Approximately Relativistic Equations for Nuclear Particles

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Cosmic-ray showers indicate that at high energies interaction between nuclear particles is concerned with the creation and destruction of matter. It may, therefore, be expected that a complete relativistic theory of nuclear forces will involve explicit reference to the phenomena described at present as the electron-neutrino field. Theories of this kind are still too incomplete and self-contradictory to be reliable in practical work. It is, nevertheless, possible to set up equations which are relativistically invariant for transformations involving low velocities. Such equations form the subject of the present report. They are restricted to energies of relative motion that are small in comparison with the rest mass. By means of them it should be possible to discuss relativistic effects for ordinary nuclear energy levels. Possible forms of classical equations contain an interaction energy between two particles in the form given by Eq. (13.2). Here a, b are arbitrary real constants. The vector from particle 1 to particle 2 is $\mathbf{r}$, the velocities of the particles are $\mathbf{v}_1$, $\mathbf{v}_2$, the velocity of light is c. If $a=b=1$, one obtains a generalization of Darwin's equation, which describes the motion of electrically charged particles. For $a=-1$, $b=1$ particle 1 acts on particle 2 approximately as though it produced a scalar potential field responsible for the acceleration of particle 2. The requirement of invariance for wave equations of particles with spin (Pauli types) makes it necessary to have spin-orbit coupling which should give rise to the fine structure of nuclear levels. For ordinary interactions the spin-orbit energy may have the form given by Eq. (15.4), where b is an arbitrary real constant, $\mathbf{p}$ is the momentum and $\boldsymbol{\sigma}$ is Pauli's spin matrix. For $b_{ij}=-1$ one obtains the type of coupling taking place between extranuclear electrons. If $b_{ij}=1$ each particle interacts only with its own orbit as though it were moving in a scalar field. It is the latter hypothesis that is simplest and corresponds to $a=-1$, $b=1$ of the classical equation. Extensions of the above classifications have been made to the Majorana [Eqs. (15.7), (15.8)] and the Heisenberg [Eq. (15.9)] exchange interactions. The simplest type in the Majorana case appears to be in satisfactory agreement with experiment for Li^7 and agrees in order of magnitude with other cases. Extensions to Dirac's types of equations have been made. They lead one to expect coupling between spins of nuclear particles in apparent qualitative but not quantitative agreement with experiment for the deuteron. This agreement is not sufficiently good to establish a form of interaction energy but indicates a possibility of doing so in the future.

It has been noticed by Inglis[1] that the apparent inversion of doublets in nuclei can be explained as a result of taking into account the effect introduced by Thomas in the discussion of the fine structure in atomic spectra. A nuclear particle, such as a neutron, is subjected to forces the actual nature of which is not known. There is no reason to consider these forces as being representable by an electromagnetic field and it is more likely that they owe their origin to the electron-neutrino phenomena. The direct influence of the electric field on the spin which is important for the fine structure of atomic spectra is thus presumably absent in nuclei. Very schematically one may consider a single nuclear particle as moving in the combined field of the other particles. In this rough approximation one may attempt to regard a single nuclear particle as exposed to the field of a scalar potential due to the remainder of the nucleus. This potential will be provisionally supposed to be a function of space-time and to be independent of the time in the coordinate system in which the nucleus is at rest. For such a model it follows that the Thomas term is the only one that exists for a neutron and that for a proton it should be combined with the effect of the electric field of the nucleus that is expected from the Dirac equation for a particle in a central field.

A proof of this has been already given by Furry,[10] who saw the possibility of inserting the scalar as an addition to the mass term mc^2 in Dirac's equation. The same result can be obtained by considering the transformation properties of wave functions. The latter method is desirable in order to bring out the connection with the treatment of many particles as well as in order to establish the interactions independently of special Dirac equations. It will be given in Section 1.

The neutrons and protons in a nucleus have approximately equal masses. In this respect the nucleus differs qualitatively from an atom which is composed of a heavy nucleus and several light electrons. Even in an atom the spin-orbit interactions are describable by a central field only somewhat accidentally. The correct description must be made by considering interactions between all pairs of particles.[2] Calculation shows[3] that the naive central field picture is approximately correct mainly because electrons in inner shells are so firmly bound that their action on a valence electron is comparable with that of a static distribution of charge. No such simplicity is apparent in a nucleus. It is therefore desirable to understand what kind of spin-orbit and orbit-orbit interactions may be postulated. In Dirac's theory spin effects may usually be considered as of the order v^2/c^2 in comparison with the kinetic energy, where v is the velocity of the particle and c is the velocity of light. It is, therefore, reasonable to require that the theory of nuclear particles be relativistically invariant to this order. It is presumably impossible to have a completely relativistically invariant theory for the interaction between several particles with a finite range of force and without explicit reference to a field. It is superfluous, therefore, and probably fruitless in the discussion of the present problem to require complete invariance. In Section 2 possible classical interactions between particles will be discussed and enumerated. It will be found that those among them that have special physical interest form a two-parameter set. Two types deserve special mention. One of them is the most immediate extension of Darwin's equation for two charged particles interacting according to the laws of classical electrodynamics. The other corresponds to each particle setting up a scalar field in a reference system in which it is at rest. It is this type that is related most closely to the interaction discussed in Section 1. In Section 3 possible types of wave equations will be discussed. It will be seen that: (a) The presence of spin-orbit interactions is a general consequence of the transformation properties of wave functions; (b) Their exact form is not uniquely determined by the transformation properties, but all possible forms of physical interest may be considered as linear combinations of electromagnetic interactions with pure Thomas terms; (c) A possible extension of the scalar one-particle picture gives only the Thomas terms. The effect of Majorana and Heisenberg exchange will be also considered in this section.

1. One Particle

It will be supposed that the particle in its free state is describable by Dirac's equation. The fact that the magnetic moment of nuclear particles is not that given by Dirac's equation in its original form does not contradict this supposition, because it is possible to introduce in Dirac's equation direct interactions between the magnetic field and the magnetic moment in the manner proposed by Pauli.[4] The transformation properties of Dirac's equation are the same whether a true magnetic moment of the Pauli type is introduced or not. It is on the other hand questionable whether Dirac's equation or any of its modifications gives an exact account of the properties of nuclear particles. In fact high energy collisions involving relative velocities of nuclear particles comparable with c indicate from experiment,[5] in the light of Heisenberg's theory[6] of showers, the importance of processes involving the creation and destruction of particles. For this reason it is preferable to base theories of nuclear spin-orbit interactions on corrections of the order v^2/c^2 rather than on an exact form of a single particle equation.

The original discussion of Thomas does not apply directly to particles whose spin is describable by means of a Pauli or a Dirac equation. For a Dirac particle the spin is a secondary characteristic. As has been pointed out by Schroedinger,[7] a Dirac particle is unable to move uniformly in a straight line when left to its own devices. This property of the particle gives rise to the spin. It is, therefore, necessary to consider the Thomas term directly from the point of view of the wave equation rather than in terms of a classical analogy.

According to Dirac[8] the Lorentz transformation

$$\begin{gathered} x' = x \operatorname{ch}\theta - ct \operatorname{sh}\theta, \qquad ct' = ct \operatorname{ch}\theta - x \operatorname{sh}\theta, \\ p_x' = p_x \operatorname{ch}\theta - p_0 \operatorname{sh}\theta, \quad p_0' = p_0 \operatorname{ch}\theta - p_x \operatorname{sh}\theta, \\ \operatorname{ch}\theta = (1-v^2/c^2)^{-\frac{1}{2}}, \quad \operatorname{sh}\theta = (v/c)(1-v^2/c^2)^{-\frac{1}{2}}; \\ p_0 = E/c \quad (\operatorname{ch}\theta \equiv \cosh\theta,\ \operatorname{sh}\theta \equiv \sinh\theta) \end{gathered} \tag{1}$$

induces the transformation

$$\psi' = e^{\alpha_1 \theta/2}\psi, \quad \psi'^* = \psi^* e^{\alpha_1 \theta/2} \tag{2}$$

on the wave function. This means that the transformed wave function ψ' satisfies the Dirac equation in the transformed system K' and that all physical quantities calculated by means of ψ' in K' are related by relativistic formulas to corresponding quantities calculated by means of ψ in the original system K.

