dots) \quad (101a)^*$$

for $\xi \ll 1$ (cf. 72, 72a);

$$\frac{\Gamma_r}{E_g} = \frac{\sigma_c}{\sigma_g} \frac{\pi}{8^{\frac{1}{2}}} \left(\frac{M}{m} \right)^{\frac{1}{2}} (1 - (8/\pi)^{\frac{1}{2}} \xi - \dots). \quad (101b)^*$$

More generally, we may consider (100) as an equation for ξ

$$\frac{\chi(\xi)}{\xi^2} = \frac{8}{\pi^{\frac{1}{2}}} \frac{m}{M} \frac{\sigma_g}{\sigma_c} \left(\frac{kT}{E_g} \right)^{\frac{1}{2}}. \quad (102)^*$$

Instead of the ratio of the absorption coefficients, we may also use the ratio of the activations C . Introducing (100) into the expression for the thermal activation (89), or also expressing the ratio of the absorption coefficients in (97) by (100), we get a very simple relation between the ratio of the activations and ξ , *viz.*:

$$\xi = \left(\frac{\pi M}{4 m} \right)^{\frac{1}{2}} / C. \quad (103)^*$$

We have now 3 relations for ξ , the general relation (98) and the two relations (102) and (103) which make use of the assumption (100). Of these 3 relations, however, only two are independent, as for instance by dividing (98) by (103) we get (102). We may find out in each case how far the thermal cross section is connected with the level causing the resonance absorption, by comparing the ξ found from (103) with the ξ supplied by the general formula (98). In this connection attention must be paid to the fact, that the ratio of the two ξ 's depends rather sensitively on the value of C ; e.g., for $\xi \gg 1$, ξ is proportional to C (cf. 98), while ξ_{act} [so we shall denote from now on the ξ determined from (103)] is, proportional to $1/C$, so that, ξ/ξ_{act} is proportional to C^2 .

§7. Discussion of experimental data on slow neutrons

A. $E_g \gg kT$. Complete data on energies, absorption coefficients and activations are available for the elements Rh, Ag (22'' period, group A) and I. The values for the resonance absorption coefficients κ_g and the resonance activations A_g are not very certain, owing to the difficulties of correction for finite thickness of the indicators and absorbers used. The following discussion should therefore be considered as demonstrating how the preceding formulae should be used in the analysis of the experiments, rather than as a derivation of final results.

1. *Rhodium (group D)*.—For the energy, determined by the boron method, we take $E_g = 1.1$ volts.⁴⁷ This corresponds to a Doppler width of 0.066 volt. The absorption coefficient for C neutrons is 0.7 cm²/g,⁴⁸ corresponding to a cross section of $120 \cdot 10^{-24}$ cm². For the ratio of the activations, one finds from Amaldi-Fermi's data $C = 0.77$.⁴⁹ From these data, we get for the neutron width $\Gamma_n = 8.6 \cdot 10^{-5}$ volt and the reduced neutron width $\Gamma'_n = 8.2 \cdot 10^{-5}$ volt.

The resonance absorption coefficient κ_g is given as 4.0 cm²/g by v. Halban,⁵⁰ while Amaldi and Fermi find $\kappa_g = 1.8$. From (97) and (98) we have (the constants following from the latter value of κ_g are put in brackets) $\Gamma_r/\chi = 0.15$ (0.33), $\xi = 0.90$ (2.4), $\Gamma_r = 0.060$ (0.16).

Supposing that the cross section for neutrons of thermal energy is entirely caused by the resonance level, to which the constants now determined refer, we would get from the activations according to (103)* $\xi_{\text{act}} = 11.5$, and from the

⁴⁷ Goldsmith-Rasetti, Phys. Rev. 50, 328 (1936).

⁴⁸ Amaldi and Fermi, reference 30.

⁴⁹ This value is obtained by using Amaldi and Fermi's correction for finite thickness, based on the assumption of a constant resonance absorption coefficient. The correction is numerically rather important, it increases the measured value of C by a factor 1.55 in the present case. The consideration of the fact that due to the variation of the capture cross section in the resonance region—the apparent absorption coefficient of resonance neutrons is lowered by their passage through thick layers, would tend to diminish the corrective factor and thus diminish the value of C . The correct procedure would be to use Amaldi-Fermi's value as a zero approximation in (97) and (98), recorrect it by introducing the line-shape thus determined, and recalculate the constants with the recorrected value of C . The lowering of the above value of C , caused by this effect, may well be upset, if Amaldi-Fermi's value for κ_g , used also for the correction of C , should prove to be too low, as would be the case according to v. Halban (see below).

⁵⁰ Private communication.

cross sections, cf. (102)*, $\xi_\sigma = 3.2$ (5.2). If, therefore, the observed resonance level were the only one contributing to the thermal cross section, the thermal activation and cross section would be by the ratio ξ_{act}/ξ , i.e., 12.8 times (4.8 times), smaller than the observed values.

If we assume, therefore, that the main resonance absorption is caused by a single level at 1.1 volts, we may conclude that—as far as we can trust the measurements and especially if v. Halban's value for the resonance cross section proves to be right—the chief contribution to the thermal cross section does not come from this resonance level, but from other (higher or negative) levels.^{50a}

2. *Silver*.—The resonance neutrons of Ag (22''-period) have been divided into two groups. About two-thirds of the resonance activation are absorbed in a thin sheet of Ag and called group *A*. The other third (group *B*) seems to consist of much harder components.

For the energy of group *A*, we take $E_g = 2.5$ volts,⁴⁷ which gives the Doppler width $\Delta = 0.096$ volt.

The absorption coefficients for thermal and resonance neutrons are³⁰ $\kappa_g = 20 \text{ cm}^2/\text{g}$, $\kappa_c = 0.25 \text{ cm}^2/\text{g}$, the ratio of the activations $C = 1.52$.

Now we must take account of the fact that Ag consists of two isotopes of nearly equal abundance, of which only one absorbs group *A*, giving rise to the 22'' period. Thus we have for the cross section σ_g obtained from κ_g : $\sigma_g = 2\kappa_g M = 7000 \cdot 10^{-24} \text{ cm}^2$. To the cross section in the thermal region both isotopes contribute. The ratio of the thermal activations for the short and the long period was found to be 3 : 1.³⁰ The thermal cross section of the isotope absorbing group *A* is therefore $\sigma_c = \frac{2}{3}\kappa_c M = 70 \cdot 10^{-24} \text{ cm}^2$. The thermal activation measured by Amaldi and Fermi refers to both periods. Therefore, the absorption coefficients to be introduced in (98) are the directly measured absorption coefficients, while in (102) enter the actual cross sections of the capturing isotope. If we use in all the formulae the measured absorption coefficients (or correspondent apparent cross sections $\sigma = M\kappa$) and the thermal activation referring to both periods, then the value of Γ_n and Γ'_n obtained from (94) and (95) have to be multiplied by 2, χ/ξ^2 from (102) by 4/3 and ξ from (103) by $\frac{2}{3}$.

^{50a} *Note added in proof*: The calculations described are very sensitive to the assumptions about the mean free path of thermal and faster neutrons in paraffin. Reasons can be given (reference 19) for the assumption that l_0 is only 0.8 to 0.9 cm rather than 1.1 cm as Amaldi and Fermi assume. This improves the agreement considerably so that it can, at the moment, not be excluded that the chief contribution to the thermal cross section of Rh comes from the "D" level.

Taking these factors into account, one finds $\Gamma_n = 7 \cdot 10^{-4}$ volts, $\Gamma'_n = 4 \cdot 10^{-4}$ volts.

From (98) we get $\chi/\xi = 2.02$. From this, we cannot calculate ξ , because the maximum value of χ/ξ for $\xi = 0$ is $(\pi/2)^{\frac{1}{2}} = 1.25$. We may only conclude that ξ is probably smaller than 1. In other words: the effective width Γ_r/χ determined from (97) turns out to be smaller than the Doppler width calculated from the boron determined energy. Indeed (97) gives $\Gamma_r/\chi = 0.047$ volt, which, if the line has pure Doppler form, would correspond to a Doppler width of 0.060 volt, while the Doppler width from the boron energy is 0.096 volt. This difference may well be due to experimental error.

