Statistical Analysis of Counter Data

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(Received February 24, 1936)

A general relation is derived for the number of counts registered by a Geiger-Müller tube counter or similar electrical counting device exposed to a radioactive source whose strength varies arbitrarily with the time when the counter has a finite, constant resolving time. This is applied specifically to the case of an exponentially decaying source superposed on a uniform background, the solution of the resulting formula being represented nomographically. The number of spurious coincidences observed in a set of P counters used coincidentally is calculated when the resolving times of the individual counters are neglected in comparison with the resolving time of the combining electrical circuit for coincidences. This general expression is applied to the cosmic-ray "telescope" and the double coincidence magnetic spectrometer such as that of Henderson and Alichanow. The constancy of the resolving time of a single counter and the justification for neglecting individual resolving times in comparison with the coincidence resolving time are discussed.

1. Introduction

It is well known that Geiger-Müller tube counters and similar electrical counting devices fail, because of their finite resolving time, to register a certain fraction of the ionizing particles that pass through them. Moreover, when several such counters are arranged to respond only to particles that traverse all of them, a certain number of spurious coincidences will be recorded due to the finite resolving time of the combining electrical circuit. The purpose of the following investigation is to show how these effects can be corrected for in some cases of interest.

2. The Distribution Law

To begin with we require a knowledge of the distribution in time of the arriving particles. Because of the finite solid angle subtended by the counter at the source, not all of the particles emitted by the source pass through the counter. Now it is convenient in the present discussion to consider only those atoms of the source which when they disintegrate will each give off a particle that goes through the counter. The number of these atoms is always proportional to the total number of atoms in the source and will be used throughout as the definition of number of atoms present at a particular time. In this paper we shall exclude chain disintegrations from consideration, since the time and space relations between successively emitted particles of a chain will complicate the following theory; however, we believe that the theory can be modified to deal with this case.

For any one of the M atoms present at $t=0$, let the chance that it disintegrate between times t and $t+s$ be $f(t, s)$. Then the chance that N' of the original M disintegrate between t and $t+s$ and that $M-N'$ do not is given by the binomial law:

$$[M!/N'!(M-N')!]f^{N'}(1-f)^{M-N'}.$$

This expression reduces exactly to Fry's formula[1] when we give $f(t, s)$ the form corresponding to exponential decay. Now we can never know the number of atoms present at the beginning of any particular trial, but only (under ideal conditions) the average number for several trials. Suppose that the source is an artificially radioactive material, prepared by exposure to a substance of very long mean life or to an artificially accelerated beam of particles. In either case the number of bombarding particles appearing in any interval is governed by Poisson's law.[2,3] Hence the numbers of new radioactive atoms produced during the time of bombardment in different trials under identical conditions are distributed according to Poisson's law. This argument is readily extended to the case in which the bombarding material has a relatively short mean life (for if we trace back far enough, we will eventually find an original or parent source of effectively infinite mean life, and starting

[1] T. C. Fry, *Probability and Its Engineering Uses* (1928), p. 237, Eq. (114). See also A. E. Ruark and L. Devol, Phys. Rev. **48**, 772 (1935).
[2] See D. J. Struik, M. I. T. Jour. Math. and Phys. **9**, 151 (1930) for an excellent discussion of Poisson's distribution law.
[3] Reference 1, Fry, p. 214 *et seq.*

from this we can use Eq. (1) repeatedly to obtain the desired result). Then the chance that the initial number of atoms is M when the average number present initially is N is: $(N^M/M!)e^{-N}$. Thus the chance that N' atoms disintegrate between times t and $t+s$ when N atoms are present on the average at $t=0$ is:

$$\sum_{M=N'}^{\infty}\left\{\frac{N^M e^{-N}}{M!}\cdot\frac{M!}{N'!(M-N')!}f^{N'}(1-f)^{M-N'}\right\}=\frac{(Nf)^{N'}e^{-Nf}}{N'!}. \quad (1)$$

Eq. (1) is simply what Poisson's law gives for the chance that N' events occur in the interval t, $t+s$ when $Nf(t, s)$ occur on the average. It should be noted that the number of atoms $Nf(t, s)$ that disintegrate on the average between t and $t+s$ may be taken as the average number of particles that arrive at the counter in this interval, regardless of origin, and includes the counter background as well as particles from the source without affecting the mathematical argument. We shall denote this number in the future by $F(t, s)$.

By defining only such quantities as possess experimental significance and therefore assuming in the above derivation that the number of atoms present initially is not a definitely known or determinable quantity, we have obtained in Eq. (1) a distribution law that is much simpler to work with than the binomial form[1, 4] used by Fry for the β-emission problem.