The Dirac equations can be written as

$$\begin{aligned}(p_0+Mc)\Phi+(\boldsymbol{\sigma}\mathbf{p})\Psi=0,\\ (p_0-Mc)\Psi+(\boldsymbol{\sigma}\mathbf{p})\Phi=0,\end{aligned} \tag{3}$$

where the σ are Pauli spin matrices, Ψ is the two-component Pauli wave function, and Φ is also a two-component wave function. Both Φ and Ψ are column matrices. The Dirac function ψ is a column matrix having the two elements of Φ in first and second place and the two elements of Ψ in third and fourth place. From (3) one obtains in sufficient approximation

$$\Phi = -(\boldsymbol{\sigma}\mathbf{p})\Psi/2Mc. \tag{3$'$}$$

This approximation is valid for interactions of both the electromagnetic and the scalar types. The occurrence of terms in the vector potential in (3) leads only to negligible corrections in (3′) because these terms can be taken to be of the order e^2v/c^2r, where r is the distance between nuclear particles and v is the velocity. This quantity is of the order of mv for $r\sim e^2/mc^2$, where m is the electronic mass, while the operations involved in $-i\hbar\partial/\partial x$ give rise to Mv, where M is the nuclear particle mass. Similarly the addition of a scalar interaction energy to Mc^2 or of an electrostatic potential to p_0 gives rise to insignificant corrections for forces having the approximate range e^2/mc^2.

Using (3′) in (2)

$$\Psi' = \{\mathrm{ch}\,(\theta/2) - (2Mc)^{-1}(\mathbf{p}+i[\mathbf{p}\times\boldsymbol{\sigma}])_x\,\mathrm{sh}\,(\theta/2)\}\Psi. \tag{4}$$

Let now the wave equation be of the form

$$\left\{\frac{\hbar}{i}\frac{\partial}{\partial t}+Mc^2+\frac{p^2}{2M}-\frac{p^4}{8M^3c^2}-J+Q\right\}\Psi=0. \tag{5}$$

Substituting Ψ in terms of Ψ' and expressing all other quantities in K in terms of quantities in K', the wave equation should return to its original form. It will be supposed that J is a scalar. It will thus transform into itself. The quantity Q is introduced into Eq. (5) in order to correct the remaining terms for lack of invariance. Instead of considering all of Eq. (4), it will be first approximated by

$$\Psi = S\Psi'; \quad S = 1+i\mathbf{v}[\mathbf{p}\times\boldsymbol{\sigma}]/4Mc^2. \tag{4$'$}$$

This approximation neglects some effects of order v^2/c^2. However, an inspection of Eq. (4) shows that effects of order v^2/c^2 are additive and no essential error is committed.[9] Substituting Eq. (4′) into Eq. (5) and multiplying by S^{-1} from the left one obtains on account of the presence of J the term

$$J - S^{-1}JS = (\hbar/4Mc^2)[\nabla J\times\boldsymbol{\sigma}]\mathbf{v}. \tag{6}$$

This term must be canceled by the introduction of a suitable term into Q. It will be supposed that Q contains only terms of order v^2/c^2. Therefore one may take $S^{-1}QS=Q$. Thus in the approximation of Eq. (4′) there is a part Q_0 of Q such that $Q_0'-Q_0$ must be equal to the right side of Eq. (6). This can be accomplished by

$$Q_0 = -(\hbar/4M^2c^2)[\nabla J\times\boldsymbol{\sigma}]\mathbf{p}. \tag{6$'$}$$

This is the Thomas term which must, therefore, be included in order to enforce invariance. In addition it is possible to postulate the presence of other terms in $\boldsymbol{\sigma}$ entering Q. Such terms must form an invariant to the order v^2/c^2. It is impossible to form such an invariant by having ∇J, $\boldsymbol{\sigma}$, and $\mathbf{p}$ entering the expression once. There exists, in consequence, no term which can be added to Q which has an essential relation to Q_0.

Transforming Eq. (5) to variables in K' one finds that

$$\left\{-E'+Mc^2+\frac{p'^2}{2M}-\frac{p'^4}{8M^3c^2}-J -\frac{v}{4Mc^2}(3p_x'J+Jp_x')-\frac{v^2}{2c^2}J +\frac{\hbar v}{4Mc^2}[\nabla' J\times\boldsymbol{\sigma}]_x+Q\right\}\Psi'=0. \tag{7}$$

The terms containing v have to be compensated by $Q-Q'$. A possible Q, determined in a manner analogous to that in Eq. (6′), gives an invariant form by substitution into Eq. (5)

$$\left\{-E+Mc^2+\frac{p^2}{2M}-\frac{p^4}{8M^3c^2}-J+\frac{p^2J+\mathbf{p}J\mathbf{p}}{4M^2c^2}\right.$$
$$\left.-\frac{\hbar}{8M^2c^2}([\mathbf{p}\times\nabla J]-[\nabla J\times\mathbf{p}])\boldsymbol{\sigma}\right\}\Psi=0. \quad (8)$$

The term in $\boldsymbol{\sigma}$ is here symmetrized so as to be Hermitean. The terms in p^2J and $\mathbf{p}J\mathbf{p}$ form a non-Hermitean operator. This is due to the fact that the total probability of finding the particle is not given by the volume integral of $|\Psi|^2$ but by

$$\int\psi_\mu{}^*\psi_\mu d\tau=\int\Psi_\alpha{}^*(1+p^2/4M^2c^2)\Psi_\alpha d\tau \quad (8')$$
$$(\mu=1,2,3,4;\ \alpha=1,2).$$

It is possible, however, to transform the wave function Ψ so as to have a Hermitean operator. Thus if

$$\Psi^{(1)}=(1+p^2/8M^2c^2)\Psi \quad (9)$$

Eq. (8) becomes

$$\left\{-E+Mc^2+\frac{p^2}{2M}-\frac{p^4}{8M^3c^2}-J+\frac{p^2J+2\mathbf{p}J\mathbf{p}+Jp^2}{8M^2c^2}\right.$$
$$\left.-\frac{\hbar}{8M^2c^2}([\mathbf{p}\times\nabla J]-[\nabla J\times\mathbf{p}])\boldsymbol{\sigma}\right\}\Psi^{(1)}=0, \quad (9')$$

which is Hermitean. The modified function $\Psi^{(1)}$ satisfies the simple normalization condition

$$\int\Psi_\alpha{}^{(1)*}\Psi_\alpha{}^{(1)}d\tau=\int\psi_\mu{}^*\psi_\mu d\tau=1, \quad (9'')$$

which follows from Eqs. (8′) and (9). An elegant treatment of the single particle problem due to Furry[10] can be carried out introducing the scalar $-J$ into the Dirac equation as an addition to the rest mass energy Mc^2. The equation in the Dirac form is

$$\{p_0+(\boldsymbol{\alpha}\mathbf{p})+\beta(Mc-J/c)\}\psi=0. \quad (10)$$

Expressing the components Φ in terms of Ψ by means of Eq. (3′) and substituting into the second Eq. (3) modified by $Mc\rightarrow Mc-J/c$, it is found that Ψ satisfies Eq. (8). This is a natural result inasmuch as Furry's Eq. (10) is strictly invariant. It should be noted that the transition from Eq. (7) to Eq. (8) is not unique. Thus it is possible to use instead of $p^2J+\mathbf{p}J\mathbf{p}$ in this equation the quantity $(3p^2J+Jp^2)/2$ without changing the transformation properties, and more generally an arbitrary constant times $p^2J+Jp^2-2\mathbf{p}J\mathbf{p}$ can be added to $p^2J+\mathbf{p}J\mathbf{p}$. There is thus a greater freedom in the forms of two-component equations invariant to order v^2/c^2 than is apparent from Eq. (10). The equations obtained in this manner are identical, however, from the point of view of classical analogy, and in the absence of another clue it is impossible to give reasons for preferring one to another. The particular form (8) may perhaps be advocated on the grounds that it is equivalent to Eq. (10).

It is possible to add to the operators in Eqs. (8), (9′) any quantity invariant to order v^2/c^2. Thus J^2/Mc^2 multiplied by a reasonably small number may be added. The addition of a term in this quantity is, however, only of trivial interest in the present case because such a term may be considered as being incorporated in J. The addition of commutator-like terms such as $(p^2J+Jp^2-2\mathbf{p}J\mathbf{p})/Mc^2$ amounts to adding terms in derivatives of J and also offers no essentially new possibility. Terms in p^2J and $\mathbf{p}J\mathbf{p}$ can occur only as in Eq. (8) to within the freedom of adding a commutator. Using p^2 and p^4 it is impossible to have an invariant except that already occurring in the equation. Within the above discussed set of the simplest possibilities, Eqs. (8) and (9′) are thus essentially unique.

