and W_{rs} denotes the molecular weight of the constituent. The N's corresponding to a given f are calculated with Eqs. (7) and (8), the mass ratios with Eq. (10), and, finally, χ_{12} with Eq. (9). The values of W_{rs} adopted in the present calculation are $W_{11}=18.015$, $W_{12}=19.021$ and $W_{22}=20.027$.

The calculated value of χ_{HDO} corresponding to $f=0.2$ is -0.6807×10^{-6}. When this value is substituted in Eq. (9) the calculated mass susceptibilities of the remaining three mixtures are $\chi_{0.4}=-0.6885$, $\chi_{0.6}=-0.6739$, and $\chi_{0.8}=-0.6600$, all $\times10^{-6}$. These agree with the observed data of Table II with a maximum discrepancy of 2 in the fourth figure.

The present value of the molar susceptibility, χW, of D_2O is $-12.95\pm0.01\times10^{-6}$ compared with the value -12.97×10^{-6} for H_2O. Other investigators have obtained the values 12.90,[9] 12.76,[10] and 12.96±0.02,[11] all $\times -10^{-6}$. It should be noted that the difference of amount 0.02×10^{-6} here found between the molar susceptibilities of H_2O and D_2O rests upon the adoption of the value 1.1058 for the density of D_2O at 21.3°C. An earlier measurement of this quantity by Lewis and Luten[12] yielded the value 1.1035, and the molar susceptibility of D_2O calculated from the present data with this figure is -12.97×10^{-6}. The experimental verification of the additivity law subsists whichever value is chosen.

In conclusion the writer gratefully acknowledges his indebtedness to Dr. W. B. Ellwood, of the Bell Telephone Laboratories, for his assistance in securing the magnet; to Dr. H. C. Urey, who supplied the mixtures of H_2O and D_2O; to Dean G. B. Pegram, in whose home in New York City this research was conducted; and to Dr. A. P. Wills, who suggested the problem and followed the progress of the work with helpful counsel and encouragement.

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[9] Selwood and Frost, J. Am. Chem. Soc. **55**, 4335 (1933).

[10] Cabrera and Fahlenbrach, Naturwiss. **22**, 417 (1934).

[11] J. H. Cruickshank, Nature **268**, 135 (1935).

[12] Lewis and Luten, reference 7.

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The Piezodielectric Effect and Electrostriction in Anisotropic or Isotropic Media

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A phenomenological theory of the piezodielectric effect in homogeneous isotropic or anisotropic solids is presented. This theory rests upon the assumption that the changes in the dielectric constants ϵ_{ij} are linear in the six components of applied mechanical stress. It is shown that 36 piezodielectric moduli are required for a complete description of the piezodielectric effect. The number of independent moduli is two in isotropic solids, eight in alpha-quartz and tourmaline, and twelve in Rochelle salt. The matrices for the piezodielectric and piezooptical moduli are identical in all crystal classes which we have considered. The term electrostriction is applied to the case in which the components of strain are quadratic functions of the applied electrical field. The character of electrostriction in any solid is predicted by considering the piezodielectric effect together with the first and second laws of thermodynamics. The relation of electrostriction to the piezodielectric effect is very similar to the relation of the converse piezoelectric effect to the direct piezoelectric effect. The equations for the components of "electrostrictive strain" in isotropic solids and alpha-quartz are given in detail. General expressions for "electrostrictive stress" are derived. The equations of motion of a perfectly elastic, piezoelectric and electrostrictive solid which is subjected to mechanical and electrical fields are then stated. From considerations of crystalline symmetry and Neumann's hypothesis it can be seen that every solid may exhibit electrostriction whereas relatively few solids are piezoelectric.