3. The Resolving Time

In order for a counter to register a particle, that particle must be preceded by a certain small interval during which no particle arrives at the counter. The length of this blank interval is called the resolving time of the counter. Before proceeding further, we must consider the constancy of this resolving time for a particular counter. Danforth[5] has found experimentally that the time of recovery from a pulse for the counters on which he worked depended on the magnitude of the change in voltage across the counter in that pulse, which in turn depended on the time that had elapsed since the previous pulse. This means that the resolving time of the counter depends somewhat on its past history. Ramsey and Lipman,[6] on the other hand, found that in the counters which they examined the voltage peak in a pulse had always the same magnitude, and was such that the extinction voltage (the voltage across the counter at peak) was equal to the threshold voltage (the minimum voltage across the counter necessary for the production of counts). Under these conditions, the resolving time is constant. According to Ramsey,[7] these characteristics vary from counter to counter, and while the extinction voltage is always less (sometimes by two to three hundred volts) than the threshold voltage, the difference may be very small for many counters operating over certain voltage ranges. Skinner[8] has considered a statistical distribution of resolving times for a source of constant strength, basing his assumptions on Danforth's observations. However, it is readily seen that a single statement concerning the resolving time that will apply to all counters cannot be made; when there are amplifying and recording circuits associated with the counter the assumption of a constant value is probably just as valid as that of a statistical distribution.[9] This is true especially when the value used for the constant resolving time can be measured under the conditions of the experiment by a method to be described in section 5 (this procedure automatically gives the right sort of averaging in case the resolving time is not

[4] Recently A. Ruark and L. Devol, Phys. Rev. **49**, 355 (1936) have written a theory of fluctuations in radioactive disintegration based on the binomial type of distribution function. It should be noted that their Eqs. (12) and (16) can be written down immediately (they are of the binomial form) by using the argument at the beginning of section 2 of the present paper. Thus, putting $t=T_1$, $s=T_2$, $M=N$, $N'=n$, and $f(t, s)=gA[e^{-\lambda t}-e^{-\lambda(t+s)}]$, the expression at the beginning of our section 2 becomes the right-hand side of their Eq. (16). Their Eq. (12) can be obtained similarly with even less algebra, using $f(t, s)=e^{-\lambda t}-e^{-\lambda(t+s)}$.

[5] W. E. Danforth, Phys. Rev. **46**, 1026 (1934).

[6] W. E. Ramsey and M. R. Lipman, Rev. Sci. Inst. **6**, 121 (1935).

[7] A private communication, for which the writer is indebted to Mr. Ramsey.

[8] S. M. Skinner, Phys. Rev. **48**, 438 (1935).

[9] For a counter whose threshold and extinction voltages are equal, Skinner's tube parameter d is equal to unity, and his theory predicts one hundred percent counting efficiency for all counting rates. This disagrees with the well-known phenomena of decreasing counting efficiency and eventual paralysis because the effect of the amplifying and recording circuits has not been taken into account. This effect should be considered even at small counting rates.

constant). In addition, the hypothesis of constant resolving time has the considerable advantage of being amenable to mathematical analysis in the case of decaying sources, while the statistical distribution of resolving times leads to quite complicated results even in the relatively simple case of a source of constant strength.[8] In view of all these considerations, we shall assume throughout that the resolving time of a particular counter is constant and has the value that would be determined experimentally by the method of section 5.

4. General Theory for a Single Counter

From the definition of resolving time given at the beginning of section 3 we see that a particle arriving at the counter between times t and $t+dt$ is recorded only if it is preceded by an interval τ in which no particle arrives at the counter. Suppose that the expectation that a particle arrive at the counter between t and $t+dt$ is $\Lambda(t)dt$, where $\Lambda(t)$ depends on the nature of the source and on the counter background. The number of particles expected on the average between times $t-\tau$ and t is then

$$F(t-\tau,\ \tau)=\int_{t-\tau}\Lambda(t)dt,$$

and from Eq. (1) the chance that no particles appear in this interval is $e^{-F(t-\tau,\ \tau)}$. Then of the particles arriving between t and $t+dt$, we expect to count the fraction $e^{-F(t-\tau,\ \tau)}$. Hence in an observation period T_1, T_2, where τ is very small compared to the time of observation $T=T_2-T_1$ (this must hold if we are to get useful data), we expect to *count*

$$n=\int_{T_1}^{T_2}\Lambda(t)e^{-F(t-\tau,\ \tau)}dt, \tag{2}$$

where $$F(t-\tau,\ \tau)=\int_{t-\tau}^{t}\Lambda(t)dt.$$

This is to be compared with the expected number of *particles* that passed through the counter

$$n_0=\int_{T_1}^{T_2}\Lambda(t)dt. \tag{3}$$

5. Constant Source; Determination of the Resolving Time

For a source of constant strength, giving off n_0 particles that go through the counter in time T (this includes the counter background), we obtain immediately from Eq. (2) (setting $\Lambda(t)\equiv n_0/T$):

$$n=n_0e^{-n_0\tau/T} \tag{4}$$

for the number n of counts registered in the time T. This result was obtained by Volz[10] in a somewhat less general way. He went on to show how the resolving time of any particular counter could be determined with the aid of Eq. (4).