From the ratio of the activations (103) one finds $\xi_{\text{act}} = 6.0$, from the ratio of the cross sections $\xi = 1$. As in Rh, the one level formula gives too high values for ξ . This means, as already mentioned, that if the resonance level with the width determined from (98) would alone be effective, the activation and cross section in the thermal region would be smaller than observed. The discrepancies are much more striking than in the case of Rh and cannot be removed by considerations as those described in reference 50a.

The existence of higher levels in the Ag isotope which absorbs group *A* is proved by the existence of group *B*, the mean energy of which is estimated by the boron method to be 4.5 volts.⁴⁷ The fact that this group is much less absorbable in Ag seems to show that either the neutron widths of the respective levels are smaller or the radiation width larger than those of group *A*. If the latter is the case, the effect of these levels upon the thermal cross section would be more important than the contribution of the level corresponding to group *A*.

3. *Iodine*.—The data for iodine are the least accurate ones. The resonance energy lies high: $E_g \sim 80$ volts.⁴⁷ The thermal activation has not been given by Fermi and Amaldi, because they eliminated q in this case with the help of the thermal cross section and activation for Rh. Herefrom and from the thermal absorption coefficient $\kappa_c = 0.024 \text{ cm}^2/\text{g}$ ⁵¹ the thermal activation can be recalculated, which gives $C = 2.16$. For the resonance absorption coefficient Amaldi and Fermi give $\kappa_g = 0.38 \text{ cm}^2/\text{g}$.

⁵¹ Frisch, private communication.

The constants derived from these values, together with the constants for Rh and Ag, are shown in Table II. The very high values for the neutron as well as for the radiation width⁵² of iodine suggest, that the resonance activation of iodine is due to a number of levels of about equal effect (case *c*, beginning of §6), in which case the observed width would be something like the sum of the widths of the single levels.

In toto, the whole situation may be compatible with the following picture:

If we consider the series of elements Rh, Ag, In, I, which, being not very different as to their mass and all containing an even number of neutrons and an odd number of protons, should exhibit a certain similarity as to their mean density of energy levels²⁰ although individual variations are by no means excluded. We note that for Rh, Ag and In the boron method gives resonance energies of a few volts, while for iodine the resonance energy lies much higher. It may therefore be that the mean distance of levels in all these elements is of the order of a few volts. For Rh, Ag and In, then, the energy of the first level is of the same order as the mean distance, and the resonance activation of these elements consists therefore of a large effect of one level, the position of which coincides roughly with the measured resonance energy, and smaller contributions of a number of higher levels.

In iodine, on the other hand, the energy of the first levels would be by chance several times larger than the mean distance. Therefore, no single level has a prominent influence upon the resonance activation. The energy measured is then a mean value over many levels, which may extend from about 20 or 30 volts upwards, and the measured widths are the weighted sum of the widths of these levels. It may even be that the actual width of most of these levels is already practically Doppler width.

How far this picture coincides with the facts may be ascertained to a certain extent by the accurate study of the absorption curves with self-indication and the refinement of the boron method by its combination with varied filterings.

⁵² According to recent measurements of Frisch, κ_p seems to be much higher than given by Amaldi and Fermi, i.e., about 2–3 cm²/g. This would reduce considerably the value of the radiation width and also create with respect to the ratio ξ_{act}/ξ a situation more similar to Ag and Rh.

B. $E_0 \approx kT$.

4. *Cadmium*.—The absorption coefficient in boron, measured with boron indication, of the neutrons penetrating cadmium is about 7 times smaller than the boron absorption coefficient for *C* neutrons.⁵³ From this it follows, that Cd has a resonance level at an energy considerably lower than one volt. This energy may, according to these data, be positive as well as negative. Rasetti, Segrè, Fink, Dunning, Pegram⁴¹ have shown that the absorption of a Cd wheel is lowered when it rotates with a velocity component in the direction of the velocity of the incident neutrons. From this it follows that the capture probability in the thermal region increases with increasing energy. This proves, that the level has a positive energy and lies above the thermal energy. Still a little more follows from a quantitative discussion.

The relative change in the absorption measured in this experiment is directly equal to the relative change in the capture probability $\Delta p/p$. We have:

$$\frac{\Delta p}{p} = \frac{1}{p} \frac{dp}{dE} \cdot \Delta E = \frac{2(E_0 - E)}{(E_0 - E)^2 + (\frac{1}{2}\Gamma)^2} \cdot \Delta E \quad (104)$$

(ΔE is the change in the effective energy of the neutrons caused by the rotation of the wheel). We may now define an energy u by

$$u = p/\Delta p \cdot \Delta E. \quad (105)$$

Then we have from (104)

$$u = \frac{(\frac{1}{2}\Gamma)^2 + (E_0 - E)^2}{2(E_0 - E)} \quad (106)$$

From (106) we see, that, if $u > 0$

$$E_0 - E < 2u, \quad (107a)$$

$$\Gamma < 2u, \quad (107b)$$

$$(E_0 - E)^2 + (\frac{1}{2}\Gamma)^2 < 4u^2. \quad (107c)$$

From the measurements, u comes out to be 0.10 volt, hence

$$E_0 - E < 0.20 \text{ volt}, \quad (108a)$$

$$\Gamma < 0.20 \text{ volt}, \quad (108b)$$

$$((E_0 - E)^2 + (\frac{1}{2}\Gamma)^2)^{\frac{1}{2}} < 0.20 \text{ volt}. \quad (108c)$$

⁵³ Frisch and Placzek, reference 33, Collie, reference 38.

We may now try to get further information by using Amaldi-Fermi's cadmium absorption curve measured with Rh indication. From this curve, the Cd absorption coefficient for the Rh resonance (D) neutrons may be estimated to be about 270 times smaller than for the C neutrons. Assuming that the Cd cross section for D neutrons is entirely determined by the resonance level near thermal energy, which is of course very doubtful, especially in the light of the evidence of section A, we have from (61):

$$\frac{\sigma_D}{\sigma_{th}} = \frac{E_{th}^{\frac{1}{2}} (E_g - E_{th})^2 + (\frac{1}{2}\Gamma)^2}{E_D^{\frac{1}{2}} (E_D - E_g)^2} \quad (109)$$

Combining (109) with (106), we find $E_g = 0.14$ volt, $\Gamma = 0.20$ volt. With these values, the cross section (cf. Fig. 1) would have a very flat minimum at about 0.05 volt and a very flat maximum at 0.12 volt, the maximum being only about 12 percent higher than the minimum.

To find the neutron width, we must decide to which of the many Cd isotopes the resonance level is to be ascribed. The most probable assumption is, that it belongs to an isotope with odd mass, as these isotopes have the largest level density.²⁰ As both odd isotopes (111 and 113) have about equal abundance (12 percent and 10 percent, respectively), it does not matter very much for the cross section, to which of the two isotopes the level belongs.

Inserting the thermal cross section, which—corrected for the abundance with this assumption—is $25,000 \cdot 10^{-24}$ cm², and the above values for E_g and Γ_r into (61), we find for the cross section at resonance $\sigma_{0N} = 25,000 \cdot 10^{-24}$ cm², i.e., the same value as at thermal energy. This gives for the neutron width $\Gamma_n = 5 \cdot 10^{-4}$ volt and the reduced neutron width $\Gamma'_n = 1.5 \cdot 10^{-3}$ volt.

§8. Fast neutrons⁵⁴

The present theory is quantitatively applicable to fast neutrons only as long as the mean distance of the compound levels remains large compared to the neutron and the radiation width,^{54a} i.e., $\Gamma_n \ll D$, $\Gamma_r \ll D$.

