$\frac{8(a+2b)(a+b)(2a+b)}{11a+10b}$, $\cdots$; and in defect (in each case the minimum is for $p=0$), b^2, $(a+b)^2$, $\frac{(a+2b)(2a+b)^2}{3a+2b}$, $\frac{(a+b)(a+2b)(3a+2b)(3a+b)^2}{19a^3+35a^2b+21ab^2+4b^3}$, $\frac{(a+2b)(3a+2b)(4a+b)^2(19a^3+35a^2b+21ab^2+4b^3)}{633a^5+1559a^4b+1525a^3b^2+736a^2b^3+174ab^4+16b^5}$,

Table I shows the convergence for $b=0$, $b=a$, $a=0$.

It is hoped to publish in the near future approximate solutions by this method of a number of problems for which approximations with known limits of error are not otherwise easy to obtain. The iteration for these problems has to be carried out numerically.

The author wishes to take this opportunity to thank Professor Bohr for the hospitality of his Institute last spring when he spent two months in Copenhagen. During that time the author's ideas about the above method took their present form.

TABLE I. *Convergence for $b=0$, $b=a$, $a=0$.*

$b=0$	$b=a$	$a=0$	REMARKS
$2\ a^2$ $1.5\ a^2$ $1.454\ a^2$	$12\ a^2$ $7.5\ a^2$ $6.857\ a^2$	$4\ b^2$ $2\ b^2$ $1.6\ b^2$	Approximations in excess
$1.4458\ a^2$	$6.592\ a^2$	b^2	Actual value[5]
$1.441\ a^2$ $1.421\ a^2$ $1.333\ a^2$ a^2 0	$6.380\ a^2$ $6.077\ a^2$ $5.4\ a^2$ $4\ a^2$ a^2	b^2 b^2 b^2 b^2 b^2	Approximations in defect

[5] Jahncke and Emde, *Funktiontafeln*, p. 238.

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Nuclear Spins and Magnetic Moments in the Hartree Model

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From the results of Feenberg and Wigner for the wave function and term character of the ground state of light nuclei (mass number between 6 and 16), the nuclear spins are determined. For those nuclei which contain (or lack) a single proton (or neutron) and an even number (singlet state) of particles of the other kind the considerations of Inglis suffice to determine the spin. For those nuclei which contain a half-filled p shell in one kind of particle it is necessary to calculate the fine structure splitting explicitly. From the spins thus found and with the experimental values for the magnetic moments of the proton and neutron, the nuclear magnetic moments are calculated. The effects on the nuclear moment of the Heisenberg forces and of the motion of the $1s$ shell are considered. The moment of Li^7 which is of particular interest, is calculated to be 3.07 nuclear magnetons. This is in agreement with the measured value of 3.20 n. m.

1. INTRODUCTION

It has been pointed out by Bethe and Bacher[1] that the individual particle model (Hartree model) affords one the opportunity to construct a rational theory of nuclear spins and magnetic moments for light nuclei. On the basis of this model, by assigning quantum states and individual wave functions[2] to each nuclear particle, one can calculate, in the same manner as in atomic spectra, the energy of the various terms which arise from any given configuration of neutrons and protons. From the spin and orbital momenta of the nucleus in the ground state found in this way and from considerations as to the coupling of these momenta the magnetic moment of the nucleus may be calculated. It is to be expected that the model will break down for all but light nuclei[3] and that even for these light nuclei one can obtain only roughly correct

[1] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 82 (1936). In §36 these authors have treated the case of the Li^6 nucleus.

[2] These are determined by a suitable auxiliary central field which may be assumed to be the same for each particle. Following the usual procedure we shall take for this field an oscillator potential.

[3] The model may be expected to give relatively reliable results for nuclei up to O^{16} where the $2p$ shell is just completed.

TABLE I. *Ground states of light nuclei as given by Feenberg and Wigner.*

Nuclei	Ground State Term	Wave Function
He^6 C^{14}	^1S	
Li^6 N^{14}	^3S	
Li^7 *Be^7* N^{13} *C^{13}*	^2P	$C_1(^1D^2P)+C_2(^1S^2P)$
Li^8 *B^{12}*	^3P	$C_1(^2P^2P)+C_2(^2D^2P)$
Be^8 *C^{12}*	^1S	$C_1(^1S^1S)+C_2(^1D^1D)$
Be^9 B^9 B^{11} *C^{11}*	^2P	$C_1(^1S^2P)+C_2(^1D^2P)+C_3(^1D^2D)$
Be^{10}	^1S	$C_1(^1S^1S)+C_2(^1D^1D)$
B^{10}	^3S	$C_1(^2P^2P)+C_2(^2D^2D)$
N^{15} O^{15}	^2P	

quantitative results for the energy. Nevertheless, it is plausible that the order of the levels will be given correctly and it is only this feature of the results that need be used here.

The calculation of the term energies for nuclei of mass number between 6 and 16 has been given by Feenberg and Wigner.[4] On the assumption of Russell-Saunders coupling, which is almost certainly valid because the magnetic forces are very small compared to the nuclear forces, and neglecting Heisenberg forces,[5] in which case both the spins of the neutrons and protons are good quantum numbers, they find for the ground state the results given in Table I.