1. The Theory of the Piezodielectric Effect

The term *piezodielectric effect* is here applied to those changes in the dielectric constants ϵ_{ij} of a homogeneous solid which result from the application of mechanical stress. The following considerations are limited to the case in which the changes are linear in the applied stress. Direct experimental evidence for this linearity has been recently obtained by A. H. Scott.[1]

Let the x, y and z components of the electric

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[1] A. H. Scott, Nat. Bur. Stand. J. Research **15**, 13 (1935).

field intensity E and of the electric induction D be denoted respectively by E_1, E_2, E_3 and D_1, D_2, D_3. The dielectric constants ϵ_{ij} in the initial or the unstrained state are now defined by the relations

$$D_i=\sum_{j=1}^{3}\epsilon_{ij}E_j,\quad i=1,2,3, \tag{1}$$

in which it can be shown that

$$\epsilon_{ij}=\epsilon_{ji}. \tag{2}$$

The change which is brought about in ϵ_{ij} by the application of stress is denoted by e_{ij}. It is assumed that the dielectric constants in the stressed state are $\epsilon_{ij}+e_{ij}$, in which

$$e_{ij}=\sum_{k=1}^{6}g_{ijk}X_k \tag{3}$$

or

$$e_{ij}=\sum_{k=1}^{6}b_{ijk}x_k, \tag{4}$$

where X_k and x_k are, respectively, the components of mechanical stress and strain after the notation of Goranson.[2]

From arguments similar to those used to show that $\epsilon_{ij}=\epsilon_{ji}$ it can be shown that in the stressed state $\epsilon_{ij}+e_{ij}=\epsilon_{ji}+e_{ji}$. Hence

$$e_{ij}=e_{ji}$$

and

$$g_{ijk}=g_{jik}. \tag{5}$$

Since i or $j=1, 2, 3$, and $k=1, 2\cdots 6$, 36 constants g_{ijk} or b_{ijk} are thus required to describe the piezodielectric effect in crystalline media. The constants g_{ijk} are called the *piezodielectric moduli*. These are by assumption independent of X_k. They are in general a function of the temperature. In nonisothermal processes Eq. (3) is more general when written in the form

$$e_{ij}=\sum_{k=1}^{6}\int_0^{X_k}g_{ijk}dX_k, \tag{6}$$

where the initial state is unstressed.

The electrostatic energy E per unit volume of an anisotropic solid is usually assumed to be[3] given by

$$2E=\sum_{i=1}^{3}\sum_{j=1}^{3}\epsilon_{ij}E_iE_j=\sum_{j=1}^{3}D_jE_j. \tag{7}$$

It is here assumed that the electrostatic energy ξ per unit volume in the strained state is likewise given by

$$2\xi=\sum_{i=1}^{3}\sum_{j=1}^{3}(\epsilon_{ij}+e_{ij})E_iE_j. \tag{8}$$

Let ϕ represent the change in the electrostatic energy per unit volume due to the piezodielectric effect. Then,

$$2\phi=2\xi-2E=\sum_{i=1}^{3}\sum_{j=1}^{3}e_{ij}E_iE_j. \tag{9}$$

When ϕ is referred to the crystallographic axes, it can be shown[4] that for most of the crystalline groups $\epsilon_{ij}=0$, when $i\neq j$.

ϕ may be written in terms of the components of stress by substituting for e_{ij} in (9) the corresponding values given by (3). Thus

$$2\phi=\sum_{k=1}^{6}X_k\sum_{i=1}^{3}\sum_{j=1}^{3}g_{ijk}E_iE_j. \tag{10}$$

In a given crystalline group the number of independent moduli which are different from zero can be reduced from considerations of crystalline symmetry and Neumann's hypothesis.[5] This reduction can be made by means of an energy method through the energy function ϕ. A primed Cartesian coordinate system is related to the unprimed, rectangular crystallographic axes by a set of linear transformations which correspond to rotation through an angle θ about an axis or to reflection across a plane. Thus with respect to the primed system

$$2\phi'=\sum_{k=1}^{6}X_k'\sum_{i=1}^{3}\sum_{j=1}^{3}g'_{ijk}E_i'E_j'. \tag{11}$$

When the linear transformations

$$X_k'=F_k(X_p,\theta)\quad\text{and}\quad E_s'=G_s(E_r,\theta) \tag{12}$$

are substituted into (11), ϕ' becomes an explicit function of X_k, E_m, E_n and θ. Let the value of θ or

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[2] R. W. Goranson, *Thermodynamic Relations* (1930).

[3] Voigt, *Lehrbuch der Kristallphysik* (1910), p. 413

[4] Reference 3, p. 414.