Even in the energy region where this condition still holds, the cross section will in general exhibit no maxima because of the Doppler effect. At an energy of 2 million volts, the Doppler width for an element of mass 100 will be about 100 volts, which is probably more than the mean level distance of most of the elements in this region. Apart from this fact, the actual inhomogeneity of fast neutron beams will, even with artificial production, not be lower than a few thousand volts, so that resonance phenomena, even if present, may only be detected if D is larger than this energy inhomogeneity. In all other cases, a mean absorption will be observed, given by⁵⁵

$$\sigma = \frac{\pi^2 R^2 \Gamma_n}{2i+1 D} \quad (110)$$

Γ_n is the mean neutron width of the levels in the energy region concerned, and R the nuclear radius. The formula is obtained by considering the influence of the various orbital momenta of the incident particle. (110) comprises capture and scattering except potential scattering. As already emphasized by Bohr,¹ most of the scattering (except the potential scattering) will be inelastic.⁵⁶

⁵⁴ For a detailed discussion of the properties of fast neutrons we refer to Bohr and Kalckar, reference 1.

^{54a} For the case of large width, cf. reference 19.

⁵⁵ Cf., Bethe, Phys. Rev. 50, 332 (1936); Bethe and Livingston, reference 19.

⁵⁶ For the experimental evidence, cf. Werthenstein, Nature 135, 747 (1935); Ehrenberg, Nature 136, 870 (1935).

TABLE II.

Element	E_g	K_g	K_c	C	$\sigma_g \cdot 10^{24}$	$\sigma_c \cdot 10^{24}$	Δ	Γ_r	$\Gamma_n \cdot 10^4$	$\Gamma'_n \cdot 10^4$	ξ	ξ_{act}
Rh	1.1 ^a	1.8 ^b	0.7 ^b	0.77	310	120	0.066	0.16	0.86	0.82	2.4	11.5
Ag(22'')		4.0 ^c	0.25 ^b	1.52	680			0.060			6.6	
I	2.5 ^a	20 ^b	0.024 ^d	2.2	7100	67	0.096	<0.05	500	56	<<1	6
Cd	80 ^a	0.38 ^b			80	5	0.50	5.5	5	15	11	4.6
	0.14		13.5			25,000		0.2				

^a Goldsmith-Rasetti, reference 46.

^b Amaldi-Fermi, reference 30.

^c v. Halban, reference 38.

^d Frisch, reference 51.

Two kinds of inelastic processes may occur. Either the compound level created by the capture of a fast neutron may emit a neutron of lower energy, leaving the final nucleus in an excited state, which then emits a γ quantum (ordinary inelastic scattering). Or the compound level originally formed may first go over by γ -ray emission to a lower state of the compound nucleus, which then decays further under emission of a neutron (inelastic scattering of the second kind). The latter type of processes must be taken into account for the calculation of the capture cross section: It has a lowering effect on the probability of capture because not all radiative processes lead to capture. However, it is likely that the probability of the inelastic scattering of the second kind is smaller than the capture cross section, because the γ -rays, emitted have lower frequency in the case of the scattering of the second kind, and the emission probability is *cet. par.* proportional to the third power of the frequency of the γ -ray. Only if the neutrons have very high energies, will the inelastic scattering of the second kind be important.

Experimentally, not very much is known about the capture of fast neutrons. The earlier conception, according to which the observed capture effects should be attributed to the effect of small admixtures of slow neutrons,⁵⁷ seems not to agree very well with numerosity considerations, and it is more likely, that at least a part of the observed effects is to be ascribed to the fast neutrons. Conclusions about the ratio of neutron and radiation width in the range of high neutron energies, based on the comparison of capture and scattering cross section, are therefore hardly possible.

In this state of affairs, it seems more promising to estimate the neutron width at high energies by simply extrapolating the data from slow neutron experiments. Assuming the $E^{1/2}$ law for the neutron width which will hold as long as the wave-length is large compared to nuclear dimensions, and supposing, as in §4, that there are no systematic differences between the matrix elements for high and for low energies, we would get from the slow neutron data for Rh and Ag a neutron width of 0.1–1 volt for energies of some million volts. This would mean, that in that region neutron and radiation width have about the same order of magnitude. However, it must be kept in mind that this extrapolation gives only the *partial* neutron width corresponding to a process in which the final nucleus is left in the *ground* state. If the neutron energy is sufficient for inelastic scattering, i.e., if the final

nucleus may be left in excited states after the reemission of the neutron, the total neutron width will be much larger, approximately in proportion to the number of possible final levels. Then the inelastic scattering will be much more probable than the capture.

§9. Elastic scattering of neutrons

The theoretical discussion of the scattering has been given in §4. Experimentally, the scattering of slow neutrons, of thermal as well as higher energies, has been investigated.⁵⁸ The cross sections observed with various elements are of the order of magnitude of nuclear dimensions. No case of resonance scattering has been found thus far.

For fast neutrons, it is again difficult to separate inelastic⁵⁶ from elastic scattering. Assuming that the "total absorption" of fast neutrons observed⁵⁹ is due to elastic scattering, Rabi⁶⁰ found good agreement with the theoretical potential scattering for hard spheres. This corresponds to the assumption about the potential which we showed in §4 to be most likely correct.

§10. Transmutations involving charged particles

If the particle Q is charged, the matrix element $U^r_q J_q$ involves two factors, firstly a factor due to the electrostatic potential barrier between the particle and the nucleus, and secondly, a factor giving the probability of the concentration of the nuclear energy on the particle Q . The latter factor is quite analogous to the case of neutrons, and is the more interesting part. In order to obtain it from the experimental data, the first (penetration) factor must be split off, using the well-known formulae for the penetration through a potential barrier. A convenient form for the penetration factor is¹⁹

⁵⁸ Dunning, Pegram, Fink and D. P. Mitchell, Phys. Rev. **48**, 265 (1935). Wick and Pontecorvo, Ric. Sci. **2**, 3/4 and 4/5 (1936). A. C. G. Mitchell and Murphy, Phys. Rev. **48**, 653 (1935). A. C. G. Mitchell, Murphy and Longer, Phys. Rev. **49**, 401 (1936). A. C. G. Mitchell, Murphy and Whittaker, Phys. Rev. **50**, 132 (1936).

⁵⁹ Dunning *et al.* reference 58.

⁶⁰ Rabi, Phys. Rev. **43**, 838 (1933). It may seem satisfactory that Rabi deduced from the scattering effective radii somewhat larger than the then accepted nuclear radius whereas we have shown (§4) that the cross section of the potential scattering should be slightly smaller than the geometrical cross section of the nucleus. This difference would compensate for the increase in the nuclear radius necessitated by the Bohr theory (cf. reference 55).

⁵⁷ Bethe, Phys. Rev. **47**, 797 (1935).

$$P_Q = \exp[-2g\gamma(E_Q/U_0)] \quad (111)$$

where $g = 0.38(M_Q Z_Q)^{1/2} Z_B^{1/2} M_B^{1/6}$, (111a)

$$\gamma(x) = x^{-1/2} \arccos x^{1/2} - (1-x)^{1/2} \quad (111b)$$

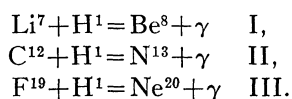
with M_Q and M_B the atomic weights of the particle Q and the nucleus B ($O^{16} = 16$), Z_Q and Z_B the respective nuclear charges and 0.38 a constant derived from the observed radii of heavy nuclei. A curve for $\gamma(x)$ is given in the report by Bethe and Livingston.¹⁹ E_Q is the absolute kinetic energy of the incident particle, and U_0 the height of the potential barrier,

$$U_0 = 0.70 Z_Q Z_B (M_B + M_Q) / M_B^{4/3} \text{ MV.} \quad (111c)$$

Since the penetrability (111) falls very rapidly with decreasing particle energy, charged particles can only appear in nuclear reactions if their energy is not too small compared to the height of the potential barrier. (For quantitative data, cf. reference 19.) Because of the considerable energy required, it is not possible to define the energy of the incident particle accurately enough to observe resonances if the resonance levels are closely spaced. Therefore *no resonance effects will be observable with charged particles and heavy nuclei* (cf., however, the case of Pt below).