In the third column of the table the first term in the bracket gives the parent term of the neutrons and the second that of the protons except in the cases of the nuclei in italics for which neutrons and protons are to be interchanged. As is indicated, in most cases the wave function for the ground state is a mixture of wave functions arising from different parentages in the neutrons and protons. The coefficients C_1, C_2 and C_3 with which the various parent wave functions are multiplied depend, of course, on the specific nuclear forces though not very sensitively.[6] In addition, in the absence of Heisenberg forces between unlike particles, they are independent of the exchange properties of the forces between like particles.[7]

[4] E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).

[5] The Heisenberg operator can be written as $P_H = \frac{1}{2}P_M(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ where P_M is the Majorana operator and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the spin operators referring to the two particles. The part $\frac{1}{2}P_M$ can be incorporated into the Majorana operator and the part $\frac{1}{2}P_M(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ neglected.

[6] See §2 reference 12 below.

[7] See appendix.

TABLE II. *Coefficients for the parent wave functions.*

Row	C_1	C_2	C_3
Li^7	0.681	0.732	
Li^8	0.785	0.619	
Be^9	0.731	−0.344	−0.589

For the unlike particle interaction we have (cf. reference 5)

$$V_{\nu\pi}=A_{\nu\pi}(1-g/2)e^{-r^2/a^2}P_M, \qquad (1)$$

P_M being the Majorana exchange operator, with $g=0.22$, $A_{\nu\pi}=37$ MV and $a=2.93\times10^{-13}$ cm.[8] For the like particle forces we take only

$$V_{\nu\nu}=V_{\pi\pi}=A_{\pi\pi}e^{-r^2/a^2} \qquad (2)$$

with $A_{\pi\pi}=21$ MV. Then the coefficients are as given in Table II.

If the Heisenberg forces are not neglected the resultant spins of both the neutrons and protons will no longer be good quantum numbers. Thus the ^3P term of the p^2 configuration and the ^4S term of the p^3 configuration will interact with the other terms (i.e., ^1S, ^1D and ^2P, ^2D, respectively). Then the wave function for Li^7, e.g., has an additional parent wave function $C_3(^3P^2P)$. We shall consider the effect of this addition to the wave function in §4(a).

2. The Nuclear Spins

We now consider the total angular momentum J of the nucleus in the ground state. For the nuclei whose ground state is an S term we can, of course, obtain J at once from Table I. However for the nuclei whose ground state term is ^2P, i.e., those with odd mass number, and for Li^8 and B^{12} whose ground state term is ^3P, we have to consider the spin-orbit forces which split the levels of the multiplet.

In the case of Li^7 the fine structure splitting is due to the single p proton, the two p neutrons entering in a singlet state. Here we can apply

[8] This range of the forces has been chosen by Feenberg and Wigner (reference 4), somewhat larger than that deduced from the theory of binding energies of H^2, H^3 and He^4 (namely, $a=2.3\times10^{-13}$ cm) in an attempt to compensate for the correlation forces between the particles which the present model fails to take into account. The particular value chosen for a makes the calculated energy of O^{16} agree with experiment.

the considerations of Inglis[9] who has pointed out that the Thomas part of the spin-orbit interaction will preponderate over the magnetic interaction. This is due to the fact that the Thomas term comes from the specifically nuclear forces while the magnetic interaction is due to the weak electrostatic forces. The nuclear forces being attractive, this leads to an inverted doublet for Li^7 and therefore a spin of 3/2.

In the absence of magnetic forces there is complete symmetry between neutrons and protons. Therefore the spins will be the same for any pair of isobars which may be obtained from each other by the interchange of neutrons and protons. Thus the doublet in Be^7 is inverted ($J=3/2$) with the same fine structure splitting as for the Li^7 doublet. Because of the opposite signs of the magnetic moments of the proton and neutron, the magnetic forces will slightly enhance the splitting in Li^7 as compared to that in Be^7.

The nuclei N^{13} and C^{13} are the images of Li^7 and Be^7 with respect to the half-closed shell of neutrons *and* protons; that is, the p shell for the former nuclei lack the particles which are contained in the latter nuclei. Moreover, just as in atomic spectra, the spin-orbit energy for the completely filled p shell is zero. Therefore the splitting for N^{13} and C^{13} is of opposite sign to that of Li^7 and Be^7. These nuclei should then have a regular doublet and spin $\frac{1}{2}$. Obviously O^{15} and N^{15}, lacking a single p particle, should also have regular doublets and spin $\frac{1}{2}$.

For Li^8, beside the single p proton, it is necessary to consider the contribution to the splitting of the half filled shell of neutrons. The Thomas part of the spin-orbit interaction may be written as

$$U=-\frac{\hbar^2}{2M^2c^2}\sum_i\left(\frac{1}{r}\frac{\partial V}{\partial r}\right)_i(\mathbf{l}_i\cdot\mathbf{s}_i), \tag{3}$$

in which $\mathbf{l}_i$ and $\mathbf{s}_i$ are the orbital and spin momenta (in units of $\hbar$) of the ith particle.