[5] Love, *Mathematical Theory of Elasticity* (1920), p. 155.

the nature of the reflections correspond to a "covering operation."[6] Since the energy ϕ is invariant with respect to the coordinate system, the necessary equations for determining the number of independent piezodielectric moduli are obtained by equating the coefficients of E_mE_n in (10) to the corresponding coefficients in ϕ'. This is similar to the manner in which the elastic constants are reduced by Love.[7]

The reduction to the number of independent piezodielectric moduli has been carried out in the above manner by the writers for (1) the isotropic case and for those crystals which in Voigt's[8] notation possess the symmetry (2) $A_z^3A_x^2$ (alpha-quartz), (3) $A_z^3E_x$ (tourmaline), (4) $A_z^2A_x^2$ (Rochelle salt), and (5) $A_z^2E_x$. The following matrices state the relations which were found to exist among the piezodielectric moduli. For the sake of ready comparison of these matrices with those which are already known to exist for the piezooptical effect, the writers have reduced g_{ijk} to a second-order tensor by means of the convention $g_{ijk}=g_{hk}$ (h and $k=1, 2\cdots 6$) in which the integers h, i and j are related by the scheme

$$\begin{array}{cccccccccc} h & 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ i & 1 & 2 & 3 & 3 & 2 & 3 & 1 & 2 & 1 \\ j & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2. \end{array} \tag{13}$$

Thus, for example, $g_{32k}=g_{23k}=g_{4k}$. With this convention (2) and (3) are conveniently written in the form

$$\epsilon_{ij}=\epsilon_h \tag{14}$$

and
$$e_{ij}=e_h=\sum_{k=1}^{6} g_{hk}X_k, \quad h=1, 2\cdots 6. \tag{15}$$

When the elements g_{hk} of these matrices are known from physical measurements, the piezodielectric effect is calculated by means of Eqs. (13) and (15).

Isotropic case:

$$\|g_{hk}\|=\left\|\begin{array}{cccccc} g_{11} & g_{12} & g_{12} & 0 & 0 & 0 \\ g_{12} & g_{11} & g_{12} & 0 & 0 & 0 \\ g_{12} & g_{12} & g_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right\|. \tag{16}$$

$A_z^3A_x^2$ and $A_z^3E_x$:

$$\|g_{hk}\|=\left\|\begin{array}{cccccc} g_{11} & g_{12} & g_{13} & g_{14} & 0 & 0 \\ g_{12} & g_{11} & g_{13} & -g_{14} & 0 & 0 \\ g_{31} & g_{31} & g_{33} & 0 & 0 & 0 \\ g_{41} & -g_{41} & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{44} & 2g_{41} \\ 0 & 0 & 0 & 0 & g_{14} & (g_{11}-g_{12}) \end{array}\right\|. \tag{17}$$

$A_z^2A_x^2$ and $A_z^2E_x$:

$$\|g_{hk}\|=\left\|\begin{array}{cccccc} g_{11} & g_{12} & g_{13} & 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 & 0 & 0 \\ g_{31} & g_{32} & g_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{66} \end{array}\right\|. \tag{18}$$

A phenomenological theory of the *piezooptical effect* has been studied by Pockels,[9] Günther[10] and Szivessy.[11] The reduction of the piezooptical moduli q_{ij} was made by the so-called geometrical method since a suitable energy function was not available. It will be observed that the matrices[12] for the piezooptical and the piezodielectric moduli are identical in form. This may be expected since the indices of refraction are closely related to the dielectric constants. To predict the piezodielectric effect in the remaining 27 crystal classes not considered in (16), (17) or (18) one may thus employ the known piezooptical matrices. It is, however, preferable to determine the piezodielectric constants by the energy method of Eqs. (11) or (12) since the writers do not present proof that the energy and geometrical methods necessarily lead to identical matrices for all crystal classes.

The writers have not succeeded in finding the relations which exist between the piezooptical and the piezodielectric moduli. A knowledge of these relations would enable one to determine the piezodielectric moduli from measurements upon the piezooptical effect as well as from measurements upon the piezodielectric effect and electrostriction.

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[6] Reference 5, p. 150.

[7] Reference 5, p. 153.

[8] Reference 3, p. 97.

[9] F. Pockels, *Lehrbuch der Kristalloptik* (1906), p. 498.

[10] N. Günther, Ann. d. Physik **405**, 783 (1932).