On the other hand, if the energy is high enough to make the process observable, there is still a rapid increase of the probability with increasing particle energy. This increase is apt to mask resonance effects if the resonance is broad and not very pronounced. Consequently, the best chance for observing resonance with charged particles is if the resonance is narrow. The process most suitable for resonance is therefore the simple capture of particles with the emission of γ -rays, provided that no other process is possible for the given energy and angular momentum of the incident particle.

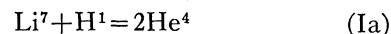
In fact, the most striking resonance effects have been observed for the simple capture of protons by nuclei. The processes studied in detail are⁶¹



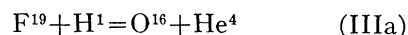
⁶¹ Hafstad and Tuve, Phys. Rev. **47**, 506 (1935). Hafstad, Heydenburg and Tuve, Phys. Rev. **50**, 504 (1936). For detailed theoretical discussions, see Hafstad, Heydenburg and Tuve; Bohr and Kalckar, reference 1.

For process I and III, the width of the resonance has been measured and found to be small, for II the resonance is also known to be narrow. The reactions have been studied by measuring the γ -rays.

In case II, the capture reaction is the only one which can occur on energetic grounds. All other reactions, e.g., $\text{C}^{12} + \text{H}^1 = \text{B}^9 + \text{He}^4$, $\text{C}^{12} + \text{H}^1 = \text{C}^{11} + \text{H}^2$, $\text{C}^{12} + \text{H}^1 = \text{N}^{12} + n^1$ would be very highly endoergic and can therefore not take place with protons of relatively low energy (below 1 MV). In case I and III, other reactions are energetically allowed, *viz.* in the first case,



and in case III



However, there are very strict selection rules for parity and angular momentum, particularly in case I.

The wave function of the α -particle in its ground state will certainly be even. The same will be true of the wave function describing the relative motion of the two α -particles formed in reaction Ia, because they obey Bose statistics. Therefore only *even* states of the compound nucleus Be^8 can disintegrate into two α -particles. Similarly, the spin of the α -particle is zero. The wave function of the relative motion of two α -particles contains only even spherical harmonics, again because it is symmetrical in the two particles (Bose statistics). Therefore a state of Be^8 must have even angular momentum J in order to disintegrate into two α -particles. For compound states with odd parity or odd J , reaction Ia is impossible, such states can only disintegrate into the original particles $\text{Li}^7 + \text{H}^1$, or into a light quantum plus a Be^8 nucleus in a lower state.

In case IIIa, the situation is not quite so clear. It is true that the ground state of O^{16} , just as that of He^4 , is even and has spin zero. But the wave function describing the relative motion of O^{16} and He^4 has no particular parity and contains in principle terms of all angular momenta because there are no symmetry requirements. Possibly, the compound state responsible for the resonance observed in case III has very high angular

momentum so that its disintegration probability into $O^{16} + He^4$ is small though not negligible.

The observed width of the resonance level in reaction I, is 11 kv. The contribution of the emission of γ -rays to the width is negligible, from estimates of the absolute yield of the capture process it turns out to be a few volts⁶¹ (see below) which is compatible with the neutron evidence (§7). The observed width is therefore to be attributed to the process $Be^8 \rightarrow Li^7 + H^1$. To obtain the width free from the influence of the Coulomb field, we divide the observed width by the penetrability of the potential barrier. Inserting $Z_Q = 1$, $Z_B = 3$, $M_Q = 1$, we find $g = 0.83$ and for the height of the potential barrier $U_0 = 1.25$ MV. With $E = 0.44$ MV for the energy of the incident proton, we have $x = 0.35$ and $\gamma(x) = 0.78$ and, according to (111), the penetrability becomes $P = e^{-1.66 \cdot 0.78} = 1/3.6$. Therefore the width of the level would be $3.6 \cdot 11,000 = 40,000$ volts if the potential barrier were absent.

The "width for one volt energy without potential barrier"

$$\gamma' = \gamma E^{-\frac{1}{2}} \quad (112)$$

is, in our case, $\gamma' = 40,000/340,000^{\frac{1}{2}} = 70$ volts.⁶² This value is, of course, very much (about 100,000 times) larger than the width of neutron resonance levels in heavy nuclei (§7) corresponding to the much larger average distance between levels, which, for Be^8 , may be of the order of one MV or more, as compared to a few volts in heavy nuclei.

In the neighborhood of the resonance, the influence of other levels of the compound nucleus can certainly be neglected so that the one-level formula (12) may be used. The angular momenta are in our case $s = \frac{1}{2}$ (proton), $i = 3/2$ (Li^7) and very probably $J = 1$ (excited Be^8). The width $\gamma'_{p^J_p}$ corresponds to the emission of protons such that the residual Li^7 is left in the ground state. But with the energy available, it is impossible that protons can be emitted by the compound nucleus and at the same time the Li^7 nucleus be left in an excited state. Therefore $\gamma'_{p^J_p}$ is identical with the total proton width of the level and thus practically identical with the total over-all

⁶² $340,000 = 440,000 \cdot (7/8)^2$ volts is the relative kinetic energy of the proton with respect to the center of gravity.

width γ , because the γ -ray width is negligible and other particles cannot be emitted.

We sum (12) over all possible excited states g in which the nucleus may be left after emission of the γ -ray,⁶³ insert the values for s , i and J and obtain

$$\sigma_c = \frac{3}{2} \pi \lambda^2 \frac{\gamma \gamma_r}{4(E - E_0)^2 + \gamma^2}, \quad (113)$$

where γ_r is the radiation width, γ the proton width and E_0 the energy of the resonance level. At exact resonance,

$$\sigma_{res} = \frac{3}{2} \pi \lambda^2 \gamma_r / \gamma. \quad (114)$$

The experimental cross section is not well known; a crude determination⁶¹ gave 10^{-27} cm². Since the resonance energy is 440 kv, we have $\lambda = 7.8 \cdot 10^{-13}$ (reduced proton mass!) and

$$\gamma_r = \frac{10^{-27}}{280 \cdot 10^{-26}} \gamma = 3.5 \cdot 10^{-4} \gamma = 4 \text{ volts.} \quad (115)$$

The scattering of protons will also be influenced by the resonance level; we have

$$\sigma_{sc} = \frac{3}{2} \pi \lambda^2 \frac{\gamma^2}{4(E - E_0)^2 + \gamma^2}. \quad (116)$$

At exact resonance, $\sigma_{sc} = (3/2) \pi \lambda^2$. With the given resonance energy of 440 kv, we have therefore $\sigma = 2.8 \cdot 10^{-24}$ cm². This is to be compared to the Rutherford scattering in the Coulomb field which is about $1 \cdot 10^{-24}$ cm² for backward scattering and $4 \cdot 10^{-24}$ cm² for scattering at right angles. Thus the increase in the scattering cross section at resonance should be observable.

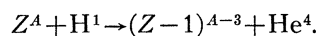
The apparent width of the resonance in the capture of protons by fluorine (case III above) in the experiments of Halfstad and Tuve is equal to the inhomogeneity of the proton beam, i.e., about 4 kv for the resonance at 330 kv, and about 15 kv for the levels at 890 and 940 kv.

⁶³ In the particular case of the capture of protons by Li^7 , there seems to be only one state in which the final nucleus may be left, *viz.* the ground state. At least the most recent experiments (Fowler, Delsasso and Lauritsen, Phys. Rev. 50, 389 (1936)) make it probable that only a single γ -ray line of 17 MV energy is emitted in the reaction. *Note added in proof:* This conclusion is again made doubtful by the results of Crane, Phys. Rev. 51, 49 (1937).