We must calculate the matrix element of this interaction with the ground state wave function for Li^8.

$$\Psi(^3P)=C_1\psi_1(^2P^2P)+C_2\psi_2(^2D^2P). \tag{4}$$

[9] D. R. Inglis, Phys. Rev. **50**, 783 (1936).

The P wave functions compounded from neutron and proton wave functions with the magnetic quantum number[10] $M_L=M_L^\pi+M_L^\nu=1$ for any multiplicities are

for $L_\nu=1$, $L_\pi=1$:

$$\psi_1=2^{-\frac{1}{2}}\{(P^0P^1)-(P^1P^0)\}, \tag{5.1}$$

for $L_\nu=2$, $L_\pi=1$:

$$\psi_2=10^{-\frac{1}{2}}\{6^{\frac{1}{2}}(D^2P^{-1})+3^{\frac{1}{2}}(D^1P^0)-(D^0P^1)\}, \tag{5.2}$$

writing the M_L for each kind of particle, M_L^π and M_L^ν, as a superscript. For reference we give the wave functions in the m_lm_s scheme. For the three-neutron wave functions with $M_S^\nu=\frac{1}{2}$ one obtains[11]

$$\begin{aligned}^2P^1&=2^{-\frac{1}{2}}[(1^+0^+0^-)-(1^+1^--1^+)],\\ ^2P^0&=2^{-\frac{1}{2}}[(1^-0^+-1^+)-(1^+0^+-1^-)],\\ ^2P^{-1}&=2^{-\frac{1}{2}}[(0^+0^--1^+)-(1^+-1^+-1^-)],\end{aligned} \tag{6.1}$$

$$\begin{aligned}^2D^2&=(1^+1^-0^+),\\ ^2D^1&=-2^{-\frac{1}{2}}[(1^+1^--1^+)+(1^+0^+0^-)],\\ ^2D^0&=6^{-\frac{1}{2}}[(1^+0^+-1^-)\\ &\qquad+(1^-0^+-1^+)-2(1^+0^--1^+)].\end{aligned} \tag{6.2}$$

In (6) the numbers in the round bracket give the value of m_l for each neutron and the symbols $+$ and $-$ refer to the sign of the spin component in a given direction z. Each round bracket, of course, is a normalized antisymmetrical (determinantal) wave function and orthogonal to all the others. If we take for the proton wave function (m_l^+), the wave function (4) as given obviously refers to the substate $M=2$, $J=2$. The coefficients C_1 and C_2 may be taken from Table II.[12]

In the calculation of the spin-orbit energy it will be sufficient for our purposes to assume that the particles in the p shell move in a central field. For any central field (one-particle inter-

[10] $M_L^\pi=\sum_{\text{protons}} m_l$, $M_L^\nu=\sum_{\text{neutrons}} m_l$.

[11] For notation see Condon and Shortley, *Theory of Complex Spectra* (Cambridge Press), p. 169. The angular part of the single particle wave function for $m_l=1$, 0 and -1 contains $2^{-\frac{1}{2}}(x+iy)$, z and $2^{-\frac{1}{2}}(x-iy)$, respectively.

[12] In calculating these coefficients the wave functions (5) and (6) with the interactions (1) and (2) are used. As an example of the insensitivity of the coefficients with regard to the magnitude of the force constants it may be mentioned that if we take $A_{\nu\nu}=A_{\nu\pi}$ we obtain $C_1=0.791$ and $C_2=0.612$.

action) the matrix elements of (3) are simply proportional to the matrix elements of

$$-\sum_i \mathbf{l}_i\cdot\mathbf{s}_i = -(\mathbf{l}_\pi\cdot\mathbf{s}_\pi + \sum_{\text{neutrons}} \mathbf{l}_\nu\cdot\mathbf{s}_\nu).$$

For the single proton, of course, only the diagonal terms $(\psi_1|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|\psi_1)$ and $(\psi_2|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|\psi_2)$ give a nonvanishing result. We find

$$\begin{aligned}(\Psi|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|\Psi) &= \tfrac{1}{2}C_1{}^2(1|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|1)\\ &+\frac{1}{10}C_2{}^2\{6(-1|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|-1)+(1|\mathbf{l}_\pi\cdot\mathbf{s}_\pi|1)\}\\ &= \tfrac{1}{4}(C_1{}^2-C_2{}^2)=0.058.\end{aligned} \quad (7)$$

The numbers in the matrix element refer to the value of $M_L{}^\pi$

For the three neutrons we may transform the neutron wave functions to the JM scheme according to[13]

$$\psi(L,S,M,M_S=\tfrac{1}{2}) = \left(\frac{L+M+\frac{1}{2}}{2L+1}\right)^{\frac{1}{2}}\varphi({}^2L^M{}_{L+\frac{1}{2}}) - \left(\frac{L-M+\frac{1}{2}}{2L+1}\right)^{\frac{1}{2}}\varphi({}^2L^M{}_{L-\frac{1}{2}}), \quad (8)$$

in which the subscripts in φ refer to the values of J and the superscript ν on the quantum numbers has been suppressed. For the neutrons we have $M=M_L+\frac{1}{2}$. We find that only the cross term gives a nonvanishing result.