We shall now give what is essentially Volz' second method for the evaluation of τ. If we solve Eq. (4) for n in terms of n_0/n (the ratio of number of particles to number of counts) we obtain:

$$n\tau/T=[\ln(n_0/n)]/(n_0/n), \tag{5}$$

which is plotted in Fig. 1. This has a maximum at $n\tau/T=1/e$, $n_0/n=e$. Thus to determine τ, we bring a source up to the counter until it reads at its maximum rate $n_{\max}$, whence:

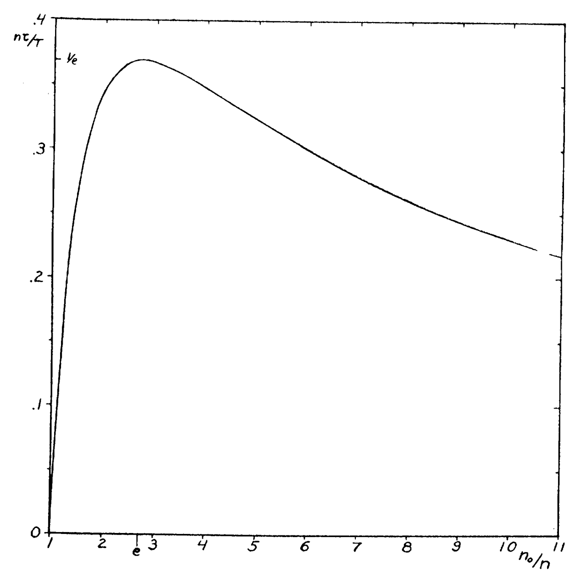

FIG. 1. Counting rate expressed as average number of counts per resolving time of the counter plotted against the ratio of number of particles to number of counts; this is for a source of constant strength.

[10] H. Volz, Zeits. f. Physik **93**, 539 (1935).

$\tau = T/e \cdot n_{\max}$. Fig. 1 permits us to find immediately the true number of particles from the observed number of counts; however, it must be remembered that this includes the counter background.

For an example, suppose that the maximum counting rate of a given counter is $n_{\max}/T = 630$ counts per minute $= 10.5$ per second; then $\tau = 1/10.5e = 0.035$ second. Now with the same counter, we count an average of 480 counts per minute from a particular source, and an average of 50 counts per minute with the source removed (background). The first figure gives a value of $480 \times 0.035/60 = 0.28$ for $n\tau/T$ (average number of counts in a resolving time), and the second a value of $50 \times 0.035/60 = 0.03$. Referring to Fig. 1, we see that the source reading is to be multiplied by 1.5, giving 720 for the number of particles per minute, while the background reading is so small that the correction is negligible. Thus the actual number of particles per minute due to the source alone is 670. It will be observed that the ordinate $n\tau/T = 0.28$ gives a value for n_0/n of 7.1 as well as 1.5. The former figure corresponds to a particle frequency of 3400 per minute. The larger figure indicates that the counter is almost completely paralyzed. To distinguish between the two in case of doubt, we simply remove the source slightly. If the counter is nearly paralyzed this will result in an increased counting rate, while if it is operating normally the counting rate will fall off. For ordinary counter usage the lower figure for the particle frequency is the correct one. This method is applicable to sources of constant or slowly varying strength.

6. Determination of Decay Constant; Peierls' Method

Peierls[11] has shown how the decay constant λ of a radioactive element can be determined with the least possible error, using a counter exposed to the source and influenced by its natural background. His method requires that the observation period be divided into intervals of equal lengths d less than $0.3/\lambda$, and that the number of particles arriving in each of these intervals be measured. When the intervals d are of length equal to or less than $0.2/\lambda$, we can find the number of particles from the number of counts by using Fig. 1 (and subtracting the background counts), with an error of less than one percent. If it happens that the intervals d must be taken much larger than $0.2/\lambda$ the method of section 5 should not be used; rather the results of section 7 are to be applied. However, the application of these results requires a knowledge of the decay constant, which is just what we are trying to determine. Therefore an approximate value for λ must be assumed; from this the number of particles in each interval is calculated by the method of the next section. These particle numbers are then used to get a closer approximation to λ. Usually, however, it will be feasible (and it is always preferable) to take $d \leq 0.2/\lambda$, in which case section 5 may be applied directly.

