A Symmetry Theorem in the Positron Theory

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(Received October 26, 1936)

In the positron theory considerable interest attaches to the consideration of processes in which the occurrence of electrons and positrons is transitory only, such as the scattering of light by a Coulomb field (Delbrück), and the scattering of light by light (Euler and Kockel). Calculations of such effects can frequently be simplified on account of cancellations brought about by the distribution's symmetry between electrons and positrons. An abstract proof is here presented for the theorem which predicts the appearance of such cancellations in the general case. Certain modifications are found to be required when interactions other than the usual electric forces are introduced.

I. INTRODUCTION

AN interesting feature of the Dirac theory of the positron is the possibility of processes in which electrons and positrons, though they are not actually created so as to be experimentally observable, are able to play the part of catalytic agents producing effects which otherwise could not appear. Examples of such effects are the scattering of light from a Coulomb field, suggested by Delbrück,[1] and the scattering of light by light, suggested by Halpern[2] and by Debye and calculated by Euler and Kockel.[3] The linear field equations on which the present electrodynamics is based do not provide any direct interactions between different components of the field such as would produce such scatterings; but the transitory or virtual presence of charged particles provides a means of mediating indirect interactions. Thus for instance the possibility of the Delbrück scattering may be seen by supposing that the incident light quantum produces a (virtual) pair, that these particles are scattered in the Coulomb field, and that in being destroyed they emit the scattered light quantum. The transitory presence of the pair is possible even when there is not enough energy available to produce an actual observable pair.

It is to be expected that the symmetry between the two kinds of particles can be used to simplify calculations of such effects; in particular, since electrons and positrons are induced in equal numbers and are oppositely affected by electric fields, it would seem likely that cancellations should occur. Immediately after the Delbrück effect was suggested, it was remarked by Oppenheimer that, on account of such a cancellation, the transition probability for such scattering is of order $(Z\alpha)^4$ instead of $(Z\alpha)^2$. Such reasoning has been applied in some other cases. Neither a statement nor a proof of the general theorem involved has been published, however, and it evidently is by no means universally known. The writer has recently been interested in obtaining a proof so formulated that one can readily see how the result has to be modified in the case of hypothetical particles which obey the Dirac equations and are subject to other sorts of forces than the usual electric ones.[4] In this note such a proof is presented.

II. STATEMENT OF THEOREM AND OUTLINE OF PROOF

Theorem: In calculations using plane wave functions as a basis ("Born approximation") for processes in which the appearance of electrons and positrons is transitory only, the odd order contributions vanish identically.

This statement applies to calculations actually possible with the existing theory. The way in which it would have to be modified if nonelectric forces were introduced will be made clear in the discussion of Section IV.

It is evident that to any set of transitions through intermediate states by means of which a given process can be brought about, there corre-

[1] M. Delbrück, Zeits. f. Physik **84**, 144 (1933).
[2] O. Halpern, Phys. Rev. **44**, 885 (1934).
[3] H. Euler and B. Kockel, Naturwiss. **23**, 246 (1935); H. Euler, Ann. d. Physik **26**, 398 (1936).
[4] W. H. Furry, Phys. Rev. **50**, 784 (1936); Abstract No. 28, American Physical Society, New York, October, 1936.

sponds another possible set in which the roles of electrons and positrons are reversed. The proof of the theorem consists in showing that the corresponding contributions to the probability amplitude are equal and opposite if the order of the transition-scheme is odd.

A typical nth order term in the probability amplitude is N/D, with

$$N=\sum_{\text{spins}}\cdots(u^{+}(\mathbf{k},\sigma,\tau)\mathbf{O}u(\mathbf{k}',\sigma',\tau'))\cdots \quad (n \text{ factors}), \tag{1}$$

$$D=(\Delta E)_1\cdots(n-1 \text{ factors}). \tag{2}$$

Here $u(\mathbf{k}',\sigma',\tau')$ is the four-component amplitude of the Dirac plane wave function of an electron of momentum $\mathbf{k}mc$, spin σ, and energy $\tau\epsilon mc^2(\epsilon=(1+k^2)^{\frac{1}{2}}$, $\tau=\pm1)$; $\mathbf{O}$ is a four-rowed matrix characteristic of the type of interaction regarded as producing the transition from $(\mathbf{k}',\sigma',\tau')$ to $(\mathbf{k},\sigma,\tau)$; and the (ΔE)'s are the energy differences between the initial state of the total system and the various intermediate states. If (1) were written out in full, the various $\mathbf{k}$'s, σ's, τ's and $\mathbf{O}$'s would have to be distinguished by subscripts. Since only a typical factor has to be discussed, a simpler notation is used to save writing.