$$\begin{aligned}(\Psi|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|\Psi) &= 2C_1C_2(\psi_1|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|\psi_2)\\ &= -2(20)^{-\frac{1}{2}}C_1C_2\{3^{\frac{1}{2}}(1|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|1)\\ &\quad+(0|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|0)\}.\end{aligned} \quad (9)$$

From (8) we have

$$(L=1,M_L|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|L=2,M_L) = -[(4-M_L{}^2)/15]^{\frac{1}{2}}({}^2P_{3/2}|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|{}^2D_{3/2}) \quad (10)$$

and taking the value for the matrix element from Condon and Shortley[14] this becomes

$$(1,M_L|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|2,M_L) = -\tfrac{1}{2}[(4-M_L{}^2)/3]^{\frac{1}{2}}. \quad (11)$$

Thus we obtain

$$(\Psi|\sum_\nu \mathbf{l}_\nu\cdot\mathbf{s}_\nu|\Psi) = \tfrac{1}{2}(5/3)^{\frac{1}{2}}C_1C_2 = 0.313. \quad (12)$$

Comparing (7) and (12) we see that the contribution to the spin-orbit energy of the three neutrons is of the same sign as the contribution of the single proton.[15] Thus with a positive value for the matrix element of $\sum_i \mathbf{l}_i\cdot\mathbf{s}_i$ it follows from (3) that the spin-orbit displacement of the level $J=2$ is negative. Therefore the multiplet is inverted and the lowest state of Li^8 has total angular momentum $J=2$.

Neglecting magnetic forces, the spin-orbit energy of a half-filled shell of protons is also given by (12). (Moreover, the magnetic forces give a contribution with the same sign as that of the half-filled shell.) Thus in B^{12} the main contribution to the spin-orbit energy will arise from the half-filled shell. The contribution of the five p neutrons, although of opposite sign, is much smaller (cf. (7) and (12)) and will not reverse the order of the levels. Therefore we have again an inverted multiplet with ground state $J=2$.[16]

For Be^9, B^9, B^{11} and C^{11} we have to consider only the spin-orbit energy of a half-filled shell in one kind of particle, the other kind of particle entering in a singlet state. A calculation similar to that given above leads to the result that these nuclei should have inverted doublets, ground state $J=3/2$.

It may be noted that the attribution of the major part of the spin-orbit interaction to the nuclear forces leads to the conclusion that the members of the isobaric pairs Be^7 Li^7, C^{11} B^{11},

[13] The phases are chosen so that the nondiagonal matrix elements of L_z are positive. Cf. Condon and Shortley, reference 11, p. 66 and §14.

[14] Reference 11, p. 268. We set $\xi(r)$ (their notation) $=1$ and therefore $\zeta_p=1$.

[15] We are indebted to Professor Wigner for pointing out an error previously made in this calculation.

[16] Although it is true that the central field model used here is somewhat crude it ought certainly to give the correct sign for each contribution to the spin-orbit energy. Furthermore, in view of the result that the contribution to the energy of the half-filled shell is about five times as large as that of a single particle, it seems safe to conclude that the sign of the spin-orbit splitting obtained here is correct.

N^{13} C^{13} and O^{15} N^{15} are completely analogous in the sense that they have the same ground state J. If only magnetic forces were considered one must conclude that the doublet of one member of the isobaric pair would be regular and that of the other member inverted. Therefore, in the β transformations between the two members of the pair, one would expect a spin change $\Delta J=0$ in the former case and $\Delta J=1$ in the latter. The former value is in agreement with the known fact that the observed β transformations (the last three pairs mentioned) belong to the first Sargent curve.[17]

The Russell-Saunders coupling, which has been assumed in our considerations, leads for many nuclei to a fairly narrow doublet of which the lower level is the ground state. Experimental evidence for this has been obtained by Rumbaugh and Hafstad[18] who observed a doublet fine structure in the proton group from $Li^6+H^2=Li^7+H^1$. The intensity ratio of the two groups was about 1 : 2, that is, the ratio of the statistical weights of the two levels of the doublet in Li^7. It may be suggested that similar fine structure may be observable in the proton groups from the reactions

$$B^{10}+H^2=B^{11}+H^1,$$
$$C^{12}+H^2=C^{13}+H^1,$$
$$N^{14}+H^2=N^{15}+H^1,$$

if thin targets are used. The best studied of these is the carbon reaction for which the range curve for the 14 cm protons has been given by Cockcroft and Lewis.[19] These authors also gave for comparison the range curve for the protons from the D–D reaction at the same deuteron energy (560 kv). The straggling of the D–D protons might be expected to be larger than that of the C–D protons for two reasons. Firstly, for the D–D protons there is a greater variation in energy with angle, the recoil of H^3 being larger than that of C^{13}. Secondly, the excitation function for the D–D reaction is flatter (smaller potential barrier) so that deuterons of a larger range of energies would be effective. Actually the straggling is somewhat less for the D–D protons. This might be taken as a rather weak argument for the existence in the C–D reaction of two proton groups superimposed on each other with an appreciable difference in range, say of the order of one cm.

[17] H. A. Bethe and R. F. Bacher, reference 1, Table XV, p. 195.