[11] G. Szivessy, *Handbuch der Physik*, Vol. 21 (1929), p. 832.

[12] Compare (16), (17) and (18) with matrices found in reference 11, p. 841.

2. Electrostriction

When solids are subjected to electric fields, they suffer deformations in which the components of strain may be linear and quadratic in the x, y and z components of the applied field. The case in which the components of strain are linear functions of the components of the electric field is known as the converse piezoelectric effect.[13] The term *electrostriction* is here applied to the case in which the strain components are *quadratic* functions of the applied field. A phenomenological theory of electrostriction does not appear to have been developed. It will be shown that electrostriction is related to the piezodielectric effect in a manner similar to that in which the converse piezoelectric effect is related to the direct piezoelectric effect. If the piezodielectric effect and electrostriction are thus assumed to exist, the energy function ϕ together with the first and second laws of thermodynamics may be used to predict the nature of electrostriction in any medium from its piezodielectric properties.

When a body is subjected to mechanical stress or to electrical fields, the resulting physical changes depend upon the thermodynamical conditions under which these changes take place. We suppose that during the change a quantity of heat dQ crosses the boundary of the small volume element under consideration and that mechanical work dW is done by the volume element. So long as energy transfers are limited to mechanical work and heat, the corresponding change in the intrinsic energy u is.

$$du=dQ-dW, \tag{19}$$

in which it will be understood that du, dQ and dW are referred to some path. If the volume element possesses a piezodielectric effect, energy may appear in the electrostatic form. For the purpose of the following argument only the piezodielectric portion of the electrostatic energy need be considered. Consequently, over any path

$$du=dQ-dW+d\phi. \tag{20}$$

If the change is assumed to be reversible, Eq. (20) can be written in the form

$$du=TdS+\sum_{k=1}^{6} X_k dx_k+\tfrac{1}{2}\sum_{i=1}^{3} E_i dD_i, \tag{21}$$

where S is the entropy and T is the Kelvin temperature. This potential function u is suitable for discussing the adiabatic case $dS=0$.

It is convenient to define the potential function

$$\zeta\equiv u-TS-\sum_{k=1}^{6} X_k x_k. \tag{22}$$

$$\text{Thus}\quad d\zeta=-SdT-\sum_{k=1}^{6} x_k dX_k+\tfrac{1}{2}\sum_{i=1}^{3} E_i dD_i. \tag{23}$$

The thermodynamic state is usually determined by three variables (v, p, T) or (x_k, X_k, T) for which there is assumed the existence of a function

$$F(x_k, X_k, T)=0. \tag{24}$$

The state of piezodielectric substances depends also upon E_i or D_i. We assume, after (24), the existence of a function

$$G(x_k, X_k, T, E_i)=0. \tag{25}$$

Any three of the four variables in this equation may be taken as independent. Let T, X_k and E_i be chosen as independent variables. The portion of D_i which is due to the piezodielectric effect is

$$D_i=\sum_{j=1}^{3}\sum_{k=1}^{6} g_{ijk}X_k E_j. \tag{26}$$

$$\therefore\quad dD_i=\sum_{j=1}^{3}\sum_{k=1}^{6}\left[g_{ijk}(E_j dX_k+X_k dE_j)+(\partial g_{ijk}/\partial T)X_k E_j dT\right]. \tag{27}$$

By substituting (27) into (23), it is found that in isothermal processes $(dT=0)$

$$d\zeta=\sum_{k=1}^{6}\left(-x_k+\tfrac{1}{2}\sum_{i=1}^{3}\sum_{m=1}^{3} g_{imk}E_i E_m\right)dX_k+\tfrac{1}{2}\sum_{n=1}^{6}\sum_{i=1}^{3}\sum_{j=1}^{3} g_{ijn}E_i X_n dE_j. \tag{28}$$

Since ζ is exact $\left(\text{or provided } \dfrac{\partial^2\zeta}{\partial X_k\partial E_j}=\dfrac{\partial^2\zeta}{\partial E_j\partial X_k}\right)$,

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[13] See reference 3, p. 817.