Turning now to the analogous term in which the roles of electrons and positrons are interchanged, we find that its relation to our original term is described as follows:

(*a*) All energy differences (ΔE) are unchanged, because the dependence of energy on momentum is the same for both kinds of particles.

(*b*) Our new term is perhaps to be prefixed with the opposite sign to that given the old, on account of the different exchange characteristics of the two sets of transitions.

(*c*) Apart from such a possible intrinsic change of sign, the change in numerator is given by replacing N as given by (1) by:

$$N'=\sum_{\text{spins}}\cdots(u^{+}(-\mathbf{k}',\sigma',-\tau')\times\mathbf{O}u(-\mathbf{k},\sigma,-\tau))\cdots(n \text{ factors}). \tag{1'}$$

We have to reverse the direction of all transitions, change the signs of all energies (in the *electron* wave functions as used), and reverse all momenta. This corresponds to the fact that a positron of momentum $\mathbf{k}mc$ is equivalent to the absence of a negative energy electron of momentum $-\mathbf{k}mc$.

The theorem will be established by proving the following two lemmas concerning the changes noted under (*b*) and (*c*), respectively:

Lemma I: An intrinsic difference of sign on account of different exchange characteristics appears if and only if the number n of transitions is odd.

Lemma II: Apart from such an intrinsic difference of sign, the two numerators are equal $(N'=N)$, *and indeed the separate factors are equal respectively in pairs.*

It obviously follows from these lemmas that the contributions cancel for odd n in such a way as to make the theorem correct. However, before we go on to the formal proofs we should mention a certain difficulty which arises in connection with these cancellations. The actual probability amplitudes for these effects are sums of integrals of such terms as we have written, the integrals being taken over one or more momentum spaces.[5] Although two convergent integrals must cancel if their integrands are identically equal and opposite, this is not in any absolute sense true of divergent integrals: by choosing different origins or different coordinate systems for the comparison of the two, one can obtain indeterminate, and frequently infinite, discrepancies. The integrals met with in calculating the sort of transition probabilities under consideration will quite commonly be divergent. The resulting situation is one which is usual in calculations in positron theory: to obtain the result, one has to find a reasonable method of excluding the infinite and indeterminate contributions, and use what is left.[6] In the present case the reasonable procedure is to match the coordinate systems according to the physically sensible idea of interchanging the roles of the two kinds of particles. When this is done, the result is zero as stated in the theorem.

III. Proofs of the Lemmas

Lemma I: This is readily seen by inspection in some of the simpler cases. It may be proved in general by using quantized amplitudes. The

[5] Conservation of momentum holds for the intermediate states, but it fixes only the *total* momentum of the pair field. Since this field contains more than one particle, there remain undetermined one or more momentum *differences*, over which one must integrate.

[6] Cf. R. Serber, Phys. Rev. **49**, 545 (1936).

transition scheme (1) can be regarded as characterized by a product of amplitudes

$$A=\cdots a^+(\mathbf{k},\sigma,\tau)a(\mathbf{k}',\sigma',\tau')\cdots. \tag{3}$$

Here

$$a_i^+a_j^++a_j^+a_i^+=a_ia_j+a_ja_i=0, \tag{4}$$

$$a_i^+a_j+a_ja_i^+=0,\quad i\neq j,$$

$$a_i^+a_i+a_ia_i^+=1. \tag{5}$$

When such a product is applied to the situation postulated in the "hole" theory, in which all levels $\tau=+1$ are empty and all levels $\tau=-1$ are full,[7] we can replace (5) by

$$\begin{aligned} &a^+(\mathbf{k},\sigma,+1)a(\mathbf{k},\sigma,+1)=0,\\ &a(\mathbf{k},\sigma,+1)a^+(\mathbf{k},\sigma,+1)=1,\\ &a^+(\mathbf{k},\sigma,-1)a(\mathbf{k},\sigma,-1)=1,\\ &a(\mathbf{k},\sigma,-1)a^+(\mathbf{k},\sigma,-1)=0. \end{aligned} \tag{6}$$

The fact that the appearance of particles is transitory only assures us that if $a(\mathbf{k},\sigma,\tau)$ appears in (3), $a^+(\mathbf{k},\sigma,\tau)$ also appears, and *vice versa;* and the fact that the transition scheme chosen is a possible one means that when interchanges of adjacent factors are made so that each factor finds its partner, a nonvanishing product will be obtained. This product will be $+1$ if the number of interchanges required is even, -1 if it is odd.