Therefore, it can only be said that the actual width of the resonance levels must be smaller than, say, 3 and 10 kv, respectively. The widths should, at least for the lower resonance level, in fact be expected to be much smaller than for the $\text{Li}^7 + \text{H}$ -reaction; firstly because of the higher potential barrier of F, and secondly because the energy levels of the heavier compound nucleus Ne^{20} are denser and therefore would be narrower even without the higher potential barrier. For fluorine, we have (cf. (111)) $g = 1.77$ and $U_0 = 2.50$. Therefore the penetrability becomes $P = 1/5000$ for 330 kv and $P = 1/13$ for 920 kv (average of 890 and 940). Thus we conclude from the experimental results that the widths without potential barrier would be smaller than $3 \cdot 5000 = 15,000$ kv and $10 \cdot 13 = 130$ kv, respectively. The very large result obtained for the level at 330 kv means that this level must actually be very much narrower than 3 kv, it is probably only a few volts wide. The second figure obtained would correspond to a γ' (cf. 112) of $130,000/830,000^{\frac{3}{2}} = 140$ volts which also seems to be too large compared to the width $\gamma' = 80$ volts of the Be^8 level (cf. above); it should be expected that γ' is smaller for the compound nucleus Ne^{20} than for Be^8 ; therefore the actual width should be less than one-half the upper limit given by the experiments of Hafstad, Heydenburg and Tuve. Thus the experiments are compatible with simple theoretical considerations. It must, however, be kept in mind that the theoretical results were obtained under the assumption that the width is entirely due to the proton, which need not be true (cf. above, selection rules).

Apart from the three cases mentioned, simple capture of protons has also been observed for B^{10} (detected by the radioactivity of C^{11})⁶⁴ and for Be^9 and B^{11} (detected by the γ -rays).⁶⁵ The process does no doubt occur with practically every initial nucleus, and will presumably show resonance features in most cases. However, in some cases no resonance level in the experimentally investigated region may be available; in other cases there may be other processes allowed besides the simple capture, and these

other processes may lead to the emission of fast particles and may make the resonance very broad and accordingly the maximum cross section very small. The alternative reaction most likely to occur is the type giving α -particles, according to the scheme

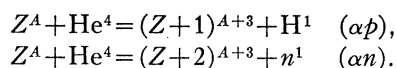


No reaction of this type is energetically possible for N^{14} and O^{16} because the resulting nuclei C^{11} and N^{13} would have too high internal energies. Therefore the simple capture of protons by N^{14} and O^{16} should show sharp resonances provided there are suitably situated resonance levels in the respective compound nuclei O^{15} and F^{17} .

Simple capture of other projectiles, notably deuterons and α -particles, has never been observed and should be hard to observe. In particular deuterons can, because of their large internal energy, cause a great variety of more probable reactions with almost every nucleus, *viz.* the reactions leading to the emission of neutrons, protons and α -particles, respectively. This prevents the existence of sharp resonance levels. Moreover, the three reactions mentioned will, in many cases, leave the residual nucleus in an excited state, so that γ -rays are given off after the reactions. This makes it difficult to identify the simple capture process by observing the γ -rays except if very accurate measurements of their energy are available. Finally, the nucleus produced by the simple capture will usually not be radioactive so that the radioactivity is also not available for the detection of the simple capture process.

The same arguments apply to the simple capture of α -particles. However, in this case there are two nuclei for which the simple capture might be observable, *viz.* C^{12} and O^{16} . These two nuclei cannot be disintegrated by α -particles in any other way. The greatest difficulty with α -particles is the small intensity available.

Resonance phenomena have also been observed for processes other than the simple capture, *viz.* for reactions produced by α -particles and giving protons and neutrons, according to the schemes



⁶⁴ Crane and Lauritsen, Phys. Rev. **45**, 497 (1934).

⁶⁵ Crane, Delsasso, Fowler and Lauritsen, Phys. Rev. **47**, 782 and **48**, 102 (1935).

For the αp reaction, resonance has been observed⁶⁶ with B¹⁰, N¹⁴, F¹⁹, Na²³, Mg, Al²⁷. In the case of the αn reaction, the only case where resonance has been observed with certainty, is Be⁹+He⁴→C¹²+n¹. The case best investigated is the αp -reaction Al²⁷+He⁴=Si³⁰+H¹. Seven resonance levels are known.⁶⁷ The average spacing is about 300,000 volts which seems very reasonable. The width of the levels apparent in the experiments is about half this amount, but no doubt the greater part of this figure is due to the inhomogeneity of the incident beam and the actual width of the levels is probably quite small.

A fluctuation of the yield with energy has also been reported for the radioactivity induced in Pt by deuteron bombardment.⁶⁸ This is certainly not a resonance phenomenon. For, with deuterons of about 4 MV energy, the energy of the compound nucleus is at least about 15 MV above the ground state. Considering the high atomic weight of Pt, the energy levels of the compound nucleus must be exceedingly dense, their spacing should be only a small fraction of a volt.²⁰ On the other hand, the reported maxima and minima in the yield are spaced several hundred thousand volts apart. However, it is quite possible that there are slow fluctuations of the cross section, due to fluctuations in the density of levels or in the magnitude of the matrix elements.

§11. Phenomena produced by γ -rays

As mentioned already in §3, there are two phenomena which may be caused by γ -rays: The photodissociation, and the scattering of γ -rays by the nucleus.

If the photodissociation were the inverse of the capture of particles, it would be exceedingly probable. E.g., for the capture of resonance neutrons we have found (§7) cross sections up to the order of 10⁻²⁰ cm² the neutron energy being around 2 volts. According to (42), the probabilities of two mutually inverse processes are apart from simple statistical factors of the order unity, proportional to the squares of the

wave-lengths of the respective incident particles. Now for neutrons of 2 volt energy, λ is of the order 3·10⁻¹⁰ cm, whereas for γ -rays of 10 MV energy (corresponding to the binding energy of neutrons in heavy nuclei) we have $\lambda_\gamma \approx 2 \cdot 10^{-12}$ cm. Therefore $(\lambda_\gamma/\lambda_n)^2 = 1/20,000$, and the cross section for photodissociation would turn out to be of the order 10⁻²⁵ to 10⁻²⁴ cm², i.e., an exceedingly large cross section compared to other phenomena produced by γ -rays.

In reality, however, the photodissociation is not the inverse of the neutron capture, as already mentioned briefly at the end of §3. When a slow neutron is captured, any of a great number of different γ -rays may be emitted, corresponding to different excitation states q of the final nucleus B . Only for a very small fraction of all capture processes, the final nucleus will be left in the ground state. In the photodissociation, we start from a nucleus B in the ground state and irradiate it with γ -rays of the appropriate energy. The photodissociation is thus the inverse of that part of the capture process in which the final nucleus is left in the ground state. A simple estimate shows that the cross section for photodissociation is thereby reduced by a factor of about 10⁵ which would make it practically unobservable for the neutron energy of 2 volts used above.

If only one level of the compound nucleus contributes, the cross section for the photoelectric dissociation is (cf. 12)

$$\sigma = \frac{\pi \lambda_\gamma^2 (2J+1)}{2(2i+1)} \frac{\gamma_{r0} \gamma_n}{(h\nu - E_0)^2 + \frac{1}{4}(\gamma_r + \gamma_n)^2}, \quad (117)$$

where λ_γ is the wave-length of the γ -ray, i and J are the angular momenta of the initial nucleus in the ground state and in the excited ("compound") state to which it is raised by absorbing the γ -ray and $h\nu$ and E_0 are the energies of the γ -ray and the compound state. γ_n is the width of the compound level corresponding to the emission of neutrons, or generally of the particles which are formed in the photodissociation, γ_r the total γ -ray width and γ_{r0} that part of the γ -ray width which corresponds to a transition to the ground state. For not very light nuclei, the spacing between the resonance levels is so narrow that it is impossible to define the energy

⁶⁶ For references to the very extensive literature, cf. the report by Bethe and Livingston, reference 19.

⁶⁷ Chadwick and Feather, *Internat. Conf. on Physics* (London, 1934), p. 95.

⁶⁸ Cork and Lawrence, *Phys. Rev.* **49**, 788 (1936).

of the γ -ray accurately enough to obtain resonance. Therefore we must average (117) over an energy region large compared to the spacing D between levels, and obtain

$$\sigma_{av} = \pi^2 \lambda_\gamma^2 \frac{(2J+1)_{av}}{2i+1} \frac{\gamma_{r0}\gamma_n}{(\gamma_r + \gamma_n)D}. \quad (118)$$

Since J may have the values $i-1$, i or $i+1$, we may put the average value of $(2J+1)/(2i+1)$ equal to unity.