[11] R. Peierls, Proc. Roy. Soc. **A149**, 467 (1935). See particularly p. 475.

7. Exponentially Decaying Source with Constant Background

Consider a counter exposed to a source of decay constant λ, on which is superposed a constant background or natural effect of strength β particles per unit time. Let N be the average number of atoms of the decaying material present at $t=0$ (using number of atoms in the sense defined at the beginning of section 2), and N_1 and N_2 be the numbers present on the average at times T_1 and T_2, respectively, where these are the limits of the observation time. Then in Eq. (2) we put

$$\Lambda(t) \equiv \beta + N\lambda e^{-\lambda t} \tag{6}$$

and obtain, on performing the integrations,[12]

$$\begin{aligned} n = (e^{-\beta\tau}/e^{\lambda\tau}-1)\{&\exp[-N_2(e^{\lambda\tau}-1)] \\ &-\exp[-N_1(e^{\lambda\tau}-1)]\} + (\beta/\lambda)e^{-\beta\tau} \\ \times\{&Ei[-N_1(e^{\lambda\tau}-1)] - Ei[-N_2(e^{\lambda\tau}-1)]\}, \end{aligned} \tag{7}$$

where $n_0 = N_1 - N_2$ is the expected number of particles due to the source alone that passed through the counter in the time of observation

[12] $\exp(-z) = e^{-z}$; $Ei(-z) = -\int_z^\infty e^{-v}/v dv$. For tables of the exponential integral and related functions with argument going from zero to one by intervals of 0.001, see W. L. Miller and T. R. Rosebrugh, Trans. Roy. Soc. Canada **9**, 73 (1903); reprinted as University of Toronto Studies, Papers from the Chemical Laboratories, no. 43.

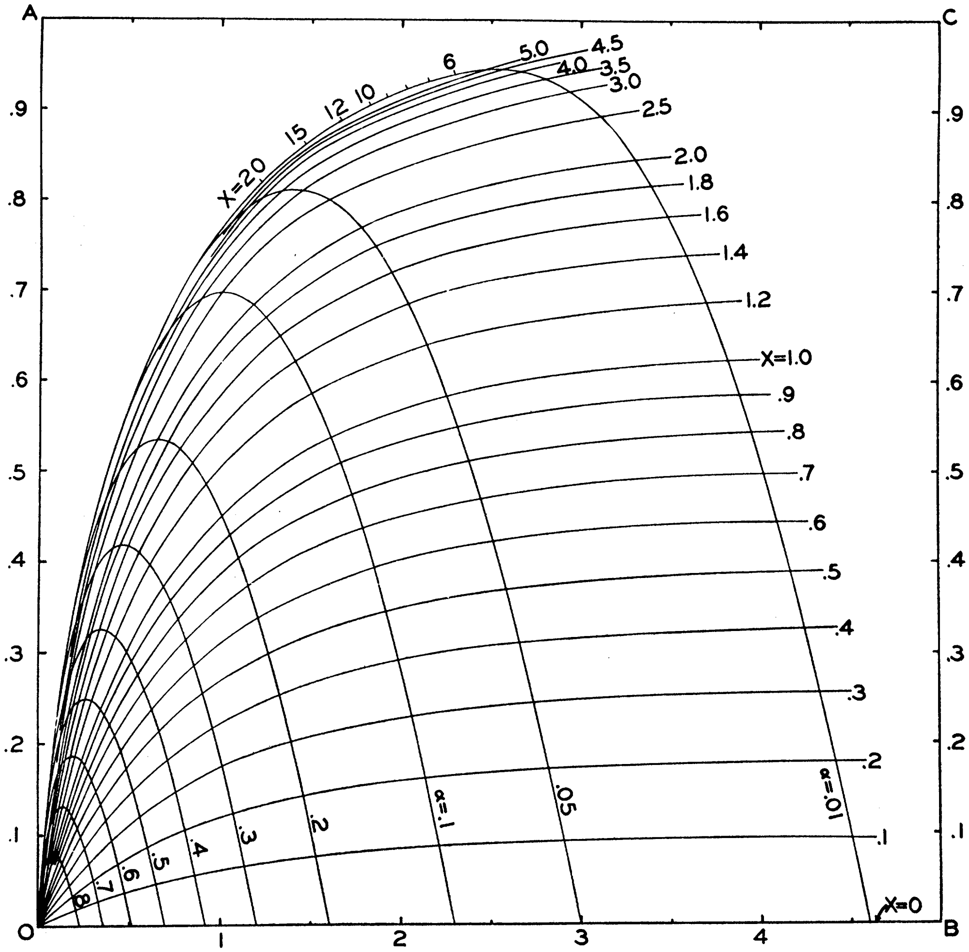

FIG. 2. Nomogram for the solution of Eq. (8). $a=n\lambda\tau e^{\beta\tau}$ is plotted along the OA scale, and $b=a-5\beta\tau$ is plotted along the BC scale; a straight line joining points a and b intersects the curve for a particular value of α at values of x which are solutions of Eq. (8).