Exactly the same situation obtains for the product

$$A'=\cdots a^+(-\mathbf{k}',\sigma',-\tau')a(-\mathbf{k},\sigma,-\tau)\cdots, \tag{3'}$$

which corresponds to (1′) as (3) does to (1). Apart from irrelevant changes in the designations of the factors, the only significant difference between (3′) and (3) consists in n interchanges of adjacent factors. Thus

$$\begin{aligned} &A'=A, && n \text{ even},\\ &A'=-A, && n \text{ odd}. \end{aligned} \qquad Q.E.D.$$

Lemma II. Given

$$M=(u^+(\mathbf{k},\sigma,\tau)\mathbf{O}u(\mathbf{k}',\sigma',\tau')) \tag{7}$$

we want to show that to every $u^+(\mathbf{k},\sigma,\tau)$ we can correlate a $u(-\mathbf{k},\bar{\sigma},-\tau)$[8] and to every $u(\mathbf{k},\sigma,\tau)$ a $u^+(-\mathbf{k},\bar{\sigma},-\tau)$, in such a way that if the original functions form a complete normalized orthogonal set the correlated ones do too, and that if

$$M'=(u^+(-\mathbf{k}',\bar{\sigma}',-\tau')\mathbf{O}u(-\mathbf{k},\bar{\sigma},-\tau)) \tag{7'}$$

we shall have $M'=M$. Let us try to do this by finding a matrix A such that

$$\begin{aligned} u(-\mathbf{k},\bar{\sigma},-\tau)&=\overline{\{u^+(\mathbf{k},\sigma,\tau)A^+\}},\\ u^+(-\mathbf{k},\bar{\sigma},-\tau)&=\overline{\{Au(\mathbf{k},\sigma,\tau)\}}. \end{aligned} \tag{8}$$

(The bar indicates transposition.)

The requirements as to completeness, orthogonality, and normalization are satisfied provided

$$\begin{aligned} &(u^+(-\mathbf{k}',\bar{\sigma}',-\tau')u(-\mathbf{k},\bar{\sigma},-\tau))\\ &\qquad=(u^+(\mathbf{k},\sigma,\tau)u(\mathbf{k}',\sigma',\tau')). \end{aligned} \tag{9}$$

Substituting from (8) in the left member of (9) we get

$$(\overline{\{Au(\mathbf{k}',\sigma',\tau')\}}\ \overline{\{u^+(\mathbf{k},\sigma,\tau)A^+\}}).$$

Since this is a mere number, we do not change its value by transposition, which gives

$$(u^+(\mathbf{k},\sigma,\tau)A^+Au(\mathbf{k}',\sigma',\tau')).$$

Thus (9) is equivalent to the requirement

$$A^+A=1; \tag{10}$$

i.e. A must be unitary.

The requirement that the new functions so formed be solutions of the Dirac equations means that

$$\begin{aligned} &(\tau\epsilon+\boldsymbol{\alpha}\cdot\mathbf{k}+\beta)u(\mathbf{k},\sigma,\tau)=0,\\ &u^+(\mathbf{k},\sigma,\tau)(\tau\epsilon+\boldsymbol{\alpha}\cdot\mathbf{k}+\beta)=0 \end{aligned} \tag{11a, b}$$

must imply

$$\begin{aligned} &\overline{\{Au(\mathbf{k},\sigma,\tau)\}}(-\tau\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}+\beta)=0,\\ &(-\tau\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}+\beta)\overline{\{u^+(\mathbf{k},\sigma,\tau)A^+\}}=0. \end{aligned} \tag{12a, b}$$

[7] The changes in notation which might be made to cover up the baldness of this statement and replace (6) by a pair of symmetrical rules would have no effect here except to make the proof slightly more cumbersome.