$$\frac{\partial}{\partial E_j}\Big(-x_k+\tfrac{1}{2}\sum_{i=1}^{3}\sum_{m=1}^{3}g_{imk}E_iE_m\Big)$$

$$=\tfrac{1}{2}\frac{\partial}{\partial X_k}\sum_{n=1}^{6}\sum_{i=1}^{3}g_{ijn}X_nE_i. \tag{29}$$

$$\therefore\qquad -\frac{\partial x_k}{\partial E_j}+\sum_{i=1}^{3}g_{ijk}E_i=\tfrac{1}{2}\sum_{i=1}^{3}g_{ijk}E_i$$

or

$$\frac{\partial x_k}{\partial E_j}=\tfrac{1}{2}\sum_{i=1}^{3}g_{ijk}E_i,$$

$$k=1,2\cdots 6;\quad j=1,2,3. \tag{30}$$

In carrying out the partial derivatives of (29), it should be remembered that $g_{imk}=g_{mik}$.

Suppose that the components of strain are linear in X_n and quadratic in E_i and E_j such that

$$x_k=\sum_{n=1}^{6}s_{kn}X_n+\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ijk}E_iE_j \tag{31}$$

where s_{kn} are the elastic moduli. If the electrostrictive effect exists, a comparison of (30) and (31) shows that the coefficients a_{ijk} are related in the isothermal case to the piezodielectric moduli according to the law

$$a_{ijk}=\tfrac{1}{2}g_{ijk}. \tag{32}$$

In the absence of mechanical stress ($X_n=0$)

$$x_k=\tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}g_{ijk}E_iE_j\quad (k=1,2,\cdots 6). \tag{33}$$

These components of strain x_k will be called the components of "electrostrictive strain."

It will be seen from the above argument that although g_{ijk} remain fixed during an isothermal process, they depend upon the temperature at which the process takes place. Adiabatic processes are more complicated theoretically than isothermal processes since the temperature (and hence in general g_{ijk}) is variable in the adiabatic case. The theory of adiabatic processes is not included in this paper, but it may be observed that provided g_{ijk} are sufficiently slow functions of T or provided T remains sensibly constant during adiabatic processes, the corresponding components of electrostrictive strain may be computed with considerable accuracy from (33).

An examination of (3) and (33) shows that in any solid a large piezodielectric effect is associated with large components of electrostrictive strain. Further, the piezodielectric moduli may be measured either from the piezodielectric effect or from electrostriction. It seems likely that these moduli can be measured with greater ease and accuracy from the piezodielectric effect since small changes of capacity are more easily measured than the deflections due to electrostriction. When the relations between the piezo-optical and the piezodielectric moduli become known, a third method becomes available for measuring the piezodielectric moduli.

The components of electrostrictive strain in the isotropic solid are

$$\begin{aligned}
2x_1&=g_{11}E_1^2+g_{12}(E_2^2+E_3^2);\\
2x_2&=g_{11}E_2^2+g_{12}(E_1^2+E_3^2);\\
2x_3&=g_{11}E_3^2+g_{12}(E_1^2+E_2^2);\\
x_4&=x_5=x_6=0.
\end{aligned} \tag{34}$$

Electrostriction cannot, therefore, give rise to the shear components of strain in isotropic substances. In alpha-quartz and tourmaline

$$\begin{aligned}
2x_1&=g_{11}E_1^2+g_{12}E_2^2+g_{31}E_3^2+2g_{41}E_2E_3;\\
2x_2&=g_{12}E_1^2+g_{11}E_2^2+g_{31}E_3^2-2g_{41}E_2E_3;\\
2x_3&=g_{13}(E_1^2+E_2^2)+g_{33}E_3^2;\\
2x_4&=g_{14}(E_1^2-E_2^2)+2g_{44}E_2E_3;\\
2x_5&=2g_{14}E_1E_2+2g_{44}E_1E_3;\\
2x_6&=2(g_{11}-g_{12})E_1E_2+4g_{41}E_1E_3.
\end{aligned} \tag{35}$$

In these solids the shear components of electrostrictive strain are not zero. It will be noted that the strain components involve cross products of the components of the electric field. The terms in these cross products change sign when the sign of one member E_s is reversed. This aspect of electrostriction does not appear to have been considered or tested.