The parts of the operators occurring in Eqs. (8), (9′) which do not contain $\boldsymbol{\sigma}\cdot$ can be interpreted as corresponding to the wave equation of a particle without spin, invariant to order v^2/c^2. The verification is analogous to the step from Eq. (7) to Eq. (8). A slight difference from Eq. (8) is found in the order of factors. Allowing the wave function to transform itself by

$$\Psi^{0\prime}=(1+v^2/4c^2-vp_x/2Mc^2)\Psi^0 \quad (11)$$

it is found that

$$\left\{-E+Mc^2+\frac{p^2}{2M}-\frac{p^4}{8M^3c^2}-J+\frac{\mathbf{p}J\mathbf{p}}{2M^2c^2}\right\}\Psi^0=0 \quad (11')$$

is an invariant form.

The transformation of Eq. (11) gives

$$[\Psi^{0*}\Psi^0]'=\left(1+\frac{v^2}{2c^2}\right)\Psi^{0*}\Psi^0 -\frac{\hbar v}{2iMc^2}\left(\Psi^{0*}\frac{\partial\Psi^0}{\partial x}-\Psi^0\frac{\partial\Psi^{0*}}{\partial x}\right).$$

One may, therefore, interpret $\Psi^{0*}\Psi^0$ as the particle density and use the normalizing condition

$$\int\Psi^{0*}\Psi^0 d\tau=1. \qquad (11'')$$

In obtaining Eq. (8) it was supposed that the contributions due to time derivatives of J arising in the transformations are small. This requirement can be satisfied by supposing that there exists a frame of reference in which J is a function only of space and does not depend on time. This frame may be called the rest frame. The other frames considered above are those obtained by the application of a small translational velocity to the rest frame. For such transformations ∇J is an invariant since it changes only by terms of order $(v^2/c^2)\nabla J$ on transformation. If J is only a function of the distance r from a fixed point there is no necessity of symmetrizing the last term containing $\boldsymbol{\sigma}$ because then

$$[\mathbf{p}\times\nabla J]=-[\nabla J\times\mathbf{p}].$$

If J, instead of transforming as a scalar, transforms as the time component of a four-vector, one obtains essentially the same results as are derivable from the ordinary Dirac equation.

2. Classical Equations for Several Particles

It will be supposed that the system can be treated by means of a variational equation,

$$\delta\int Ldt=0. \qquad (12)$$

The trajectory of the dynamical system in the reference system K corresponds to segments of world lines in an interval $t_1<t<t_2$. The Lagrangian L will be supposed to be given approximately by

$$L_0=\sum_i M_i v_i^2/2+\sum_{i>j}J_{ij}(r_{ij}), \qquad (12')$$

where M_i are the masses, v_i the velocities of the particles, and r_{ij} are the distances between them. In addition to L_0 the function L will be supposed to contain terms of the order $(v^2/c^2)L_0$. These terms will now be determined by the requirement that Eq. (12) should determine the same world lines independently of the coordinate system used. For a system of noninteracting particles $(J_{ij}=0)$ there is an obvious solution corresponding to describing each particle separately. In this case one may use

$$L_0=\sum_i M_ic^2[1-(1-v_i^2/c)^{2\frac{1}{2}}].$$

Expanding each term of this sum one obtains

$$L_0=\sum_i M_i(v_i^2/2+v_i^4/8c^2) \quad (J=0)$$

to the required order. For this L_0 the integral in Eq. (12) is not invariant because: (a) each term of the sum contains $M_ic^2\int dt$ which is not invariant. However, the variation of this quantity is zero and its inclusion is harmless. (b) In the transformation $K\rightarrow K'$ it is impossible to deal with identical segments of world lines in both systems. If the end points of the world line for i are kept the same, those of j have to be changed in K' from what they were in K. This circumstance also does not matter because the variations of the coordinates of i and j are supposed to be independent of each other. Therefore the variations over that portion of the world line of j which is common to K and K' for fixed end points of the world line of i are automatically the same as long as each term of Ldt is invariant. It is thus seen that the invariance of Ldt is sufficient for the invariance of the world lines. It is obviously not a necessary condition as is seen in the case of the addition of terms in $\int dt$.

The terms J_{ij} substituted into Eq. (12) give rise to noninvariant quantities. These will now be calculated. It will be supposed that in calculating the correction terms it is legitimate to neglect the curvature of the world lines. This assumption does not follow from considerations of invariance. It simplifies the discussion by preventing the explicit occurrence of accelerations in the Lagrangian. For electromagnetic interactions it leads to neglecting radiation reactions. For nuclear forces it amounts to neglecting $x\ddot{x}/c^2 \sim J/Mc^2 \sim 20m/M \sim 10^{-2}$ in comparison with unity. Here m is the electronic mass. In terms of this estimate the approximation is a good one since the effects neglected are of the order of 10^{-2} times the correction terms in v^2/c^2.

For the transformation given by Eq. (1) the distances between particles 1,2 in K' and K are related by

$$r' = r - \frac{(\mathbf{vr})^2}{2c^2 r} + \frac{(\mathbf{vr})}{c^2 r}\frac{M_1(\mathbf{rv}_2)+M_2(\mathbf{rv}_1)}{M_1+M_2}$$

and hence

$$J(r') = J(r) + \left[-\frac{(\mathbf{vr})^2}{2c^2} + \frac{(\mathbf{vr})}{c^2}\frac{M_1(\mathbf{rv}_2)+M_2(\mathbf{rv}_1)}{M_1+M_2}\right]\frac{dJ}{rdr}. \quad (13)$$

In order that Ldt be invariant it is necessary that L should transform as ds/dt where ds is the differential of the four dimensional distance. The transformation formulas for ds/dt are different for the world lines of 1 and 2. The world line of the center of gravity of 1 and 2 gives a still different transformation of ds/dt. At this stage, therefore, the definition of the world line is still arbitrary, since one has the above three choices, each of which has some physical plausibility. This arbitrariness of the choice of the world line would destroy the possibility of having a sensible description of the system by means of Eq. (12) if it were not for the fact that the results of Eq. (12) are independent of the choice in the following sense. Let M_1, M_2 in Eq. (13) be given arbitrary values and let the same values be used for the definition of the path of the center of gravity. Let the transformation of L be similar to that of ds/dt along this path of the center of gravity. The results of Eq. (12) will be seen to be independent of the values of M_1 and M_2. This means that it is possible to use either the path of 1 or that of 2 and that the results of the two procedures are consistent.

For the path of the center of gravity for the transformation of Eq. (1)

$$\frac{ds}{dt'} = \left[1 - \frac{v^2}{2c^2} + \frac{M_1(\mathbf{vv}_1)+M_2(\mathbf{vv}_2)}{(M_1+M_2)c^2}\right]\frac{ds}{dt}. \quad (13')$$

It is now required that $J-Q$ should transform as ds/dt. Since Q contains by definition only terms of order v^2/c^2, the transformation formula for Q is determined by means of Eqs. (13), (13′). It is

$$Q - Q' = \left[-\frac{v^2}{2c^2} + \frac{(\mathbf{vv}_1)+(\mathbf{vv}_2)}{2c^2}\right]J + \left[\frac{(\mathbf{vr})^2}{2c^2} - \frac{(\mathbf{vr})}{2c^2}((\mathbf{v}_1+\mathbf{v}_2)\cdot\mathbf{r})\right]\frac{dJ}{rdr} + \frac{M_1-M_2}{2c^2(M_1+M_2)}\frac{d}{dt}[(\mathbf{rv})J], \text{ where } \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2. \quad (13'')$$

The last term in Eq. (13″) may be disregarded because in Eq. (12) it gives rise only to a function of the end points of the path. The remainder of $Q-Q'$ is independent of M_1 and M_2 and is equal to $Q-Q'$ for $M_1=M_2$. It is symmetric in the coordinates and velocities of 1 and 2. It is natural to look for possible Q which are also symmetric in 1 and 2. Unsymmetric Q are either reducible to symmetric Q by the addition of a complete differential to Ldt or else they imply a different action of particle 1 on particle 2 from that of 2 on 1. Such a law of interaction is improbable and will not be considered here. Only symmetric possibilities for Q will be discussed.