[8] By $\bar{\sigma}$ we mean not necessarily an index different in *numerical value* from σ but an index whose *significance* may be different, i.e. $u(\mathbf{k},\bar{\sigma},\tau)$ may not be the same function of $\bar{\sigma}$ that $u(\mathbf{k},\sigma,\tau)$ is of σ. There is no need to trace the correlation between the two types of spin quantization, because the spin-sums (1) and (1′) are independent of the type used, as long as we are careful always to use complete normalized orthogonal function sets.

Transposing (12, a, b) we get

$$(-\tau\epsilon-\bar{\boldsymbol{\alpha}}\cdot\mathbf{k}+\bar{\beta})Au(\mathbf{k},\sigma,\tau)=0,$$
$$u^+(\mathbf{k},\sigma,\tau)A^+(-\tau\epsilon-\bar{\alpha}\cdot\mathbf{k}+\bar{\beta})=0. \quad (13a, b)$$

(13a) is implied by (11a) provided

$$\bar{\alpha}_iA=A\alpha_i,\quad \bar{\beta}A=-A\beta. \quad (14)$$

Since $\bar{\alpha}_i$, $\bar{\beta}$ are Hermitian as well as α_i, β, by taking the adjoint of (14) we see that it is also a sufficient condition for (13b) to be implied by (11b).

On account of the use of transposition in fixing the conditions on A, it does not seem to be possible to prove that an A exists satisfying (10) and (14) without referring to an explicit form of the Dirac matrices. If we suppose that they have the form originally given by Dirac, we see that α_x, α_z, β are real symmetric, while α_y is imaginary antisymmetric. Thus for this representation (14) means that A must commute with α_x and α_z, and anticommute with α_y and β; and $i\alpha_y\beta$ is obviously a unitary matrix which does this. Having found a suitable A for this one representation, we can readily determine one for any other. This other representation must be connected with the one discussed by a unitary transformation,

$$\alpha_i'=S^{-1}\alpha_iS=S^+\alpha_iS,\quad \beta'=S^+\beta S, \quad (15)$$

and one finds that a suitable unitary matrix A' is now given by

$$A'=\bar{S}AS. \quad (16)$$

A does not transform according to the unitary transformation applied to $\boldsymbol{\alpha}$, β, and hence does not have the same expression in terms of $\boldsymbol{\alpha}$, β in different representations. This is due to the fact that the conditions (14) placed on A are not invariant under unitary transformation.

We now have only to substitute (8) into (7') and apply (14) to evaluate M':

$$M'=\bar{M}'=(\{\overline{Au(\mathbf{k}',\sigma',\tau')}\}\mathbf{O}\{\overline{u^+(\mathbf{k},\sigma,\tau)A^+}\})$$
$$=(u^+(\mathbf{k},\sigma,\tau)A^+\bar{\mathbf{O}}Au(\mathbf{k}',\sigma',\tau')). \quad (17)$$

From (10) and (14), $A^+\bar{\mathbf{O}}A=\pm\mathbf{O}$ whenever $\mathbf{O}$ is a product of Dirac matrices. Thus for such $\mathbf{O}$'s, $M'=\pm M$. In detail,

$\mathbf{O}_1=\beta$: $M'=-M$, (18a)

$\mathbf{O}_2=1;\ \alpha_i$: $M'=M$, (18b)

$\mathbf{O}_3=i\beta\alpha_i;\ i\beta\alpha_i\alpha_j$: $M'=M$, (18c)

$\mathbf{O}_4=i\alpha_1\alpha_2\alpha_3;\ i\alpha_i\alpha_j$: $M'=-M$, (18d)

$\mathbf{O}_5=\beta\alpha_1\alpha_2\alpha_3$: $M'=-M$. (18e)

The various operators have been listed according to the relativistic transformation properties of $u^+\mathbf{O}_pu$.[9] The five types of $\mathbf{O}_p$ listed give respectively a scalar, a four-vector, an antisymmetric tensor, an axial four-vector (antisymmetric tensor of the third rank), and a pseudo-scalar. As will be explained at the beginning of the next section, (18b) establishes Lemma II for the case contemplated in the original statement of the theorem.

IV. Discussion of the Various Operators

The interactions involved in all effects so far discussed are those with the radiation field and with Coulomb fields. These interactions are given by the product of the potential four-vector with the four-vector $\mathbf{O}_2$ of (18b). Thus Lemma II is established for all these effects, and with it the theorem as stated.