(118) is obviously very small if $\gamma_n \ll \gamma_r$, i.e., if the dissociation leads to the emission of slow particles. If the particles emitted are fast,⁶⁹ and particularly if the residual nucleus may be left in several excited states, γ_n will be larger than γ_r . Then (118) reduces to

$$\sigma_a = \pi^2 \lambda_\gamma^2 \gamma_{r0}/D. \quad (119)$$

In order to estimate γ_{r0} , we assume that, in the average, the partial radiation width of the compound state corresponding to the transition to *any* lower state is, in the average, simply proportional to the third power of the frequency of the γ -ray. Then the ratio of the total radiation width γ_r to the contribution of the ground state γ_{r0} is of the same order as but smaller than the number of energy levels below the compound state in question, which again is of the order of but smaller than $h\nu/D$. Thus we may write

$$\gamma_{r0}/\gamma = \kappa D/h\nu, \quad (120)$$

where κ is a numerical factor which may be of

the order 100 to 1000. Thus

$$\sigma_a = \pi^2 \kappa \lambda_\gamma^2 \gamma_r/h\nu. \quad (121)$$

With $h\nu = 10$ MV, $\lambda_\gamma = 2 \cdot 10^{-12}$, $\kappa = 500$, we have

$$\sigma_a = 2 \cdot 10^{-27} \gamma_r \text{ cm}^2. \quad (121a)$$

Assuming γ_r to be about 0.1 volt, the cross section for photodissociation would be of the order $2 \cdot 10^{-28}$ cm² which would be just observable if γ -rays of sufficient energy (10 MV) are available with an intensity of the same order as that of the Th C' γ -rays. It should be mentioned that the product nucleus may be left in an excited state if the energy of the γ -ray is sufficient. This does not increase the probability of the photodissociation.

The *scattering* cross section for γ -rays is to the cross section for photodissociation as γ_r to γ_n . Not making the assumption $\gamma_r \ll \gamma_n$, we have therefore, analogous to (121), for the average cross section

$$(\sigma_{sc})_a = 2 \cdot 10^{-27} \gamma_r^2 / (\gamma_r + \gamma_n). \quad (122)$$

This is of the order of 10^{-28} cm² if $\gamma_n < \gamma_r$, i.e., if photodissociation is either energetically quite impossible or leads to the emission of slow neutrons. (122) is still smaller if the γ -rays are energetic enough to produce fast neutrons by photodissociation so that $\gamma_n > \gamma_r$. In any case, the nuclear scattering of γ -rays is entirely negligible compared to the Klein-Nishina scattering which is of the order 10^{-26} cm².

APPENDIX

In this appendix we shall consider the influence of the angular momenta of the various nuclei involved, on the probability of the nuclear processes. We denote by J the angular momentum of the compound nucleus C , by M its component in a given (z -) direction, and by r all quantum numbers of the state of the compound nucleus except the angular momentum. Similarly, i , m , p are the total angular momentum, its z component, and a symbol abbreviating the other quantum numbers for the initial nucleus A , $i'm'q$ have the same significance for the final nucleus B . The incident particle P may be characterized by an orbital momentum l , a spin s , their components μ_l and μ_s , their resultant (total angular momentum) j and its z component

⁶⁹ According to the end of §8, the neutron width will be of the same order as the γ -ray width for neutron energies of about 1 MV or more. This is true if the residual nucleus after the emission of the neutron can only be left in the ground state. If it can also be left in an excited state, a smaller neutron energy is sufficient to make $\gamma_n = \gamma_r$.

μ . The quantities $l's'\mu_l\mu_s'j'\mu'$ refer to the outgoing particle Q . We shall also use, for incident and outgoing particle, plane waves. In this case, the state of the incident particle is defined by a unit vector $\mathbf{\kappa} = \mathbf{k}/k$ in its direction of motion, and by s and μ_s . If a plane wave is used to describe the incident particle, the z axis (axis of quantization) will be chosen parallel to the direction of motion; if a spherical wave of given l is used, the direction of z is arbitrary.

We assume first that the incident particle has a wave function corresponding to a definite j and μ . The probability of the elementary process is (cf. (1))

$$w^A_B{}^p{}_q{}^i{}_i{}^m{}_m{}^P{}_Q{}^j{}_j{}^{\mu}{}_{\mu'} = \frac{2\pi}{\hbar} \times \left| \sum_{rJM} \frac{H^A_{r^p j^i M^m P} H^r_B{}^J{}^M{}_{i'm'Q}{}^j{}^{\mu}{}_{\mu'}}{E_A + E_P - E_r J + \pi i \sum_{Q'q'i'm''j''\mu''} |H^r_B{}^J{}^M{}_{i'm''Q'}{}^j{}^{\mu}{}_{\mu''}|^2} \right|^2. \quad (201)$$

We are interested in the *total* transition probability to *all* substates $m'\mu'$ of the final system, averaged over all possible directions of the angular momentum of the initial nucleus and the incident particle, i.e. over all m and μ . This probability is

$$w^A_{B^p q^i i' P Q^j j'} = \frac{1}{(2i+1)(2j+1)} \sum_{m\mu m'\mu'} w^A_{B^p q^i i' m' P Q^j j' \mu'}. \quad (202)$$

The matrix elements may be written

$$H^r_{A^J p^M i m P j \mu} = w^r_{A^J p i P j} C^J_{i^M m j \mu}, \quad (203)$$

where the coefficients C depend only on spacial symmetry while w depends on the special properties of the system considered. The C 's fulfill the well-known orthogonality relation

$$\sum_{m\mu} C^J_{i^M m j \mu} C^{J'}_{i'^M m' j' \mu} = \delta_{JJ'} \delta_{MM'} c^J_{ij}. \quad (204)$$

The normalization coefficient c^J_{ij} is arbitrary; it is convenient to choose

$$c^J_{ij} = 1/2\pi. \quad (205)$$

This also normalizes the quantities u introduced in (203).

With (203) to (205) we get immediately

$$\begin{aligned} \gamma^r_{A^J p i P j} &= 2\pi \sum_{m\mu} |H^r_{A^J p^M i m P j \mu}|^2 \\ &= (w^r_{A^J p i P j})^2 2\pi \sum_{m\mu} |C^J_{i^M m j \mu}|^2 = (w^r_{A^J p i P j})^2, \end{aligned} \quad (206)$$

so that the last term in the denominator of (201) becomes

$$\frac{1}{2} i \gamma_{rJ} = \frac{1}{2} i \sum_{Q^i q^i i' Q^j j'} \gamma^r_{B^J q^i i' Q^j j'} = \frac{1}{2} i \sum (w^r_{B^J q^i i' Q^j j'})^2. \quad (207)$$

Considering that H is Hermitian, i.e.,

$$H^A_{r^p j^i M^m P j \mu} = H^r_{A^J p^M i^* m P j \mu}, \quad (208)$$

we have further

$$\begin{aligned} w^A_{B^p q^i i' P Q^j j'} &= \frac{2\pi}{(2i+1)(2j+1)\hbar} \sum_{m\mu m'\mu'} \\ &\times \left| \sum_{rJM} \frac{w^r_{A^J p i P j} w^r_{B^J q^i i' Q^j} C^J_{i^M m^* j \mu} C^J_{i'^M m' j' \mu'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2 \\ &= \frac{2\pi}{(2i+1)(2j+1)\hbar} \sum_{JJ'} S_J S^{*J'} \sum_{mm'\mu\mu' MM'} \\ &\times C^J_{i^M m^* j \mu} C^J_{i'^M m' j' \mu'} C^{J'}_{i'^M m' j' \mu'} C^{J'}_{i^M m^* j \mu} \end{aligned} \quad (209)$$

$$\text{with } S_J = \sum_r \frac{w^r_{A^J p i P j} w^r_{B^J q^i i' Q^j}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}}. \quad (210)$$

The last sum in (209) may be summed first over m and μ , then over m' and μ' , using the relations (204), (205). The result is

$$(2\pi)^{-2} \sum_{MM'} \delta_{MM'} \delta_{JJ'} = (2\pi)^{-2} (2J+1) \delta_{JJ'}. \quad (211)$$

This relation shows that the contributions of *compound states of different angular momentum JJ' do not interfere*. The probability w becomes

$$\begin{aligned} w^A_{B^p q^i i' P Q^j j'} &= \frac{1}{2\pi\hbar(2i+1)(2j+1)} \sum_J (2J+1) \\ &\times \left| \sum_r \frac{w^r_{A^J p i P j} w^r_{B^J q^i i' Q^j}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2. \end{aligned} \quad (212)$$

It may be mentioned that it is not necessary to average

over the magnetic quantum number μ of the incident particle but that the result (212) applies also to each magnetic state μ separately. This may be shown by a somewhat more detailed investigation of the properties of the C 's.