[18] L. H. Rumbaugh and L. R. Hafstad, Phys. Rev. **50**, 681 (1936). See also L. A. Delsasso, W. A. Fowler and C. C. Lauritsen, Phys. Rev. **48**, 848 (1935).

[19] J. D. Cockcroft and W. B. Lewis, Proc. Roy. Soc. **A154**, 261 (1936).

3. The Nuclear Magnetic Moments

The magnetic moments of the nuclei in the ground state may now be calculated according to the following procedure. The neutrons and protons enter the wave function for the ground state, as given in Table I, in either a singlet or doublet state. The part of the magnetic moment μ_S which is due to the spin of the neutrons and protons will then be

(1) 0 if both kinds of particles are in a singlet state,
(2) μ_π the proton moment, or μ_ν the neutron moment, if the protons, or neutrons, are in a doublet state,
(3) $\mu_\pi+\mu_\nu$ if both kinds of particles are in a doublet state and the nucleus is in a triplet state.

We shall take $\mu_\pi=2.85$, in units $e\hbar/2Mc$, and $\mu_\nu=-2.00$.

In addition to μ_S there is the contribution to the moment of the orbital motion of the protons. This is given by

$$\mu_L=C_1{}^2\mu_L{}^{(1)}+C_2{}^2\mu_L{}^{(2)}+\cdots, \tag{13}$$

where

$$\mu_L{}^{(i)}=\frac{\mathbf{l}_\pi{}^{(i)}\cdot\mathbf{L}}{\mathbf{L}^2}L =\frac{L(L+1)+l_\pi{}^{(i)}(l_\pi{}^{(i)}+1)-l_\nu{}^{(i)}(l_\nu{}^{(i)}+1)}{2(L+1)}, \tag{14}$$

$\mathbf{l}_\nu{}^{(i)}$ and $\mathbf{l}_\pi{}^{(i)}$ are the neutron and proton orbital momenta (units of $\hbar$) in the ith parent wave function. The total orbital momentum $\mathbf{L}$ is their sum.

The total magnetic moment μ is obtained by adding the projections of $\boldsymbol{\mu}_S$ and $\boldsymbol{\mu}_L$ on the total angular momentum $\mathbf{J}=\mathbf{L}+\mathbf{S}$.

$$\boldsymbol{\mu}=\frac{g_S(\mathbf{S}\cdot\mathbf{J})+g_L(\mathbf{L}\cdot\mathbf{J})}{\mathbf{J}^2}\mathbf{J} \tag{15}$$

or

$$\mu=\tfrac{1}{2}(g_S+g_L)J+(g_S-g_L)\frac{S(S+1)-L(L+1)}{2(J+1)}, \tag{15.1}$$

where the g factors are defined by

$$g_S S = \mu_S, \quad g_L L = \mu_L. \tag{16}$$

For the nuclei with inverted multiplets the ground state $J = L + S$ and we obtain the obvious result

$$\mu = \mu_L + \mu_S. \tag{17}$$

For the nuclei with regular multiplets the ground state $J = L - S$ and (15.1) reduces to

$$\mu = (J/J+1)[(L+1)g_L - Sg_S]. \tag{18}$$

The nuclear magnetic moments, in units of the nuclear magneton $e\hbar/2Mc$, calculated from the formulae (13), (14), (17) and (18) and the nuclear spins deduced in §2 are given in Table III.[20]

These values for the magnetic moment have not been corrected for the presence of Heisenberg forces. This correction will not pertain to those nuclei which have an S ground state (cf. §4(c)). In §4(a) we shall give the calculation of the effect of the Heisenberg forces on the moment of Li^7. We find there for the corrected value of the moment $\mu(Li^7) = 3.07$.

Experimentally the following moments are known: $\mu(Li^6) = 0.85$;[21] $\mu(Li^7) = 3.20$ from atomic beam measurements[22] with neutral Li^7 and the use of the modified Goudsmit formula[23] and 3.28 from spectroscopic measurements of hyperfine structure for Li^7 II and the use of Breit and Doerman atomic wave functions.[24]

The agreement between the calculated and measured values of the Li^6 and Li^7 moments is quite gratifying since no empirical adjustment of constants has been made in the theory. A more decisive measure of the correctness of the theory can be obtained from the comparison of the ratio of the moments since this quantity is independent of atomic wave functions. The calculated value $\mu(Li^7)/\mu(Li^6) = 3.61$ is in satisfactory agreement with the observed ratio 3.87 ± 0.03.[21] However, it must be admitted that the calculated ratio lies outside the limit of experimental error and cannot be changed sufficiently by any reasonable adjustment of the value of the proton (or deuteron) moment.[25] At the same time, the favorable comparison between the calculated and observed values of the magnitude of the moments is an indication that the atomic wave functions used for Li^7 are rather reasonable.

The general agreement obtained justifies the assumption, which is made in our calculation, that the magnetic moments of the proton and neutron are not affected appreciably when the particles are bound in the nucleus.

However, there is some experimental evidence that the magnetic moment of N^{14} is probably $\lesssim 0.2$.[26] It is difficult to reconcile the calculated moment with this value. In §4 we shall discuss the corrections caused by admixtures of other configurations due to Heisenberg forces.