A possible $J-Q$ with Q transforming itself as in Eq. (13″) is

$$J - Q = J - [(\mathbf{v}_1\mathbf{v}_2)J - (\mathbf{v}_1\mathbf{r})(\mathbf{v}_2\mathbf{r})dJ/rdr]/4c^2, \quad (13.1)$$

as is verified by applying the transformation of Eq. (1) and observing that for the transformation of Q it is sufficient to use Newtonian rather than relativistic addition of velocities and to consider r as invariant in this connection. All other Q can be obtained from Eq. (13.1) by the addition of invariants. In this way one obtains among other possibilities the following expression for $J-Q$:

$$J - Q = J - \{2a(\mathbf{v}_1\mathbf{v}_2) + (1-a)(v_1^2+v_2^2)\}J/4c^2 + \{2b(\mathbf{v}_1\mathbf{r})(\mathbf{v}_2\mathbf{r}) + (1-b)[(\mathbf{v}_1\mathbf{r})^2 + (\mathbf{v}_2\mathbf{r})^2]\}dJ/4c^2rdr, \quad (13.2)$$

where a and b are arbitrary real constants. For $a=b=1$, Eq. (13.2) agrees with Eq. (13.1). In this case the Lagrangian L_0+J-Q is a generalization of that obtained by Darwin for electromagnetic interactions, as is seen by substituting e^2/r for J. It will be recalled that Darwin's approach was that of retarded potentials and that the Lagrangian was found by: (a) working out the equations of motion for particle 1 using position and velocity of 2 as parameters, (b) finding the Lagrangian for 1, and (c) showing that there exists another symmetrical Lagrangian differing from the first one by a time derivative. It is desirable to explain why Darwin's equation is of type (13.1) rather than of some other type contained in Eq. (13.2). Obviously the distinguishing feature of the form (13.1) is that of antisymmetry with respect to reversal of sign of one of the velocities. All other types in Eq. (13.2) have another type of symmetry. In the electromagnetic case corrections for the velocity $\mathbf{v}_1$ enter for two reasons. In the first place particles 1 and 2 interact magnetically. This interaction is antisymmetric in $\mathbf{v}_1$ because it depends on $(\mathbf{v}_1\mathbf{v}_2)$. In the second place corrections for retardation must be made. These corrections can depend on the direction of motion of 1, while one discusses the motion of 2 only because 1 is moving away or towards 2. Hence again the addition to the Lagrangian must be antisymmetric in $\mathbf{v}_1$.

It can be verified by explicit calculation that the forms given by Eq. (13.2) give an expression for the force which is in agreement with relativity kinematics to the order v^2/c^2. According to relativity if $\mathbf{P}_1$ stands for $M_1\mathbf{v}_1(1-v_1^2/c^2)^{-1/2}$ then at the same point in space time the rates of change of $\mathbf{P}_1$ and $\mathbf{P}_1'$ in K and K' are connected by

$$\dot{\mathbf{P}}_1=\dot{\mathbf{P}}_1'+[-v^2\dot{\mathbf{P}}_1'+\mathbf{v}(\mathbf{v}\dot{\mathbf{P}}_1')]/2c^2 \qquad (14)$$
$$\left(\frac{dx'}{dt'}=\frac{dy'}{dt'}=\frac{dz'}{dt'}=0\right),$$

where $\mathbf{v}$ is the transformation velocity, provided terms of order higher than v^2/c^2 are neglected and provided $\mathbf{P}_1'=0$. The forces on 1 are thus determinable in terms of the force in that frame in which 1 is instantaneously at rest. The verification of Eq. (14) is straightforward but somewhat lengthy and is not reproduced here. It involves no approximations except that of neglecting all terms of order higher than v^2/c^2.

If one sets $a=b=0$ in Eq. (13.2) a simple form for $J-Q$ is obtained:

$$J-Q=J-\frac{v_1^2+v_2^2}{4c^2}J+\frac{1}{4c^2}[(\mathbf{v}_1\mathbf{r})^2+(\mathbf{v}_2\mathbf{r})^2]\frac{dJ}{rdr}. \qquad (14.1)$$

This is somewhat related to Eqs. (9′), (11′) for a single particle but does not quite correspond to them. Thus if particle 2 has a sufficiently large mass to make v_2 small at all times, the above $J-Q$ is not altogether equivalent to $J-v^2J/2c^2$ because

$$v_1^2J+(\mathbf{v}_1\mathbf{r})^2dJ/rdr=(d/dt)[(\mathbf{v}_1\mathbf{r})J]-(\dot{\mathbf{v}}_1\mathbf{r})J$$

and is not a complete differential. Working out the equations of motion for Eq. (14.1) and comparing them with the equations of motion following from Eqs. (9′), (11), no simple correspondence is found.

A simple correspondence is found, however, setting $a=-1$, $b=1$. In this case

$$J-Q=J-(\mathbf{v}_1-\mathbf{v}_2)^2J/2c^2-(\mathbf{v}_1\mathbf{v}_2)J/2c^2 +(\mathbf{v}_1\mathbf{r})(\mathbf{v}_2\mathbf{r})dJ/2c^2rdr \qquad (14.2)$$
$$(a=-1,\ b=1).$$

If $\mathbf{v}_2=0$, this agrees with Eqs (9′), (11′). Comparing the equations of motion following from Eq. (14.2) with those from Eq. (9′), it is found in both cases that in the reference system in which 2 is at rest the acceleration of 1 is the same function of its position with respect to 2 and its velocity. One may, therefore, regard Eq. (14.2) as being a generalization of Eq. (9′) and of Eq. (10) if in the latter the effect of the spin is neglected. One may think of Eq. (14.2) as giving such a motion of the two particles as corresponds to particle 2 producing in its reference system a scalar field which acts on particle 1, and the other way around.

Only two particles are explicitly considered above. The arguments which lead to Eq. (13″) for $Q-Q'$ can be generalized to any number of particles having arbitrary interactions in pairs. The calculation is more lengthy but the result is simply that, to within a term which is a complete time derivative, $Q-Q'$ is a sum over pairs of terms such as are given by Eq. (13″) for a single pair. An extension of Eq. (14.2) to many particles is obtained by using a sum over pairs of such expressions. If particles 2, 3, $\cdots$ n are tied by strong forces into a large mass approximately at rest and particle 1 moves in their field, this generalization becomes equivalent to Eq. (9′) because the time average of $\mathbf{v}_2, \mathbf{v}_3, \cdots \mathbf{v}_n$ can be considered to be 0.

The forms used in Eq. (13.2) can be generalized by the addition of other invariants. Thus one could add terms in $(\mathbf{v}_1-\mathbf{v}_2)^2 r dJ/c^2 dr$ or in $((\mathbf{v}_1-\mathbf{v}_2)\mathbf{r})^2 J/c^2 r^2$. The use of such terms cannot be eliminated by general principles but appears to be an unnecessary complication. Similarly one can add terms in higher derivatives of J, and it is possible to have approximate invariants of order of magnitude v/c which in Lorentz transformation undergo changes only of order v^3/c^3.

3. Wave Equations for Many Particles

General methods for setting up relativistic equations for composite systems have been given by Heisenberg and Pauli, Fermi, Dirac, and Dirac, Fock and Podolsky.[11] These methods do not apply directly to the problem considered here because they are concerned with exact rather than approximate invariance. The approach of Heisenberg and Pauli by means of an invariant Lagrange function can nevertheless be made use of in a modified form. Instead of extending the integration over space time as is done by them it is more natural to consider here variational integrals over the time and the configuration space because this is the simplest formulation of a nonrelativistic equation. Speaking of the motion classically, it may be expected that the interaction energy J should be treated as small. Otherwise the velocity of at least one particle will be large and an approximate relativistic treatment will have no sense. This assumption corresponds to the similar restriction of small accelerations made in the classical case when Hamilton's principle was used. Forms of

$$\delta\int \mathfrak{L}d\tau=0, \quad d\tau=dV_1dV_2\cdots dV_ndt \qquad (15)$$

will be looked for. Here $dV_1, dV_2, \cdots dV_n$ are elements of volume for each particle. If the interactions are not of an exchange type and if the particles have no spin, it may be expected that the resultant wave equations should have a possible classical form such as one of the forms in Section 2. Only the occurrence of commutators in the wave equations will, therefore, become clearer in such a discussion. The presence of terms in the spin-orbit and spin-spin interactions and of the effect of exchange is, on the other hand, obtainable from Eq. (15) and cannot be inferred from a classical discussion.

It will be supposed that if $c\to\infty$ in $\mathfrak{L}$ and the corresponding wave equation, the system will be invariant under Galilean transformations. Thus for exchange interactions it will be simplest to confine oneself to particles of equal masses. In discussing Lorentz invariance it is easiest to consider

$$\mathfrak{L}=\sum_i \mathfrak{L}_i+\sum_{i>j}\mathfrak{L}_{ij} \qquad (15.1)$$

and to require that (a) the forms $\mathfrak{L}_i$ should yield invariant equations for the separate particles and (b) all the $\int\mathfrak{L}_{ij}d\tau$ should be invariant when one substitutes into $\mathfrak{L}_{ij}$ arbitrary linear combinations of products of single particle wave functions, each single particle wave function being assumed here to obey the wave equation of a free particle. Assuming Eq. (15.1), the restriction (b) will formally give a correct treatment only for very small J. Nevertheless, it may be expected to be also applicable, to some extent, also in cases of reasonably large J, because the largest part of the interaction energy will be Galileo invariant. Regarding the interaction energy as a power series in $1/c$, the first term of it will be thus correct by assumption and the remaining term will be established only for small J. This is less than one might wish to attain but it is not much worse than the knowledge of corresponding terms for interactions between charged particles. The discussion of the classical spinless equations by means of their equations of motion showed that even though the assumption of small accelerations was made in using the variational principle, the transformation formulas for the force are formally invariant to the order v^2/c^2 independently of this assumption. Since the classical equations are better than expected it is possible that the wave equations behave similarly.