We now consider the case that the initial and final state of the particle is characterised by definite values of $l s \mu_l$ and μ_s rather than of j and μ . This case may be reduced to the one previously treated with the help of the C 's. These coefficients appear in all problems in which two angular momenta are compounded to form a resultant, such as i and j to form J in our previous problem, and l and s to form j in our present one. Accordingly, we may write

$$\begin{aligned} H^r_{A^J p^M i P l \mu_l s \mu_s} &= \sum_{j\mu} C^j_{l \mu_l s \mu_s} H^r_{A^J p^M i P l s j \mu} \\ &= \sum_{j\mu} C^j_{l \mu_l s \mu_s} C^J_{i^M m j \mu} w^r_{A^J p i P l s j}. \end{aligned} \quad (213)$$

With the help of a calculation on the same lines as our previous one, which we shall give below, it may be shown that states of different j of the incident particle do not interfere, just as states of the compound nucleus of different J do not. The same holds, of course, for states of different j' of the outgoing particle. Consequently,

(1) the total probability of the transitions to all states of the outgoing particle with given l' and s' is obtained by *summing* the probabilities for all values of j' which are possible for the given l' and s' , i.e. $j' = |l' - s'|$ to $l' + s'$,

(2) the average probability of a transition starting from any state of given l and s is obtained by *averaging* the probabilities for all possible values of j , taking account of the statistical weights. Since $2j+1$ is the statistical weight of a state j , and $(2l+1)(2s+1)$ that of all the states $l s$, we have

$$\begin{aligned} w^A_{B^p q^i i' P Q^j l' s'} &= \frac{1}{(2l+1)(2s+1)} \sum_{j'} (2j+1) w^A_{B^p q^i i' P Q^j l' s' j'}, \end{aligned} \quad (214)$$

or, inserting (212)

$$\begin{aligned} w^A_{B^p q^i i' P Q^j l' s'} &= \frac{1}{2\pi\hbar(2l+1)(2s+1)(2i+1)} \sum_{JJ'} (2J+1) \\ &\times \left| \sum_r \frac{w^r_{A^J p i P l s j} w^r_{B^J q^i i' Q^j l' s' j'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2 \end{aligned} \quad (215)$$

$$\text{with } \gamma_{rJ} = \sum_{Q^i q^i i' Q^j l' s' j'} (w^r_{B^J q^i i' Q^j l' s' j'})^2. \quad (215a)$$

Formulae (215), (215a) give the probability of the process if the incident particle has given orbital momentum l . In reality, the particle has a given direction of motion κ and must therefore be represented by a plane wave. If the plane wave is normalized per unit energy and per solid angle 4π , its expansion in terms of wave functions with given angular momenta, is⁷⁰

$$\begin{aligned} \delta(\mathbf{r}-\mathbf{r}') &= \int dE (d\Omega/4\pi) \psi_{E\kappa}(\mathbf{r}) \psi_{E\kappa}^*(\mathbf{r}')^* \\ &= \int dE \sum_{lm} \psi_{Elm}(\mathbf{r}) \psi_{Elm}^*(\mathbf{r}'). \end{aligned} \quad (216a)$$

We insert (216) in the first expression and average over the direction of κ , i.e., of the axis of the polar coordinate system to which the spherical harmonic in $\psi_{Elm} = \psi_{El}(\vartheta) Y_{lm}(\vartheta, \varphi)$ refers. The average gives, according to well-known relations between spherical harmonics:

$$\int (d\Omega/4\pi) Y_{l0}(\vartheta) Y_{l'0}(\vartheta') = \delta_{ll'} \sum_m Y_{lm}(\vartheta) Y_{lm}^*(\vartheta') / (2l+1) \quad (216b)$$

which reduces the middle expression in (a) to the right-hand expression.

$$\psi_{E\mathbf{k}} = \sum_l (2l+1)^{\frac{1}{2}} i^l \psi_{E l 0}, \quad (216)$$

where $\psi_{E l 0}$ is real, is normalized per unit energy, and has a magnetic quantum number $\mu_l = 0$, i.e., no orbital momentum around the direction \mathbf{k} . The factor $(2l+1)^{\frac{1}{2}}$ differs from the familiar $2l+1$ in the expansion of a plane wave in spherical harmonics because we use here normalized spherical harmonics, and the normalized harmonic Y_{l0} is equal to $(2l+1/4\pi)^{\frac{1}{2}} P_l$. It may then be shown along the same lines as before that different l 's do not interfere. Since the factor $2l+1$ in (216) cancels the $2l+1$ in the denominator of (215), we have for the total probability

$$w^P_{Q^p q^i} = \frac{1}{2\pi\hbar(2s+1)(2i+1)} \sum_{J l j l' j'} (2J+1) \times \left| \sum_r \frac{w^r_{A^J p i P l s j} w^r_{B^J q i' Q l' s' j'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2. \quad (217)$$

It remains to show that different j 's and l 's really do not interfere. The probability may be written

$$w^P_{Q^p q^i} = \frac{2\pi}{\hbar} \sum_{\mu_s m l' \mu_l' s' m'} \left| \sum_{l j \mu r M j' \mu'} i^l C^l_{i' 0 \mu s} C^J_{j' M \mu i m} \times \frac{w^r_{A^J p i P l s j} w^r_{B^J q i' Q l' s' j'} C^J_{j' M \mu' i' m'} C^{j' l' \mu' \mu_l' s' \mu_s'} \right|^2 \\ = \frac{2\pi}{\hbar(2s+1)(2i+1)} \sum_{l' l a i a' j a' i' j' a} S_{l j j' l'} S^*_{l a j a' i' a'} \\ \times \sum_{\mu_s m l' \mu_l' s' m' \mu_a M M a \mu' \mu'_a} C^l_{i' 0 \mu_s} C^{j a l' \mu_a} C^J_{j' M \mu i m} C^{j a i a' M a \mu'_a} \\ \times C^{j' l' \mu' \mu_l' s' \mu_s'} C^{j a i a' M a \mu'_a} C^{j' l' \mu' \mu_l' s' \mu_s'} C^{j' a i a' l' \mu'_a} \mu_l' s' \mu_s' \quad (218)$$

with $S_{l j j' l'} = i^l \sum_r \frac{w^r_{A^J p i P l s j} w^r_{B^J q i' Q l' s' j'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}}. \quad (218a)$

The second sum in (218) will be summed,
 (1) over μ'_l and μ'_s : The result will vanish unless $j'_a = j'$ and $\mu'_a = \mu'$. This proves that different values of j' do not interfere, and the sum over μ'_a reduces to a single term.
 (2) over μ'_l and m' : The result will vanish unless $J_a = J$ and $M_a = M$. Therefore different J 's do not interfere, and the sum over M_a reduces to one term.
 (3) over M and m . Since we have already proved that $J_a = J$ and $M_a = M$, this sum has the form

$$\sum_{M m} C^J_{j' M \mu i m} C^J_{j a M \mu'_a i m} = c \delta_{j j a} \delta_{\mu \mu'_a}. \quad (219)$$

This expression differs from the orthogonality relation (204) in that the sum extends over the upper and one lower index, rather than the two lower ones. But the C 's are, except for normalization, symmetrical in the three angular momenta J, j and i so that there holds the orthogonality relation (219) similar to (204). C is a constant (in our normalization $2\pi(2J+1)/(2j+1)$). (219) shows that different j 's do not interfere and that the sum over μ_a in (218) reduces to one term.
 (4) There remains the sum over μ and μ_s . In analogy to

(219), we find that it vanishes unless $l=l_a$. This proves (217).