4. Refinements to the Theory

(a) Heisenberg forces

We now wish to consider the effect on the magnetic moment of Heisenberg forces. To the unlike particle interaction (1) the interaction

$$gA_{\nu\pi} r^{-r^2/a^2} (P_H - \tfrac{1}{2} P_M) \tag{19}$$

is added (cf. reference 5). The values of a and $A_{\nu\pi}$ are the same as in the Majorana interaction and $g = 0.22$. In general the Heisenberg forces will yield only a small correction to the magnetic moment as calculated above and it will suffice to consider its effect only for the case of greatest interest, namely Li^7.[27]

The complete wave function for the ground state is

[20] The nuclei not listed in Table III have zero spin.

[21] J. H. Manley and S. Millman, Phys. Rev. **50**, 380 (1936).

[22] M. Fox and I. I. Rabi, Phys. Rev. **48**, 746 (1935).

[23] Using atomic wave functions due to James and Coolidge a value of 3.33 was calculated for $\mu(Li^7)$ by J. H. Bartlett, J. J. Gibbons and R. E. Watson, Phys. Rev. **50**, 315 (1936).

[24] L. P. Granath, Phys. Rev. **42**, 44 (1932). G. Breit and F. W. Doerman, Phys. Rev. **36**, 1262 (1930).

[25] The ratio of the Li^7 to Li^6 moments may be regarded as equal to the ratio of the proton to deuteron moments plus a small correction which is due mainly to the orbital motion of the protons in Li^7. The ratio μ_π/μ_D is known (J. M. B. Kellogg, I. I. Rabi and J. R. Zacharias, Phys. Rev. **50**, 472 (1936)) to be 3.35 with an error estimated to be 3 percent or less. The deuteron moment is given by the same authors, $\mu_D = 0.85 \pm 0.03$. The computed ratio is $\mu(Li^7)/\mu(Li^6) = 0.968\mu_\pi/\mu_D + 0.306/\mu_D$. If we take the extreme values $\mu_\pi/\mu_D = 3.45$ and $\mu_D = 0.82$ we obtain $\mu(Li^7)/\mu(Li^6) = 3.71$ in place of 3.61 given above.

[26] R. F. Bacher, Phys. Rev. **43**, 1001 (1933).

[27] Obviously in considering Li^7 we obtain at the same time the correction to the moments of Be^7, C^{13} and N^{13}.

TABLE III. *Calculated nuclear magnetic moments and spins.*

NUCLEUS	MAG. MOM.	SPIN	NUCLEUS	MAG. MOM.	SPIN
Li^6	0.85	1	B^{11}	3.50	3/2
Li^7	3.15	3/2	B^{12}	1.73	2
Li^8	0.97	2	C^{11}	−1.65	3/2
Be^7	−1.30	3/2	C^{13}	1.13	1/2
Be^9	−1.65	3/2	N^{13}	−0.75	1/2
B^9	3.50	3/2	N^{14}	0.85	1
B^{10}	0.85	1	N^{15}	−0.28	1/2
			O^{15}	0.67	1/2

$$\Psi(^2P)=C_1\psi_1(^1D\ ^2P)+C_2\psi_2(^1S\ ^2P)+C_3\psi_3(^3P\ ^2P). \quad (20)$$

The wave functions ψ_1 and ψ_3 are given in (5.1) and (5.2) and for the same substate, namely $M=1$,

$$\psi_2(^1S\ ^2P)=(^1S^0\ ^2P^1). \quad (21)$$

The two-neutron wave functions are

$$^1D^2=(1\ 1)\chi,$$

$$^1D^1=2^{-\frac{1}{2}}[(0\ 1)+(1\ 0)]\chi,$$

$$^1D^0=6^{-\frac{1}{2}}[2(0\ 0)-(1\ -1)-(-1\ 1)]\chi, \quad (22.1)$$

$$\left.\begin{aligned}^1S_0&=3^{-\frac{1}{2}}[(0\ 0)+(1\ -1)+(-1\ 1)]\chi,\\ ^3P^1&=2^{-\frac{1}{2}}[(1\ 0)-(0\ 1)]\alpha_1\alpha_2,\end{aligned}\right. \quad (22.2)$$

$$^3P^0=2^{-\frac{1}{2}}[(-1\ 1)-(1\ -1)]\alpha_1\alpha_2, \quad (22.3)$$

$$\chi=2^{-\frac{1}{2}}(\alpha_1\beta_2-\alpha_2\beta_1), \quad (23)$$

and for the proton

$$^2P^{m_l}=(m_l)\alpha_3. \quad (24)$$

The indices 1 and 2 refer to the neutrons, 3 to the proton. α and β are spin wave functions corresponding to positive and negative spin component in the z direction.