Another possible form of interaction with the electromagnetic field is that obtained by multiplying the tensor $\mathbf{O}_3$ of (18c) by the field tensor. For such interactions the theorem also holds. This form of interaction corresponds to the particle's possessing a magnetic moment differing from that given by its charge and spin. It was first invented by Pauli as a possible interaction between a neutral particle (precursor of the neutrino) and the field; also it has been suggested that the proton may possibly obey a Dirac equation in which such a term is inserted to give agreement with the observed magnetic moment. For the electron this interaction must be supposed to vanish.

The only other one of the operators (18) which can be supposed to describe an interaction of the particle with a field of force is the scalar $\mathbf{O}_1$ of (18a). The writer has recently pointed out, in connection with the question of the spin-orbit

[9] Cf. W. Pauli, *Handbuch der Physik* 24/1, 221. The rather unsymmetrical appearance of the operators listed in (18) is due to our use of the original Dirac notation α, β instead of the γ^μ notation which is better adapted for clear display of the transformation properties.

interaction of a particle moving under the action of a nonelectric force,[4] that this scalar may be associated with a scalar potential such as occurs in Nordström's special-relativistic gravitational theory. One sees from (18a) that in applying our theorem to any process in which such an interaction might play a part we must *omit* the matrix elements of this **O** in counting the number of factors in the numerator (1) to get n. This is just what is to be expected, since the reason for the alternation of relative signs from order to order in the electric case is to be found in the opposite action of the field on the two signs of charge. The result is of some interest academically, though at present there seems not to be much utility in the concept of a particle which is susceptible both to the action of nonelectric forces and to creation and destruction in pairs.

The operators $\mathbf{O}_4$ and $\mathbf{O}_5$ of (18d) and (18e) cannot be associated with any sort of interaction which has a classical analogue, on account of their unsuitable transformation properties as regards reflection. If, however, an interaction between *two matter-fields* is described by a *biquadratic* form in the Dirac amplitudes, the antisymmetry under reflection can be eliminated by squaring. In fact Wigner, and also Bethe and Bacher,[10] have remarked that these operators, quite as well as the others, can reasonably be used in neutrino theory. It is possible that the symmetry property here described may find applications in this connection.

[10] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. **8**, 190 (1926).

JANUARY 15, 1937 PHYSICAL REVIEW VOLUME 51

Energy Bands for the Face-Centered Lattice

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(Received October 2, 1936)

The Slater method of obtaining wave functions for metallic lattices is applied to the face-centered lattice. The solutions previously obtained by Krutter for this lattice were mainly for certain simple lines in momentum space. Methods are developed for obtaining more general solutions from these special ones. On this basis the entire 110 plane is worked out. For certain new lines in this plane especially simple solutions are given. An approximate method suitable for calculating energy contours in momentum space for small values of momentum is developed.

INTRODUCTION

SLATER[1] has proposed an extenison of the method of Wigner and Seitz[2] for the calculation of wave functions in the periodic field of metallic lattices. He applied his method to the body-centered lattice with metallic sodium particularly in mind. The method has been applied to lithium[3] (body-centered), to copper[4] (face-centered), and with modifications to diamond[5] and the sodium chloride lattice.[6]

Briefly, the method consists of dividing the lattice into polyhedral cells centered about each atom and containing those points of space nearer that atom than any other atom. The wave function ψ in one of these cells is expanded in terms of surface harmonics times radial functions which are numerical solutions of wave equation. For an atom at a center of symmetry this function is most conveniently handled in the form $\psi=u_g+iu_u$ where u_g is a real even function of the coordinates having the atom as a center (i.e., is expanded in harmonics of even l) and u_u is an odd real function. The functions in other cells are obtained by translating ψ and multiplying it by $e^{i\mathbf{k}\cdot\mathbf{R}}$ where $\mathbf{R}$ is the translation vector. The wave function and its normal derivative are required to be continuous from cell to cell at the midpoints of the cell boundaries.

[1] J. C. Slater, Phys. Rev. **45**, 794 (1934).
[2] Wigner and Seitz, Phys. Rev. **43**, 804 (1933) and **46**, 509 (1934).
[3] J. Millman, Phys. Rev. **47**, 286 (1935); F. Seitz, Phys. Rev. **47**, 400 (1935).
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[6] Ewing and Seitz, Phys. Rev. **50**, 760 (1935); W. Shockley, Phys. Rev. **50**, 754 (1935).