We have, in this derivation, started from particle wave functions with given l , and then constructed plane waves out of these. It is equally possible to start from plane waves. We may write

$$\sigma^P_{Q^p q^i} = \int \frac{d\Omega'}{4\pi} \frac{\pi \tilde{\lambda}^2}{(2s+1)(2i+1)} \sum_J (2J+1) \times \left| \sum_r \frac{w^r_{A^J p i P s \mathbf{k}} w^r_{B^J q i' Q s' \mathbf{k}'}}{E_A + E_P - E_{rJ} + \frac{1}{2} i \gamma_{rJ}} \right|^2 \quad (220)$$

and $\gamma_{rJ} = \sum_{Q' q' i'} \int (d\Omega'/4\pi) |w^r_{B^J q' i' Q' s' \mathbf{k}'}|^2. \quad (221)$

These relations mean just that the particle wave functions with different \mathbf{k} , as well as the functions with different lm , form a complete set. The proof is analogous to that in reference 70. The matrix element $w^r_{A^J p i P s \mathbf{k}}$ may be directly calculated using a plane wave for the particle, the normalization follows most quickly from the fact that the average of $(w^r_{A^J p i P s \mathbf{k}})^2$ over \mathbf{k} represents the total disintegration probability of state rJ with the emission of particle P with energy $E_P(P)$.

Some special considerations are required for light quanta. Corresponding to the two directions of polarization, we have to put the quantity $2s+1$ in the denominators of all formulae equal to 2. (We have to average over the two directions of polarization of the incident quantum, just as over the $2s+1$ directions of the spin of a particle.) The selections rules (§3, beginning) on the other hand, cannot be obtained satisfactorily with any value of s (see below), but the value $s=1$ comes nearest to the truth. The reason for this difference is that in the case of light there is a relation between the polarization and the direction of propagation whereas the direction of the spin of a particle is quite independent of its direction of motion. Thus the polarization of the light introduces a spherical harmonic of order one in the matrix element but only a weight factor 2 which would correspond to $s=\frac{1}{2}$. The same fact also prevents the existence of any wave function for a light quantum with total momentum $j=0$.

To investigate the selection rules, we expand a plane light wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ in spherical harmonics with the direction of \mathbf{k} as polar axis (z axis). The two possible directions of polarization x and y are perpendicular to \mathbf{k} . The operator of the radiation theory may therefore be written

$$e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial x} = \sum_L \chi_L(kr) Y_{L0}(\vartheta) \left(\sin \vartheta \cos \varphi \frac{\partial}{\partial r} + \dots \frac{\partial}{\partial \vartheta} + \dots \frac{\partial}{\partial \varphi} \right) \\ = \sum_L \chi_L(kr) [\alpha_{L+1}(Y_{L+1, 1} + Y_{L+1, -1}) \\ + \alpha_L(Y_{L-1, 1} + Y_{L-1, -1})] \frac{\partial}{\partial r} + \dots, \quad (222)$$

where the α 's are certain coefficients. For each value of L , the total momentum j may thus have the values $j=L+1$ or $L-1$. This would correspond to $s=1$ except for the fact that $j=L$ does not occur for light. Furthermore, $j=0$ which should correspond to $L=1$ does actually not occur at all in (222) because the spherical harmonics Y_{01} and Y_{0-1} do not exist,

We may rewrite (222) using the total momentum j rather than the "orbital" momentum L for labeling the terms:

$$e^{i\mathbf{k}\cdot\mathbf{r}} \frac{\partial}{\partial x} = \sum_{j=1}^{\infty} (Y_{j1} + Y_{j-1}) \times \left\{ [\alpha_j \chi_{j-1}(kr) + \alpha_{j+1} \chi_{j+1}(kr)] \frac{\partial}{\partial r} + \dots \right\}. \quad (223)$$

For light, only small values of kr are important because the wave-length of the light is generally large compared to the dimensions of the radiating system. Now, according to (18), we have $\chi_{j-1} \propto (kr)^j$ and $\chi_{j+1} \propto (kr)^{j+2}$ so that the first term in the square bracket in (223) is always much more important than the second. The terms $j=1, 2, 3, \dots$ (or $L=0, 1, 2, \dots$) correspond to dipole, quadrupole, octopole . . . radiation.

MARCH 15, 1937

PHYSICAL REVIEW

VOLUME 51

Relativistic Wave Functions in the Continuous Spectrum for the Coulomb Field

MORRIS E. ROSE

Cornell University, Ithaca, New York

(Received January 22, 1937)

For purposes of reference the continuous spectrum solutions of the Dirac wave equation for the Coulomb field are given. The solutions in the form of series and integral representations and the asymptotic behavior at large distances are included among the formulae.

THE solutions in the continuous spectrum of the Dirac wave equation for the Coulomb field have been known and used extensively for some time. However, the form in which they appear in the literature is not convenient for some purposes. It is with the intention of furnishing a reference from which one may obtain the wave functions with a minimum expenditure of time and labor that the following formulae are given.

For the sake of simplicity we adopt a system of units in which energy is measured in mc^2 , length in \hbar/mc and momentum in mc . The symbols which occur below have the following meanings:

- $\psi_1\psi_2\psi_3\psi_4$, the four components of the wave function.
- f_l and g_l , radial wave functions.
- Y_{lm} , normalized spherical harmonics (see Eq. (2)).
- j , total angular momentum quantum number.
- m , magnetic quantum number.
- l , auxiliary index characterizing the wave functions.
- (l is the orbital momentum for the electron *only*, in the nonrelativistic limit.)
- W , absolute value of the energy.
- p , absolute value of the momentum $= (W^2 - 1)^{1/2}$.
- α , fine structure constant $= e^2/\hbar c$.
- Z , nuclear charge.

The wave functions are of two types:
Type *a*, $j = l + \frac{1}{2}$,

$$\begin{aligned} \psi_1 &= i \left(\frac{l-m+\frac{3}{2}}{2l+3} \right)^{\frac{1}{2}} Y_{l+1, m-1} f_l, \\ \psi_2 &= i \left(\frac{l+m+\frac{3}{2}}{2l+3} \right)^{\frac{1}{2}} Y_{l+1, m+1} f_l, \\ \psi_3 &= \left(\frac{l+m+\frac{1}{2}}{2l+1} \right)^{\frac{1}{2}} Y_{l, m-1} g_l, \\ \psi_4 &= - \left(\frac{l-m+\frac{1}{2}}{2l+1} \right)^{\frac{1}{2}} Y_{l, m+1} g_l, \end{aligned} \quad (1a)$$

$$l \geq 0, \quad -(l+1) \leq m - \frac{1}{2} \leq l-1.$$

Type *b*, $j = l - \frac{1}{2}$,

$$\begin{aligned} \psi_1 &= i \left(\frac{l+m-\frac{1}{2}}{2l-1} \right)^{\frac{1}{2}} Y_{l-1, m-1} f_{l-1}, \\ \psi_2 &= -i \left(\frac{l-m-\frac{1}{2}}{2l-1} \right)^{\frac{1}{2}} Y_{l-1, m+1} f_{l-1}, \\ \psi_3 &= \left(\frac{l-m+\frac{1}{2}}{2l+1} \right)^{\frac{1}{2}} Y_{l, m-1} g_{l-1}, \\ \psi_4 &= \left(\frac{l+m+\frac{1}{2}}{2l+1} \right)^{\frac{1}{2}} Y_{l, m+1} g_{l-1}, \end{aligned} \quad (1b)$$

$$l \geq 1, \quad -l \leq m - \frac{1}{2} \leq l-1,$$