As in Eq. (4′) one may discuss the spin-orbit interactions separately. Corresponding to Eq. (3) one now has 2^n equations, where n is the number of particles. Instead of Eq. (2), one has for non-interacting particles a transformation in which the exponential function of Eq. (2) is replaced by a product of similar exponentials. Eq. (3′) is replaced by n equations which determine those wave function components in which the Dirac indices for just one particle are in the small group, (1, 2). The extension of Eq. (2) gives similar extensions of Eqs. (4), (4′) in which there occur sums over $[\mathbf{p}_i\times\boldsymbol{\sigma}_i]$. In this way instead of Eq. (6) one has

$$Q_0'-Q_0=\sum_{i,\,j}(\hbar/4M_ic^2)[\nabla_iJ_{ij}\times\boldsymbol{\sigma}_i]\mathbf{v}. \quad (15.2)$$

This equation is satisfied by

$$Q_0=-\sum_{i,\,j}(\hbar/4M_ic^2)[\nabla_iJ_{ij}\times\boldsymbol{\sigma}_i]\cdot\sum_k a_{ik}\mathbf{p}_k/M_k;\quad \sum_k a_{ik}=1 \quad (15.3)$$

and the a_{ik} are so far arbitrary. Since it was decided to use for $J-Q$ only interactions of particles in pairs, since the occurrence of $\mathbf{p}_k/M_i$ in the interaction energy is strange, and since finally it is simplest to suppose that the interaction energy of a pair is symmetric, the following form is sufficiently general for most purposes:

$$Q_0=-\sum_{i>j}(\hbar/4c^2)\{[\nabla_iJ_{ij}\times\boldsymbol{\sigma}_i][b_{ij}\mathbf{p}_i/M_i^2+(1-b_{ij})\mathbf{p}_j/M_iM_j]+[\nabla_jJ_{ij}\times\boldsymbol{\sigma}_j]\cdot[b_{ij}\mathbf{p}_j/M_j^2+(1-b_{ij})\mathbf{p}_i/M_iM_j]\}. \quad (15.4)$$

The b_{ij} may be different for every pair. For electromagnetic interactions[2] $b_{ij}=-1$, so that $2\mathbf{p}_2-\mathbf{p}_1$ goes with $\boldsymbol{\sigma}_1$. For $b_{ij}=1$ an extension of Eq. (9′) is obtained, as is readily verified by letting all M_j but one become infinite. Since J_{ij} is a function only of r_{ij}, Eq. (15.4) gives a Hermitean operator for Q_0.

If the interaction is of the Majorana type, it is simplest to take the masses as equal to each other because otherwise the operator must be modified in order to preserve Galilean invariance. One obtains

$$Q_0'-Q_0=(\mathbf{v}/8Mc^2)\sum_{k>l}[\mathbf{A}_{kl}\times(\boldsymbol{\sigma}_k-\boldsymbol{\sigma}_l)] \quad (15.5)$$

with

$$\mathbf{A}_{kl}=i[(\mathbf{p}_k-\mathbf{p}_l)J_{kl}+J_{kl}(\mathbf{p}_k-\mathbf{p}_l)]P_{kl}^M, \quad (15.6)$$

where P_{kl}^M is the Majorana operator exchanging the space coordinates of k and l. Possible forms for Q_0 are given by

$$Q_0=\sum_{k\gtrless l}\left(\frac{-1}{8M^2c^2}\right)[a\mathbf{p}_k+(1-a)\mathbf{p}_l][\mathbf{A}_{kl}\times\boldsymbol{\sigma}_k]=\sum_{k\gtrless l}\left\{\frac{ia}{4M^2c^2}[\mathbf{p}_k\times J_{kl}P_{kl}^M\mathbf{p}_k]\boldsymbol{\sigma}_k-\frac{i(1-a)}{4M^2c^2}[\mathbf{p}_k\times J_{kl}P_{kl}^M\mathbf{p}_k]\boldsymbol{\sigma}_l\right\}, \quad (15.7)$$

where a is a real constant. In this case J_{kl} rather than just dJ_{kl}/dr_{kl} enters into Q_0. Thus for $a=1$ and $a=0$, respectively,

$$(a=1)\quad Q_0=\sum_{k\gtrless l}\left\{\frac{\hbar}{4M^2c^2}(\nabla_kJ_{kl})[\mathbf{p}_l\times\boldsymbol{\sigma}_k]+\frac{i}{4M^2c^2}J_{kl}[\mathbf{p}_k\times\mathbf{p}_l]\boldsymbol{\sigma}_k\right\}P_{kl}^M,$$

$$(a=0)\quad Q_0=\sum_{k\gtrless l}\left\{\frac{\hbar}{4M^2c^2}(\nabla_kJ_{kl})[\mathbf{p}_k\times\boldsymbol{\sigma}_k]+\frac{i}{4M^2c^2}J_{kl}[\mathbf{p}_k\times\mathbf{p}_l]\boldsymbol{\sigma}_k\right\}P_{kl}^M. \quad (15.8)$$

For $a=1$ the value of Q_0 may be thought of as

$$(a=1)\quad Q_0=-\frac{\hbar}{4M^2c^2}\sum_i[\mathbf{p}_i\times\dot{\mathbf{p}}_i]\boldsymbol{\sigma}_i=\frac{\hbar}{4M^2c^2}\sum_i[\dot{\mathbf{p}}_i\times\mathbf{p}_i]\boldsymbol{\sigma}_i \quad (15.8')$$

and corresponds therefore most closely to the expectation on classical theory. From the point of view of Eq. (15.8′) the choice $a=1$ is the simplest. It should be noted that with Majorana interactions $\dot{\mathbf{p}}_i/M$ is only approximately the acceleration. If classical analogy were followed literally, Eq. (15.8′) would be more complicated because the operator representing the acceleration contains many terms. The expression for it can be substituted for $\dot{\mathbf{p}}_i/M$ in Eq. (15.8′). However, there appears to be little justification for such a literal use of the Thomas term, particularly since $\mathbf{p}_i/M$ does not represent the velocity. For exchange interactions involving Heisenberg's operator,

$$Q_0 = -\frac{\hbar}{4M^2c^2}\sum_{k\lessgtr l}[a\mathbf{p}_k+(1-a)\mathbf{p}_l] \cdot[\nabla_k J_{kl}\times\boldsymbol{\sigma}_k]P_{kl}^H. \quad (15.9)$$

Possible forms for Q can be discussed analogously to the classical Eqs. (13.2) introducing the momenta and spins into the equations. The discussion is somewhat simpler, however, if instead one uses equations of Dirac's type with the Hamiltonian given by

$$H = -c(\boldsymbol{\alpha}_1\mathbf{p}_1)-c(\boldsymbol{\alpha}_2\mathbf{p}_2) -(\beta_1+\beta_2)Mc^2-J+Q \quad (16.1)$$

as well as

$$H = -c(\boldsymbol{\alpha}_1\mathbf{p}_1)-c(\boldsymbol{\alpha}_2\mathbf{p}_2) -(\beta_1+\beta_2)Mc^2-\beta_1\beta_2J+Q \quad (16.2)$$

in the customary notation.