$T=T_2-T_1$. Making use of the relations $N_1=n_0/(1-\alpha)$, $N_2=\alpha n_0/(1-\alpha)$, where $\alpha=e^{-\lambda T}$, and remembering that τ must be very small compared to $1/\lambda$ to obtain data of any value at all, we may write Eq. (7) to good approximation:

$$n=(e^{-\beta\tau}/\lambda\tau)[e^{-\alpha x}-e^{-x}]+(\beta/\lambda)e^{-\beta\tau} \times[Ei(-x)-Ei(-\alpha x)], \quad (8)$$

where $x=n_0\lambda\tau/(1-\alpha)$. It is impossible to solve Eq. (8) analytically for x (or n_0) in terms of n and the other parameters, and any approximation good enough to cover the whole field of experimental application is of little value. However, as there are just four independent parameters (x, α, and two others which may be regarded as the coefficients of the bracket terms divided by n), a fairly simple nomographic representation is possible (see Fig. 2).

The nomogram consists of a set of curves of constant α and a set of curves of constant x. The studs on the curve for $\alpha=0.01$ above $x=5$ give the points of intersection with the curves (not drawn) for eight values of x from 6 to 20. The values of λT corresponding to each α are given by the intersection of that α-curve with the OB scale and are also found in Table I.

TABLE I. *Values of* $\lambda T=-\ln\alpha$ *for the ten* α*-curves of Fig. 2.*

α	0.01	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
λT	4.605	2.996	2.302	1.609	1.204	0.916	0.693	0.511	0.357	0.223

The α-curves are drawn for ten values of α ranging from 0.01 ($\lambda T=4.605$) to 0.8 ($\lambda T=0.223$). An observation time greater than $4.6/\lambda$ would hardly ever be used in practice because of the low intensity of the source towards the end of the run and the consequent increase in importance of the background. On the other hand, for $T<0.2/\lambda$ the variation of the source strength during the time of observation is so small that Fig. 1 of section 5 may be used directly with an error of less than one percent, and the procedure of this section is unnecessary. It is obvious that a choice of simple values of α rather than of λT simplifies enormously the computation of the nomogram while it does not materially affect its convenience in use, since the time of observation will depend on λ as well as λT and will not in general be a simple number.

To use the nomogram we first find the resolving time of the counter by the method of section 5, the decay constant λ of the source by Peierls' method corrected as in section 6, and the background strength β corrected as in section 5. These will be constant throughout a particular experiment. By using the observed number of counts n we then calculate two numbers a and b defined as

$$a=n\lambda\tau e^{\beta\tau} \tag{9}$$

$$b=a-5\beta\tau. \tag{10}$$

(Since the background strength is generally rather small, it will facilitate computation to note that $e^{\beta\tau}$ may be replaced by $1+\beta\tau$ with an error of less than one percent when $\beta\tau<0.14$.) We then mark off the point a on the OA scale and the point b on the BC scale of the nomogram. From the length of observation time T we find the number $\alpha=e^{-\lambda T}$, and select the corresponding curve from the set for constant α. (It is advisable to use a value of α for which a curve is drawn; otherwise it is necessary to interpolate.) We then draw a straight line through the points a and b on the OA and BC scales, respectively; this in general intersects the curve of constant α that we have chosen at two values of x which we shall call x_1 and x_2, where x_1 is the smaller. From these values of x we find two values of n_0 from the relation:

$$n_0=x(1-\alpha)/\lambda\tau. \tag{11}$$

The smaller value of n_0 thus obtained corresponds to normal operation of the counter and the larger to operation close to paralysis. In general it is safe to say that x_1 should be used to determine n_0.

An example will perhaps clarify the use of the nomogram of Fig. 2. Suppose that we have found by the methods of sections 5 and 6 that $\tau=0.023$ second, $\beta=1.83$ seconds^{-1}, $\lambda=0.0126$ second^{-1} (half-value time$=55.0$ seconds) for a particular counter and source. Now suppose that it happens that it is most convenient in this experiment for us to take readings over a period of about five minutes. This being the case, we choose $\alpha=0.01$, which gives $\lambda T=4.60$, or $T=6.09$ minutes. Our reading at the end of this time is $n=3070$ counts. We then calculate a and b from Eqs. (9) and (10) and obtain: $a=0.929$, $b=0.719$. Laying off these values on the OA and BC scales, respectively, we see that the straight line through these points intersects the curve for $\alpha=0.01$ at $x_1=1.60$, $x_2=14.2$. We find that the counter is operating normally by removing the source slightly and noting that the counting rate decreases (if the counter were nearly paralyzed a removal of the source would at first increase the counting rate). Therefore, we use x_1 and obtain $n_0=5460$, which is the number of particles from the source (exclusive of background) during the 6.09 minutes of observation.