With these wave functions the complete secular determinant of the interaction (1) plus (19) is calculated and the coefficient C_3 determined. We find, with the force constants as given above,

$$C_3=0.109. \quad (25)$$

The magnetic moment is given by

$$\mu=(1-C_3^2)\mu^{(12)}+C_3^2\mu^{(3)}, \quad (26)$$

where $\mu^{(12)}$ is the moment calculated without Heisenberg forces and $\mu^{(3)}$ is the moment associated with the state ψ_3.

$$\mu^{(3)}=\mu_S^{(3)}+\mu_L^{(3)}, \quad (27)$$

in which the moment due to spin $\mu_S^{(3)}$ is given by a formula analogous to (18)

$$\mu_S^{(3)}=\frac{g_\nu(S_\nu+1)-g_\pi S_\pi}{S_\nu-S_\pi+1}(S_\nu-S_\pi) \quad (28)$$

if $S_\nu>S_\pi$ and

$$\mu_S^{(3)}=\frac{g_\pi(S_\pi+1)-g_\nu S_\nu}{S_\pi-S_\nu+1}(S_\pi-S_\nu) \quad (28.1)$$

if $S_\pi>S_\nu$. g_ν and g_π are the g factors for the neutron and proton, respectively, and the moment due to the orbital motion of the proton $\mu_L^{(3)}$ is given by (13). For Li^7 the correction to the moment is -0.075 and we obtain the corrected moment $\mu=3.07$. The corrected moments for Be^7, C^{13} and N^{13} are -1.33, 1.14 and -0.72, respectively.

(b) Motion of the 1*s* shell

In the foregoing we have omitted all consideration of the 1*s* shell. From naive considerations it might be thought that the orbital motion of the 1*s* shell, as well as that of the *p* particles, around their mutual center of gravity would give a contribution to the magnetic moment of the nucleus. It turns out that this is not the case. To show this we carry out the calculation of the orbital momentum of all the protons in the nucleus around the center of gravity of the whole nucleus. We introduce the coordinates of the center of gravity.

$$\mathbf{R}=\sum_i \mathbf{r}_i/N, \quad (29)$$

N being the total number of particles, and coordinates relative to that point

$$\boldsymbol{\rho}_i=\mathbf{r}_i-\mathbf{R}. \quad (30)$$

The orbital momentum operator is then given by

$$\mathbf{L}=\sum_{\text{protons}}\boldsymbol{\varrho}_i\times\text{grad}_{\boldsymbol{\varrho}_i} \tag{31}$$

and the corresponding magnetic moment by

$$\boldsymbol{\mu}_L=e\hbar/2Mc\sum_{\text{protons}}\boldsymbol{\varrho}_i\times\text{grad}_{\boldsymbol{\varrho}_i}. \tag{32}$$

The wave function $\Psi(\mathbf{r}_i)$ may be transformed into the coordinates (29) and (30) with the result[28]

$$\Psi(\mathbf{r}_i)=e^{-\gamma NR^2}\,\Psi(\boldsymbol{\varrho}_i), \tag{33}$$

where γ is a constant characteristic of the oscillator potential.

In calculating the orbital momentum $(\Psi|\mathbf{L}|\Psi)$ we need only consider the proton part of the wave function. Further, since $\mathbf{L}$ is a one particle operator, any particular term in the expansion of the determinant in $\mathbf{L}\Psi$ combines only with that term of Ψ^* which is the same permutation of the particles. We may therefore consider only the principal diagonal term since all the terms give the same result. It is then readily seen that in the operator $\mathbf{L}$ only the p proton operators give a nonvanishing result, *the s shell wave functions being spherically symmetrical.* It follows that we get the same result for the orbital momentum as we would obtain if the absolute coordinates $\mathbf{r}_i$ were used; that is, the motion of the 1*s* shell does not affect the magnetic moment.

(c) Higher configurations for *S* state nuclei

In view of the discrepancy between the calculated and experimental magnetic moment of N^{14} it might be thought that the wave function of the ground state might contain an admixture of higher configurations and that these would give a different value for the moment. However, it is easily seen that all higher configurations which may mix with the ground state give the same value for the magnetic moment, namely $\mu_\pi+\mu_\nu$. For example, if we consider Majorana forces alone, the total spin of each kind of particle and the total L being good quantum numbers, the only terms of higher configurations which can interact with the $^3S(^2P^2P)$ are other ^3S states which arise from two doublets. Such states yield only the deuteron moment, $\mu_\pi+\mu_\nu=0.85$. With Heisenberg forces we may obtain admixtures of ^3S states arising from combinations of either doublet and quartet states or two quartet states. In the former case we may have either $^3S(^4L^2L)$ or $^3S(^2L^4L)$. The moments for these two states are given by (28) and (28.1). This gives $\mu(^4L^2L)=\frac{1}{2}(5\mu_\pi-\mu_\nu)$ and $\mu(^2L^4L)=\frac{1}{2}(5\mu_\nu-\mu_\pi)$. These two states will presumably have about the same energy and therefore will appear with about the same coefficients in the ground state wave function. Thus the total moment μ' is obtained as the mean of these two values which gives again $\mu'=\mu_\pi=\mu_\nu$. In the case of the state $^3S(^4L^4L)$ we have

$$\begin{aligned}\mu'(^4L^4L)&=\frac{g_\pi(\mathbf{S}_\pi\cdot\mathbf{S})+g_\nu(\mathbf{S}_\nu\cdot\mathbf{S})}{S+1}\\&=\frac{S}{2}(g_\pi+g_\nu)=\mu_\pi+\mu_\nu.\end{aligned} \tag{34}$$