It is also possible to combine the two types and to have J occur partly with $\beta_1\beta_2$ and partly by itself. It will first be shown that Eq. (16.1) is satisfactory with

$$Q=\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})(dJ/rdr). \quad (16.3)$$

Only the contributions due to $J-Q$ to the integral in Eq. (15) need be considered since the other terms give an invariant contribution. The Lorentz frame K is changed to K'. The integral in K' will be evaluated by comparison with the integral in K. The world point (1) corresponding to (t, x_1, y_1, z_1) will be kept fixed temporarily. In K, dV_2 at a world point A corresponds then to the same time t. In K', however, the point $(t, x_1, y_1, z_1)=(t', x_1', y_1', z_1')$ occurs in the integral together with points $B=(t', x_2', y_2', z_2')$ which, therefore, have values of t different from that of (1). Let the points B be associated to the points A by having $(x, y, z)_B=(x, y, z)_A$. Then

$$dx_B'=(1-v^2/c^2)^{\frac{1}{2}}dx_A, \quad y_B'=y_B, \quad z_B'=z_B;$$
$$t_B-t_A=(v/c^2)(x_2-x_1);$$
$$J(\mathbf{r}_B'-\mathbf{r}_1')=J(r)-\frac{v^2}{2c^2}\frac{(x_2-x_1)^2}{r}\frac{dJ}{dr}, \quad (16.4)$$

where quantities without suffixes refer to system K. Consider a wave function $\psi_{\mu\nu}=\varphi_\mu\chi_\nu$ where φ depends only on 1 and χ depends only on 2. Then, using the conservation theorem for particle density, it is found, on neglecting all terms of order higher than v^2/c^2, that

$$(\chi'^*\chi')_B=\left[\left(1+\frac{v^2}{2c^2}\right)(\chi^*\chi)+\frac{v}{c}(\chi^*\alpha^1\chi) +\frac{v}{c}(x_2-x_1)\frac{\partial(\chi^*\alpha^k\chi)}{\partial x_2^k}\right]_A, \quad (16.5)$$

where $(\alpha^x, \alpha^y, \alpha^z)=(\alpha^1, \alpha^2, \alpha^3)$ and $(x, y, z)=(x^1, x^2, x^3)$. The symbol $(\chi^*Y\chi)$ stands for $\chi_\mu^*Y_{\mu\nu}\chi_\nu$. From now on the suffix A is omitted where no confusion is caused. From Eqs. (16.4) and (16.5) it follows that

$$\int(\chi_B'^*J(\mathbf{r}_B'-\mathbf{r}_1')\chi_B')dV_B'$$
$$=\int\left(\chi^*\left[J-\frac{vx}{c}\frac{(\mathbf{r}\boldsymbol{\alpha}_2)}{r}\frac{dJ}{dr}-\frac{v^2}{2c^2}\frac{x^2}{r}\frac{dJ}{dr}\right]\chi\right)dV_2$$

and

$$\left[\int(\psi^*J\psi)dV_2\right]_B'=\int\left(\psi^*\left[J-\frac{vx}{c}\frac{(\mathbf{r}\boldsymbol{\alpha}_2)}{r}\frac{dJ}{dr} -\frac{v^2}{2c^2}\frac{x^2}{r}\frac{dJ}{dr}+\frac{v^2}{2c^2}J+\frac{v}{c}\alpha_1^xJ\right]\psi\right)dV_2,$$

where $\mathbf{r}=(x, y, z)=(x_2-x_1, y_2-y_1, z_2-z_1)$. (16.6)

Using Q of Eq. (13.3)

$$\left[\int(\psi^*Q\psi)dV_2\right]'=\int(\psi^*Q\psi)dV_2$$
$$+\int\left(\psi^*\left[\frac{v}{2c}(\alpha_1^x+\alpha_2^x)J+\frac{v^2}{2c^2}J-\frac{vx}{2c}((\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)\mathbf{r})\frac{dJ}{rdr} -\frac{v^2x^2}{2c^2}\frac{dJ}{rdr}\right]\psi\right)dV_2.$$

On the left side the $'$ means that *all* quantities are evaluated in K'. Combining this with Eq. (16.6) and using the transformation formula for $\varphi^*\varphi$ it is found that

$$\left[\int(\psi^*[J-Q]\psi)dV_2\right]'$$
$$=\int\left(\psi^*\left\{J-Q+\frac{v}{2c}\frac{\partial}{\partial x}[\mathbf{r}(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2)J]\right\}\psi\right)dV_2$$
$$=\int(\psi^*[J-Q]\psi)dV_2. \quad (16.7)$$

Since dV_1dt is Lorentz invariant, so is Eq. (15). The use of products $\varphi\chi$ for ψ corresponds to small accelerations. Reducing Eqs. (16.1), (16.3) it is found that if

$$\{E+J+c\boldsymbol{\alpha}_1\mathbf{p}_1+c\boldsymbol{\alpha}_2\mathbf{p}_2+(\beta_1+\beta_2)Mc^2\}\psi=0, \quad (17)$$

then the so-called "large" components of ψ satisfy an equation of the Pauli type. This equation contains a non-Hermitean operator and is of the form

$$E\Psi+\frac{1}{4M^2c^2}[\mathbf{p}_1J\mathbf{p}_1-p_1{}^2J+\mathbf{p}_2J\mathbf{p}_2-p_2{}^2J]\Psi+\cdots=0,$$

all other terms containing only Hermitean operators. The occurrence of a non-Hermitean part is analogous to that already encountered in Eq. (8). Analogously to Eqs. (8′), (9) it is convenient to transform the wave function by

$$\Psi^{(1)}=\Psi+(p_1{}^2+p_2{}^2)\Psi/8M^2c^2 \quad (17.1)$$

so as to have the density represented by $(\Psi^{(1)*}\Psi^{(1)})$. The two-component wave function $\Psi^{(1)}$ satisfies

$$\left\{E-2Mc^2+J+\frac{h^2\Delta J}{4M^2c^2}+\sum_{i=1}^{2}\left[-\frac{p_i{}^2}{2M}+\frac{p_i{}^4}{8M^3c^2}+\frac{h}{4M^2c^2}([\nabla_iJ\times\mathbf{p}_i]\boldsymbol{\sigma}_i)\right]\right\}\Psi^{(1)}=0. \quad (17')$$

$$(\Delta J=\Delta_1J=\Delta_2J).$$

This equation does not take into account Q and represents only the result of using Eq. (17). The expression for Q given by Eq. (16.3) is a generalization of a similar term[2] in the energy for electromagnetic interactions between point charges. In the latter case $J=\pm e^2/r$ and the term in Q represents the combined effect of the magnetic energy and the retardation in the electrostatic potential. The term in Q has to be handled with circumspection because it represents the last term of an expansion. It is known[12] in the electromagnetic case that that this term is only good for first-order calculations of the energy and that such calculations lead to sensible results using directly Eq. (16.1) and regarding it as a perturbed Eq. (17). It may, therefore, be expected that the same procedure applies to the Q of Eq. (16.3). The result of the calculation may be expressed in terms of the Pauli function $\Psi^{(1)}$. If

$$\int\psi_{\mu\nu}{}^*Y_{\mu\nu;\,\mu'\nu'}\psi_{\mu'\nu'}d\tau=\int\Psi^{(1)*}{}_{\alpha\beta}Y^P{}_{\alpha\beta;\,\alpha'\beta'}\Psi^{(1)}{}_{\alpha'\beta'}d\tau \quad (17.2)$$

$$(\mu,\nu,\mu',\nu')=1,2,3,4;\ (\alpha,\beta,\alpha',\beta')=3,4$$

then the operator Y^P may be called the *equivalent Pauli operator*. Using it with the Pauli function in first-order energy calculations is equivalent to using Y with the Dirac functions satisfying Eq. (17). Since the two parts of Q entering Eq. (16.3) occur also in other possible equations, their equivalent Pauli operators are given separately:

$$[\tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J]^P=\frac{1}{8M^2c^2}\left\{4J\mathbf{p}_1\mathbf{p}_2+\frac{2h}{i}f\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)+h^2(3-2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)f+h^2[r^2-r^2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2+(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]\frac{df}{rdr}+2hf([\mathbf{r}\times\mathbf{p}_2]\boldsymbol{\sigma}_1-[\mathbf{r}\times\mathbf{p}_1]\boldsymbol{\sigma}_2)\right\}, \quad (17.3)$$

$$\left[-\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})\frac{dJ}{rdr}\right]^P=-\frac{1}{8M^2c^2}\left\{4fx^ax^bp_1{}^ap_2{}^b+\frac{8h}{i}f\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)+\frac{2h}{i}\frac{rdf}{dr}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-2hf([\mathbf{r}\times\mathbf{p}_2]\boldsymbol{\sigma}_1-[\mathbf{r}\times\mathbf{p}_1]\boldsymbol{\sigma}_2)+h^2f(12+2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)+h^2r^3\frac{d}{dr}\left(\frac{df}{rdr}\right)+h^2\frac{df}{rdr}[9r^2+r^2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]\right\}, \quad (17.4)$$

where $$f=dJ/rdr,\quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2=(x^1,x^2,x^3). \quad (17.5)$$