8. General Theory for Multiple Coincidence Counters

A true P-fold coincidence is produced in a set of P coincidence counters when a particle traverses all the counters, or when the individual members of a burst traverse all simultaneously. A certain number of spurious coincidences will also be produced by particles (other than those that cause true coincidences) arriving at each of the P counters so that their arrival times are grouped within a small interval of time. The length τ' of this interval is called the resolving time of the combining electrical circuit for coincidences and is assumed to be constant. The spurious counts may be produced by distinct particles in each counter, by single particles in some of the counters and true partial coincidences in the rest, or by any combination of single particles and true partial coincidences. A true partial coincidence is defined here as a true coincidence produced by one particle traversing more than one but not all of the counters. In the following analysis we shall assume that the resolving times of the individual counters may be neglected in comparison with that of the combining circuit for coincidences. While this is not strictly true, as long as the counters are operated well below their maximum counting rate (as is usually the case in coincidence work) the principal effect of the finite resolving times of the separate counters is to increase somewhat the effective coincidence resolving time; this is automatically allowed for when τ' is measured as indicated at the end of section 9.

Let the expectation that a particle pass through only the ith of P coincidence counters between times t and $t+dt$ be $\Lambda_i(t)dt$; likewise let the expectation of a true partial coincidence in only counters $i, j, \cdots l$ between times t and $t+dt$ be $\Lambda_{i,\, j,\, \dots\, l}(t)dt$. We shall make the simplifying assumptions throughout that $\Lambda_i(t) \gg \Lambda_{ij}(t) \gg \cdots \gg \Lambda_{i,\, j,\, \dots\, l}(t)$, etc., in order of increasing number of subscripts, and that ${}_P\Lambda(t) \ll 1/\tau'$, where ${}_P\Lambda(t)$ is the expectation function for true P-fold coincidences. These conditions generally hold; however, the theory can be developed without them, although it would have an even more complicated form than that given below. It is also assumed that the various Λ's are independent; this is quite valid since a particle which causes a true partial coincidence in any group of counters at a particular time can have no effect on any of the counters at any other time. In general, these Λ's will depend only on the geometrical arrangement of sources and counters. Then according to Eq. (1) the chance that at least one particle appear in only the ith counter between t and $t+\tau'$ is $1-e^{-F_i(t,\, \tau')}$, where

$$F_i(t, \tau') = \int_t^{t+\tau'} \Lambda i(t)dt;$$

also, the chance that at least one true partial coincidence appear only in counters $i, j, \cdots l$ between t and $t+\tau'$ is $1-e^{-F_{i,\, j\, \cdots\, l}(t,\, \tau')}$, where

$$F_{i,\, j,\, \dots\, l}(t, \tau') = \int_t^{t+\tau'} \Lambda_{i,\, j,\, \dots\, l}(t)dt.$$

Now a spurious coincidence is caused by an initiating particle (or true partial coincidence) in one (or more) of the counters being followed by particles (either single or members of a true partial coincidence) through each of the other counters within the time τ'. This is provided that at least one of the counters other than that (or those) which initiated the spurious coincidence had no particle through it for a time τ' in the past; this assures that the combining circuit was clear when the initiating particle arrived. For example, with a group of six counters, it is seen that the expectation that a true double coincidence occur in counters 2 and 4 between times t and $t+dt$, followed by a single particle through counter 6 and a true triple coincidence in counters 1, 3, and 5 within a time τ' is:

$$\{1 - \prod_{i=1,\, 3,\, 5,\, 6} (1-e^{-F_i(t-\tau',\, \tau')})\}\Lambda_{24}(t) \times (1-e^{-F_6(t,\, \tau')})(1-e^{-F_{135}(t,\, \tau')})dt, \quad (12)$$

when at least one of counters 1, 3, 5, and 6 was not discharging for a time τ' before t. In order to find the expected number m of spurious coincidences observed between times T_1 and T_2, we must add together all possible combinations of the Λ's and the F's of the form of Eq. (12) that involve all of our P counters, and then integrate over the time from T_1 to T_2. It is evident that with more than three counters the result is very complicated indeed. Making the assumptions above that the Λ's decrease rapidly as the number of subscripts increases and that ${}_P\Lambda \ll 1/\tau'$, we obtain for two counters used coincidentally:

$$m_2 = \int_{T_1}^{T_2} \{e^{-F_2(t-\tau',\, \tau')}\Lambda_1(t)(1-e^{-F_2(t,\, \tau')}) + e^{-F_1(t-\tau',\, \tau')}\Lambda_2(t)(1-e^{-F_1(t,\, \tau')})\}dt. \quad (13)$$

The similar expressions for three and four counters would contain nine and thirty-six terms, respectively, instead of two as in Eq. (13). No matter how complicated, however, the general expression for any number of counters can always be written down by using the above method, and it will be shown in the next section that the results simplify considerably in an important special case.