Note added in proof: Dr. Bacher has suggested to us that a rather large correction to the moment may be expected when the two states $^3S(^4L^2L)$ and $^3S(^2L^4L)$ differ in energy even though this energy difference is not large. If E_1 and E_2 are the energies of the two states, E_0 that of the ground state and V the matrix element of the Heisenberg forces between ground state and any of the two states, the correction to the moment is

$$\begin{aligned}\delta\mu&=\frac{3}{2}(\mu_\pi-\mu_\nu)\left(\frac{V}{E_0-E_1}-\frac{V}{E_0-E_2}\right)\\&\approx\frac{3}{2}(\mu_\pi-\mu_\nu)\frac{V(E_1-E_2)}{[\frac{1}{2}(E_1+E_2)-E_0]^2}.\end{aligned} \tag{35}$$

We may estimate $\frac{1}{2}(E_1+E_2)-E_0=5$ MV, $V=1$ MV, while E_1-E_2 may be expected to be of the order of the Coulomb energy for one proton which is about 1 MV for nitrogen. This gives

$$\delta\mu\approx0.3 \tag{35.1}$$

which is of the same order of magnitude as the difference between the elementary theoretical and the experimental moments. It is satisfactory that this correction increases with increasing Coulomb energy and decreasing spacing between the levels arising from different configurations and is therefore to be expected to be much larger for N^{14} than for Li^6. Presumably the small moment of N^{14} is due to the joint action of this effect and others such as deviations from the Russell-Saunders coupling.

One of us (M. E. Rose) wishes to express his indebtedness to The American Philosophical Society for a grant.

[28] H. A. Bethe and M. E. Rose, Phys. Rev. (in press).

Appendix

Invariance of the coefficients with respect to the exchange properties of like particle forces

The most general model for the interaction between two like particles is a linear combination involving the Majorana operator P_M and a scalar (ordinary force). Heisenberg forces, because of antisymmetry of the wave function are equivalent to (repulsive) ordinary forces and, for the same reason, the Bartlett operator (exchange of spin coordinates) is equal to $-P_M$. The interaction involving the scalar product $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ of the two spin operators referring to the two particles, which has also been proposed, is equivalent to $-(1+2P_M)$. We must choose the interaction which gives the correct energy for nuclei with symmetric space wave functions (H^2, H^3, He^3 and He^4).

This is
$$V(r)(bP_M+1-b), \qquad (36)$$
in which $V(r)$ is a suitable function of the distance between the particles and b is an arbitrary constant (limited only by stability considerations). The difference between any two interactions for different values of b is obviously a multiple of $V(1-P_M)$. From this result we can show that the difference of the interaction energies of the like particles, using the various force models (36) is the same for all low terms of a p^n configuration.

For 2 or 4 particles the low terms are ^1S and ^1D. For these the space wave function is symmetrical. Therefore the operator $(1-P_M)$ gives a zero result in this case for both terms.

For 3 particles the low terms are ^2P and ^2D. Considering the particular substate $M=M_L+M_S=1$,[29] the wave functions are $\psi(^2P)=2^{-\frac{1}{2}}(\varphi_1-\varphi_2)$ and $\psi(^2D)=2^{-\frac{1}{2}}(\varphi_1+\varphi_2)$; $\varphi_1=(1^+0^+0^-)$ and $\varphi_2=(1^+1^- -1^+)$. To prove the theorem for this case we have to show that $(\varphi_2|V(1-P_M)|\varphi_1)=0$. The only set of individual particle quantum numbers which match in φ_1 and φ_2 is the first, 1^+. Therefore it is clear from orthogonality considerations that only the interaction between particles corresponding to the last two sets of quantum numbers will contribute to the matrix element. If these particles are the ith and kth, the wave function φ_1 will contain $z_i z_k$ as a factor and is therefore invariant with respect to the Majorana operator $P_M{}^{ik}$.

This shows that the difference between the matrix of the interaction energy for the various force models is a multiple of a unit matrix and this does not affect the values of the coefficients C_1C_2 and C_3.

Exceptions to the theorem

The invariance of the coefficients is no longer true if the high terms, ^3P and ^4S, of the p^n configurations are included. The like particle energy of the state ^3P will depend on the force model. The energy difference for the various force models is proportional to $(^3P|V(1-P_M)|^3P)=2(^3P|V|^3P)$, since the space part of the wave function is antisymmetrical, and this matrix element is not zero in contrast to the result for the singlet states. For the ^4S state, in the case of any two force models, the energy difference is twice that for the doublet states.[30]

Further, it is not sufficient for the truth of the theorem that the terms have the same multiplicity. In the configuration d^2 the terms ^3P and ^3F arise. In this case

$$(^3P|V(1-P_M)|^3P)=2(^3P|V|^3P)\neq 2(^3F|V|^3F),$$

so that the energy difference with the different force models depends on the L value of the term.

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[29] We have omitted the superscript on the magnetic quantum numbers but, of course, they refer to a single kind of particle and should not be confused with the symbols used before to denote the magnetic quantum numbers of the whole nucleus.

[30] E. Feenberg and E. Wigner, reference 4, Table III.