Reversibility has been assumed in the theory leading to (33). Consequently (33) does not accurately apply to viscoelastic solids of the type discussed by J. H. C. Thompson.[14] Rochelle salt is sufficiently viscoelastic that (33) can be expected to apply only after the components of electrostrictive strain have reached equilibrium with the impressed electric field. These components lag varying fields and thus give rise to hysteresis.

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[14] J. H. C. Thompson, Phil. Trans. Roy. Soc. London **231**, 358 (1932–33).

3. THE ELASTIC, PIEZOELECTRIC, ELECTROSTRICTIVE SOLID

The components of strain in a perfectly elastic solid which is piezoelectric and electrostrictive will be considered briefly. In this solid the components of strain are increased by the components which originate from the converse piezoelectric effect. Hence for this solid Eq. (31) assumes the form

$$x_k=\sum_{n=1}^{6} s_{kn}X_n+\tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} g_{ijk}E_iE_j+\sum_{i=1}^{3} d_{ik}E_i, \qquad (36)$$

where d_{ik} are the piezoelectric moduli.

4. THE EQUATIONS OF MOTION

When a body is not in equilibrium with a system of mechanical and "electrostrictive stresses," the series of states through which it passes is described by the equations of motion and the boundary conditions. When only the mechanical stresses are present, the equations of motion are[15]

$$\begin{aligned}
&\frac{\partial X_1}{\partial x}+\frac{\partial X_6}{\partial y}+\frac{\partial X_5}{\partial z}+\rho X=\rho\frac{\partial^2 u}{\partial t^2};\\
&\frac{\partial X_6}{\partial x}+\frac{\partial X_2}{\partial y}+\frac{\partial X_4}{\partial z}+\rho Y=\rho\frac{\partial^2 v}{\partial t^2};\\
&\frac{\partial X_5}{\partial x}+\frac{\partial X_4}{\partial y}+\frac{\partial X_3}{\partial z}+\rho Z=\rho\frac{\partial^2 w}{\partial t^2}.
\end{aligned} \qquad (37)$$

If (4) is substituted into (9)

$$2\phi=\sum_{k=1}^{6} x_k\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ijk}E_iE_j, \qquad (38)$$

in which b_{ijk} are the "piezodielectric constants." These are connected with the piezodielectric moduli by the relations

$$b_{ijm}=\sum_{k=1}^{6} g_{ijk}c_{km}, \qquad (39)$$

where c_{km} are the elastic constants.

With the aid of (38) the thermodynamic potential ζ may be expressed using x_k, E_j and T as independent variables. Following a method similar to that leading to (33), one can show that

$$X_k=\tfrac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} b_{ijk}E_iE_j \quad (k=1,2\cdots 6), \qquad (40)$$

wherein X_k are components of "electrostrictive stress," i.e., those components of mechanical stress which would bring about a state of strain identical with that produced by electrostriction.

Let R_k denote the sum of the mechanical, electrostrictive, and piezoelectric stresses, i.e., let

$$R_k=X_k+\tfrac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3} b_{ijk}E_iE_j+\sum_{i=1}^{3} e_{ik}E_i, \qquad (41)$$

where e_{ik} are the piezoelectric constants in the notation of Voigt. The equations of motion in elastic, piezoelectric and electrostrictive media are now obtained by replacing the purely mechanical stresses X_k in (37) by R_k from (41). The solution of the equations of motion and an appropriate set of boundary conditions determines the instantaneous states through which the medium passes.

The boundary conditions may specify the X, Y and Z[16] components of surface traction across a plane element of normal ν at the boundary of the medium. The components X_ν, Y_ν, and Z_ν, of the surface traction are

$$\begin{aligned}
X_\nu&=R_1\cos(x\nu)+R_6\cos(y\nu)+R_5\cos(z\nu);\\
Y_\nu&=R_6\cos(x\nu)+R_2\cos(y\nu)+R_4\cos(z\nu);\\
Z_\nu&=R_5\cos(x\nu)+R_4\cos(y\nu)+R_3\cos(z\nu).
\end{aligned} \qquad (42)$$

The above equations of motion and boundary conditions are accurate only in the isothermal case, but when the effects of the adiabatic temperature changes are negligibly small, the adiabatic case can be solved in the first approximation as an isothermal problem.

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[15] Reference 5, p. 85.

[16] Reference 5, p. 78.