9. Multiple Coincidences; Constant Sources

In the case where the average number of particles per unit time at any counter is independent of the time (as in the cosmic-ray "telescope"), we may put: $\Lambda_i(t) \equiv \nu_i$, $\Lambda_{i,\,j,\,\ldots\,l} \equiv \nu_{i,\,j,\,\ldots\,l}$. When the various ν's are small compared to $1/\tau'$, the expected spurious counting rates for two, three, and four counter coincidence units, respectively, become to good approximation:

$$\begin{aligned}
&m_2/T = 2\tau'\nu_1\nu_2\,;\\
&m_3/T = 3\tau'^2\nu_1\nu_2\nu_3 + 2\tau'(\nu_1\nu_{23}+\nu_2\nu_{13}+\nu_3\nu_{12})\,;\\
&m_4/T = 4\tau'^3\nu_1\nu_2\nu_3\nu_4 + 3\tau'^2(\nu_1\nu_2\nu_{34}+\nu_1\nu_3\nu_{24}\\
&\quad+\nu_2\nu_3\nu_{14}+\nu_1\nu_4\nu_{23}+\nu_2\nu_4\nu_{13}+\nu_3\nu_4\nu_{12})\\
&\quad+2\tau'(\nu_{12}\nu_{34}+\nu_{13}\nu_{24}+\nu_{14}\nu_{23})\\
&\qquad+2\tau'(\nu_1\nu_{234}+\nu_2\nu_{134}+\nu_3\nu_{124}+\nu_4\nu_{123}).
\end{aligned} \tag{14}$$

Similar expressions are readily written down for any number of counters. It must be remembered in using Eq. (14) that the various ν's represented there are the average numbers of particles per unit time that belong *only* to the classifications indicated by the subscripts. In practise it is impossible to measure these quantities directly. For example, the quantity ν_{234} in a four counter coincidence unit is given by:

$$\nu_{234} = \mu_{234} - 3\tau'^2_{234}\nu_2\nu_3\nu_4 - 2\tau'_{234}(\nu_2\nu_{34}+\nu_3\nu_{24}+\nu_4\nu_{23}),$$

where μ_{234} is the observed number of coincidences per unit time between counters 2, 3, and 4 only, τ'_{234} is the coincidence resolving time for these counters, and the ν's appearing on the right-hand side above must themselves be obtained from similar expressions. Thus the ν's can be found accurately only as the solution of a set of simultaneous nonlinear equations. This would be a very laborious process at best; however, a good approximation is generally obtained by using the observed μ's for the ν's, and neglecting the correction terms.

The experimental measurement of the coincidence resolving time τ' may be effected by arranging the P counters in space so that true coincidences are very unlikely between any combination of two or more counters. In this case all of the ν's that possess more than one subscript vanish, and it is readily seen that the spurious counting rate is given by:

$$m_P/T = P(\tau')^{P-1}\nu_1\nu_2\cdots\nu_P. \tag{15}$$

Since the counters are arranged so that all of the observed P-fold coincidences are spurious, m_P/T, ν_1, ν_2, $\cdots$ ν_P can be measured directly, and τ' is readily calculated. This method for the determination of τ' tends to compensate for neglecting the individual counter resolving times.

10. The Double Coincidence Magnetic Spectrometer

Henderson,[13] Alichanow,[14] and others have employed a magnetic spectrometer in the measurement of beta-ray and positron energy distribution spectra. The apparatus is designed to respond to coincidences produced by electrons that are deflected around a semicircular path by a magnetic field and traverse two counters placed in their path. On the other hand, gamma-rays from the source can set off either one of the counters but never both. Because of the finite coincidence resolving time, the gamma-rays from the source, and the natural backgrounds of the two counters, a certain number of spurious coincidences will be produced. To calculate this number for an exponentially decaying source we put into Eq. (13):

$$\Lambda_1(t) \equiv \Lambda_2(t) \equiv \beta_0 + N\lambda e^{-\lambda t}, \tag{16}$$

where β_0 is the constant counter background, N the average number of atoms for each counter at zero time (defined as at the beginning of section 2), and λ the decay constant of the source. Since the counters are generally similar in construction and placed symmetrically with respect to the source, β_0 and N will be the same for both.[15] Putting Eq. (16) into Eq. (13) and carrying out the integrations, we find that the result is most easily expressed in the notation of section 7:

$$m_2 = 2(n_1 - \tfrac{1}{2}n_2). \tag{17}$$

In Eq. (17), m_2 is the expected number of spurious coincidences observed in the double

[13] W. J. Henderson, Proc. Roy. Soc. **A147**, 572 (1934).