A short calculation shows that the operators of Eqs. (17.3) and (17.4) are Hermitean. The above expressions can be expressed more elegantly by means of symmetrical formulas involving J, $\mathbf{p}_1$, $\mathbf{p}_2$, $(\mathbf{p}_1\mathbf{r})$, etc. It was thought, however, to be more useful to give the expressions in the above form because the occurrence of $\mathbf{p}_1$, $\mathbf{p}_2$ only on the right, uniquely defines the expressions. Combining Eqs. (17′), (17.3), (17.4) one has as an equivalent Pauli equation in $\Psi^{(1)}$,

$$\Big\{-E+2Mc^2-J+\frac{p_1^2+p_2^2}{2M}-\frac{p_1^4+p_2^4}{8M^3c^2}+\frac{\hbar f}{4M^2c^2}([\mathbf{r}\times(2\mathbf{p}_2-\mathbf{p}_1)]\boldsymbol{\sigma}_1-[\mathbf{r}\times(2\mathbf{p}_1-\mathbf{p}_2)]\boldsymbol{\sigma}_2)+\frac{J}{2M^2c^2}$$
$$\times\mathbf{p}_1\mathbf{p}_2-\frac{f}{2M^2c^2}x^ax^bp_1{}^ap_2{}^b-\frac{3\hbar f}{4iM^2c^2}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{\hbar}{4iM^2c^2}\frac{rdf}{dr}\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{\hbar^2 f}{8M^2c^2}$$
$$\times(15+4\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-\frac{\hbar^2}{4M^2c^2}\frac{df}{rdr}[5r^2+r^2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2-(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]-\frac{\hbar^2r^3}{8M^2c^2}\frac{d}{dr}\left(\frac{df}{rdr}\right)\Big\}\Psi^{(1)}=0. \quad (17.6)$$

For $J=-e^2/r$ this equation simplifies into Eq. (48) of reference 13. In that case terms in $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, $(\boldsymbol{\sigma}_1\mathbf{r})(\boldsymbol{\sigma}_2\mathbf{r})$ combine to give a contribution to the energy

$$(\hbar e/2Mc)^2[\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2r^{-3}-3(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)r^{-5}], \quad (17.7)$$

which represents the magnetic interaction energy between spins having each a magnetic moment $\hbar e/2Mc$. The interaction energy of Eq. (17.7) vanishes to the first order for the deuteron. On the other hand[14] Eq. (17.6) contains terms which do not vanish for the S states of the deuteron and give a net effect in $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ of amount

$$-(\hbar^2/6M^2c^2)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\Delta J \quad (\mathrm{H}^2). \quad (17.8)$$

This interaction gives different energies in the singlet and triplet states, which corresponds to experience. The sign of Eq. (17.8) is such that for simple attractive potentials varying smoothly and monotonically with r, the energy is higher for triplets than for singlets. Thus ordinary, non-exchange, purely attractive interaction energies of the type of Eqs. (16.1), (16.3) are not likely to correspond to experience even though they contain a spin-spin dependence of the interaction energy. A repulsion in the region of smaller r produces a reversal of the sign of the effect of the expression (17.8). It is conceivable that such an interaction corresponds to reality, particularly since a combination of repulsions at small distances with attractions at larger distances is helpful for securing stability of heavy nuclei without the aid of exchange forces.

The presence of terms in $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ and $(\boldsymbol{\sigma}_1\mathbf{r})(\boldsymbol{\sigma}_2\mathbf{r})$ is secured above through the introduction of the form (16.3) and not directly through the requirements of invariances. A form such as that of Eq. (17.6) remains invariant if arbitrary terms in spin-spin interactions are added to the left side, provided these terms are of order v^2/c^2. There is thus nothing binding about (17.8). The only argument advanced here is that of the attractiveness of using a simple Diracian form such as that of Eq. (16.3). The verification of invariance given by Eqs. (16.4) to (16.7) is now seen to contain little more than a direct inspection of Eq. (17.6) and its comparison with possible spin-orbit interactions of Eq. (15.7) as well as possible classical forms given by Eq. (13.2). It will thus not be necessary to go through a formal proof in discussing Diracian forms.

The use of Eq. (16.2) is somewhat analogous to that of Eq. (10). It should be noted, however, that the term in $\beta_1\beta_2J$ without Q does not give an invariant contribution to the Lagrangian.[15] This can be verified either in the Dirac or the Pauli form.

Reducing Eq. (16.2) with $Q=0$ to the Pauli form one finds

$$\Big\{E-2Mc^2+J+\sum_{i=1}^{2}\Big[-\frac{p_i^2}{2M}+\frac{p_i^4}{8M^3c^2}-\frac{p_i^2J+2\mathbf{p}_iJ\mathbf{p}_i+Jp_i^2}{8M^2c^2}+\frac{\hbar}{4M^2c^2}[\mathbf{p}_i\times\nabla_iJ]\boldsymbol{\sigma}_i\Big]\Big\}\Psi^{(1)}=0. \quad (18)$$

Comparison with Eq. (15.4) shows that the spin-orbit interaction is given correctly by Eq. (18).

Comparison with Eq. (13.2) shows that the sum of the coefficients of $\mathbf{p}_1^2 J$, $\mathbf{p}_1 J \mathbf{p}_1$, $J\mathbf{p}_1^2$ is twice what it should be. The equation can be corrected for invariance either by arbitrarily modifying these terms or by looking for Diracian forms which will automatically give a correct behavior. By means of Eqs. (17.3), (17.4) such forms can be reduced to equations of the Pauli type.

It was seen that the classical Eq. (14.2) has a simpler physical interpretation than Eq. (14.1). A Diracian form agreeing with Eq. (14.2) in the sense of the correspondence principle is

$$\{E+c(\boldsymbol{\alpha}_1\mathbf{p}_1)+c(\boldsymbol{\alpha}_2\mathbf{p}_2)+(\beta_1+\beta_2)Mc^2+\beta_1\beta_2 J + \tfrac{1}{2}(\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)J+\tfrac{1}{2}(\boldsymbol{\alpha}_1\mathbf{r})(\boldsymbol{\alpha}_2\mathbf{r})(dJ/rdr)\}\psi=0. \quad (18.1)$$

Its equivalent Pauli form is found to be, by means of Eqs. (18), (17.3), (17.4), (17.5),

$$\Bigg\{-E+2Mc^2-J+\sum_{i=1}^{2}\left[\frac{p_i^2}{2M}-\frac{p_i^4}{8M^3c^2}+\frac{p_i^2J+2\mathbf{p}_iJ\mathbf{p}_i+Jp_i^2}{8M^2c^2}+\frac{h}{4M^2c^2}[\boldsymbol{\nabla}_iJ\times\mathbf{p}_i]\boldsymbol{\sigma}_i\right]-\frac{J}{2M^2c^2}(\mathbf{p}_1\mathbf{p}_2)$$
$$-\frac{f}{2M^2c^2}x^ax^bp_1^ap_2^b-\frac{h}{4iM^2c^2}\left(5f+\frac{rdf}{dr}\right)\mathbf{r}(\mathbf{p}_2-\mathbf{p}_1)-\frac{h^2}{8M^2c^2}\left[15f+10\frac{rdf}{dr}+\frac{r^3d}{dr}\left(\frac{df}{rdr}\right)\right]\Bigg\}\Psi^{(1)}=0. \quad (18.2)$$

Here the spin-orbit interactions are entirely of the Thomas type and there are no spin-spin interactions. Thus Eq. (18.1) may be looked at as a generalization of Eq. (10). It represents an interaction in which particle 2 is moving approximately in the field of a scalar due to particle 1. Eqs. (17.3), (17.4), (17.6), (18.2) apply only to interactions of a nonexchange type.