[14] A. I. Alichanow et al., Zeits. f. Physik **90**, 249 (1934); **93**, 350 (1935).

[15] Eq. (17) is readily modified if this is not the case, although the result is somewhat more complicated.

coincidence unit in the time T; if n^0 is the number of counts due to the source alone observed in this time in each counter separately (these should be about the same for the two counters), then n_1 is given by Eq. (8) when we set $\tau=\tau'$, $\beta=\beta_0$, and $n_0=n^0$, and n_2 is given by Eq. (8) with $\beta=2\beta_0$ and $n_0=2n^0$. n^0, T, and β_0 are directly observable. To calculate n_1 and n_2 rapidly from Eq. (8) without having to look up exponentials and exponential integrals, the nomogram of Fig. 2 may be used as follows. We select the curve of constant α corresponding to our value of λT (again it is advisable to choose T so that the resulting value of α is one for which a curve is drawn in Fig. 2; this removes the necessity for interpolation). We find the value of x corresponding to $n_0(=n^0)$ from the second of Eq. (8) (with $\tau=\tau'$), and mark this point on the correct α-curve. Then the ordinate of this point is the value of $[e^{-\alpha x}-e^{-x}]$, the first bracket expression in Eq. (8), and the abscissa is the value of $[Ei(-x)-Ei(-\alpha x)]$, the second bracket expression in Eq. (8). This facilitates the calculation of n_1. The bracket expressions for n_2 are found in exactly the same manner, except that $2n^0$ is used for n_0.

To illustrate, suppose that we are examining a source of decay constant $\lambda=2.87\times10^{-4}$ seconds^{-1} (half-value time$=40.2$ minutes) with a double coincidence arrangement of resolving time $\tau'=0.078$ second, and that a convenient length of observation time is about an hour. We then choose $\alpha=0.4$, which gives $\lambda T=0.916$, or $T=53.1$ minutes. The constant background for each counter is found to be $\beta_0=0.37$ count per second. We observe for 53.1 minutes, and find an average total $(=n^0+\beta_0 T)$ of 3430 single counts in each of the counters; subtracting the number of background counts gives $n^0=2250$. We now proceed to find n_1. From the second of Eq. (8) (with $\tau=\tau'=0.078$) we obtain $x=0.084$; this point on the curve of Fig. 2 for $\alpha=0.4$ has an ordinate of 0.045 and an abscissa of 0.87. Substituting these values for the bracket expressions into Eq. (8) we obtain $n_1=3040$. Similarly for n_2, we have $n_0=2n^0=4500$, $x=0.168$ with an ordinate of 0.085 and an abscissa of 0.80, giving $n_2=5510$ when we use $\beta=2\beta_0=0.74$ second^{-1}. Substitution into Eq. (17) gives immediately $m_2=570$ for the number of spurious coincidences expected under these conditions.

11. Concluding Remarks

The aim of the writer in this treatment of the statistics of electrical counting devices has been to develop the theory necessary for the correction of single and multiple coincidence counter readings for finite resolving time in as general a way as possible; and then to apply the general theory to specific cases of interest, the results being represented in such a manner as to be readily available for experimental application. Two main assumptions have been made: (1) The resolving time of a single counter is constant; (2) the resolving times of the individual counters in a multiple coincidence arrangement can be neglected in comparison with the resolving time of the combining electrical circuit for coincidences. There is no doubt that neither of these assumptions is always strictly true. On the other hand, it seems very probable that the results obtained with the aid of these assumptions when the procedures outlined are followed differ from the truth only by an amount that is of smaller order than the corrections to the actual readings introduced by the above theory. Also, assumption (1) in particular seems to be as accurate over a wide range of operating conditions as any moderately simple one that can be made. In any case, the simplifications in the mathematical theory and the resulting ease of application to any given experimental situation more than compensate for the second order errors that appear as a consequence.

The writer wishes to express his sincere appreciation to Professor Robley D. Evans for suggesting this problem and for continued interest in the progress of its solution; to Professor D. J. Struik of the mathematics department for helpful discussion of probability theory in general; and to Professor L. H. Thomas of the Ohio State University for illuminating conversations on certain points of the theory, particularly section 2.