4. Applications

According to Rumbaugh and Hafstad[16] there is a level of Li^7 at about 400 kv above the normal level. This level is probably the $^2P_{\frac{1}{2}}$ part of the ^2P normal level of Li^7. According to Hafstad and Tuve[17] the resonance of the reaction $C^{12}+H^1\rightarrow N^{13}+h\nu$ shows indications of fine structure of the order of 80 kv. According to Herb, Parkinson and Kerst[18] there are indications of fine structure in the resonances for γ-ray production of Li, B, F, Al, Na, Be when these elements are bombarded by protons. The order of magnitude of the difference between adjacent resonance levels is about 200 kv. The spin-orbit interactions must be arranged so as to account for this order of magnitude. Using ordinary interaction potentials the order of magnitude of the expected splitting of a single particle moving in a p orbit is

$$\Delta E=-\frac{3h^2}{4M^2c^2}\left[\frac{dJ_{\text{eff}}}{rdr}\right]_{\text{av}}\cong-\frac{h^2}{M^2c^2r_0^2}[J_{\text{eff}}]_{\text{av}}, \quad (19)$$

where the subscript av indicates the average value. Here J_{eff} is the effective value of the interaction energy due to all the other particles considered as producing an effective central field. The nuclear radius is r_0. For $[J_{\text{eff}}]_{\text{av}}=10$ Mev and $r_0=e^2/mc^2$ the expected ΔE is then of the order of 50 kv. This agrees with the observations on N^{13} but is too small for the 400 kv splitting in Li^7. These are fitted better with $[J_{\text{eff}}]_{\text{av}}=20$ Mev, $r_0=e^2/2mc^2$ which is not unreasonable. Eq. (19) corresponds to the use of Eq. (15.4) with $b=1$, because in such a case the spin-orbit interaction consists entirely of terms of type

$$[\boldsymbol{\nabla}_i\sum_{j\neq i}J_{ij}\times\boldsymbol{\sigma}_i]\mathbf{p}_i \quad\text{and}\quad \sum_{j\neq i}J_{ij}$$

gives directly the potential in which particle i is moving. Thus Wigner forces are readily reconcilable with present experimental indications as to the magnitude of fine structure using essentially interactions of the type of Eq. (18.1). The inversion of the fine structure in Li^7 is also in agreement with this type of interaction energy.

For Majorana forces Eq. (15.7) with $a=1$ gives an interaction which is somewhat analogous to the Wigner case, just discussed, as is seen from Eq. (15.8′). With this form the following contributions to the fine structure are found for a proton interacting with a complete shell of $2(2L'+1)$ neutrons

$$\Delta E = -\frac{(2L+1)(2L'+1)\hbar^2}{32\pi^2 M^2c^2 L(L+1)}\int \{L(L+1)\cos\theta$$
$$\times P_L[3Q_1Q_2 + r_1Q_1'Q_2 + r_2Q_1Q_2']$$
$$+\sin^2\theta P_L'[(4+L(L+1))Q_1Q_2 + 2r_1Q_1'Q_2$$
$$+2r_2Q_1Q_2' + r_1r_2Q_1'Q_2']\}$$
$$\times R_{L'}(1)R_{L'}(2)P_{L'}J_{12}d\tau_1 d\tau_2. \quad (19.1)$$

Here L = azimuthal quantum number of proton; L' = azimuthal quantum number of complete shell; P_L = Legendre function of argument $\cos\theta$, where θ is the angle between the directions of points 1 and 2; $R_{L'}$, rQ are the radial functions of proton and neutron respectively. Derivatives with respect to r and $\cos\theta$ are indicated by $'$. The normalization is such that

$$\int R_L^2 r^2 dr = 1; \quad \int Q^2 r^4 dr = 1. \quad (19.2)$$

Eq. (19.1) applies only to cases in which a proton is outside a complete shell. It does not take care of perturbations due to other states of the shell, which, according to Feenberg and Wigner,[19] are important in nuclear structure. The effect of such perturbations is not taken into account in this paper.

For Li^7 estimates of expected fine structure can be made by supposing that there are two protons in s states and one proton in a p state. Two neutrons are supposed to be in s states and two in p states. The pairs of like particles in s states form complete shells. Their effect can be computed by means of Eq. (19.1). The two neutrons in p states are considered here to be coupled into a ^1S term. They effectively form a complete shell of two particles. Eq. (19.1) applies to their effect on ΔE, provided the result of the calculation is divided by $6/2=3$. This can be verified by explicit calculation. In addition there is an interchange effect due to the fact that the proton in the p state is a particle identical with the two protons in the s shell. In the estimates, the radial wave functions of all particles with the same L were taken to be the same, independently of whether they are protons or neutrons. The radial functions used were such that

$$R_0 = N_R e^{-\mu r^2/2}, \quad N_R^2 = 4\mu^{3/2}/\pi^{1/2};$$
$$Q = N_Q e^{-\nu r^2/2}, \quad N_Q^2 = 8\nu^{5/2}/3\pi^{1/2}; \quad (19.3)$$

and the interaction potential was taken to be

$$J = Ae^{-\alpha r^2}. \quad (19.4)$$

The contributions to ΔE were found to be:

$$(\Delta E)_{s\nu} = -\frac{3\hbar^2}{4M^2c^2}\frac{A(\mu\nu)^{5/2}(\mu+4\alpha)}{[\alpha(\mu+\nu)+(\mu+\nu)^2/4]^{5/2}} \quad (19.5)$$

due to the s shell of neutrons,

$$(\Delta E)_{s\pi} = (\Delta E)_{s\nu} - \frac{3\hbar^2}{2M^2c^2}\frac{A(\mu\nu)^{5/2}\alpha}{[\alpha(\mu+\nu)+\mu\nu]^{5/2}} \quad (19.6)$$

due to the s shell of protons, and

$$(\Delta E)_{p\nu} = -\frac{\hbar^2}{4M^2c^2}\frac{A\alpha\nu^5(4\alpha\nu-3\nu^2)}{[2\alpha\nu+\nu^2]^{7/2}} \quad (19.7)$$

due to two p neutrons in a ^1S configuration.

Using $A=38\ mc^2$, $\alpha=16\ Mmc^2/\hbar^2$, and trying to fit the apparent experimental value $\Delta E=-0.8\ mc^2$, one obtains approximate agreement using $\mu=\nu \cong 40\ Mmc^2/\hbar^2$. No precise fit was attempted. The above values of μ and ν correspond to values of $(r^2)_{\text{av}}^{1/2}$ of 1.7×10^{-13} cm and 2.2×10^{-13} cm for s and p particles, respectively. The maximum in the number of p particles per unit thickness of a thin spherical shell (per $\Delta r=1$) would be at 2.0×10^{-13} cm. Since the calculation is not precise, these numbers may be considered satisfactory. It may be premature to expect an exact agreement inasmuch as even in atoms calculations of fine structure using Hartree and Fock fields do not always give correct results.

The values of μ and ν are seen to agree quite well with the calculations of Feenberg and Wigner,[19] who obtain from mass defect values of $\mu=\nu=2\alpha\sigma$ of 26 and 32 in units $Mmc^2/\hbar^2$. Increasing the value of A from $38mc^2$ to $63mc^2$ the value of $\mu=\nu$, as estimated from fine structure of Li^7, drops to $27Mmc^2/\hbar^2$. The agreement thus obtained may be partly accidental.

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$$\Delta E=\frac{8\pi}{3}\left(\frac{e\hbar}{2Mc}\right)^2\psi^2(0)[2s(s+1)-3],$$

where $s=1$ for ^3S, $s=0$ for ^1S, and $\psi(0)$ is the Schroedinger wave function representing the relative motion of proton and neutron taken at $r=0$. This formula agrees with Casimir's who obtains the same result multiplied by $\gamma_n\gamma_p/4$. Casimir's γ's are defined so that the projection of the magnetic moment on an axis is $(\gamma/2)(e\hbar/2Mc)$ when the projection of the spin is $\hbar/2$. For particles obeying Dirac's equation $\gamma/2=1$ and the factor $\gamma_n\gamma_p/4$ should thus be absent in our discussion. From Eq. (17.6), collecting and rearranging terms responsible for spin-spin interactions, one obtains as perturbation energy

$$\Delta E=\frac{\hbar^2 f}{4M^2c^2}[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-3r^{-2}(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)]-\frac{\hbar^2\Delta J}{4M^2c^2}[(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)-r^{-2}(\mathbf{r}\boldsymbol{\sigma}_1)(\mathbf{r}\boldsymbol{\sigma}_2)],$$

which shows clearly how the first part alone is inadequate for a general treatment. This last expression for ΔE can be compared with analogous expressions in the theory of hyperfine structure as, e.g., Eq. (2) in G. Breit, Phys. Rev. **37**, 51 (1931). The inclusion of Casimir's factor γ corrects for the fact that the actual magnetic moment differs from that predicted by Dirac's equation. On the other hand the possible effect of ΔJ, where J stands for the interaction energy between proton and neutron, is not included in his treatment. If Eq. (17.6) were applicable to the interaction between an electron and a neutron (or proton) one would expect a coupling of the electron to the nucleus of a nonelectromagnetic origin. The present agreement of values for the magnetic moment of the deuteron and other nuclei as obtained by different methods (Stern-Rabi) indicates that this coupling is not large. An ultimate decision must, however, be postponed until present experimental uncertainties are cleared up. Since the additional interactions discussed here would not be expected to vary in the same way with distance as for ordinary magnetic effects it may also be possible to test for their presence by a careful comparison of hyperfine structure in spectroscopic terms having different azimuthal quantum numbers.

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