quantitatively in agreement, as regards magnitudes, variation with angle, and variation with voltage, with a simple phase shift of the spherically symmetrical de Broglie wave ("S wave") due to the collision or scattering, corresponding to a new attractive force overpowering the Coulomb repulsion, and give a rather accurate measure of the "potential well" which is therefore permissible as representing the interaction. Interestingly enough, this potential well appears to be identical, within the limits of error of both determinations, with the potential well which represents the proton-neutron interaction as derived from the scattering and absorption of slow neutrons. Furthermore, the magnitude of interactions thus determined by scattering experiments is in very satisfactory agreement with that used successfully for calculations of mass defects of light nuclei.* It thus appears that a real beginning has been made toward an accurate and intimate knowledge of the forces which bind together the "primary particles" into the heavier nuclei so important in the structure and energetics of the material universe.

* A very readable discussion in this connection is given by Bethe in Rev. Mod. Phys. **8**, 82 (1936).

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Theory of Scattering of Protons by Protons*

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The experiments of Tuve, Heydenburg and Hafstad and those of White are discussed by means of the standard theory of scattering in central fields. The theoretical formulas are presented in a form convenient for numerical computation and are supplemented by tables. These are arranged so as to enable an experimentalist to compute the effect of phase shifts due to angular momenta $L=0$, $\hbar$, $2\hbar$, and to infer these phase shifts from the experimental material (Tables I, II, III, IV, V, VI, VII, VIII, IX). Tables of necessary Coulomb wave functions are also given for zero angular momentum. By means of these the interaction energy can be computed from the experimental material (Tables X, XI, XII, XIII).

Statistical fluctuations make conclusions drawn from White's data somewhat uncertain. The experiments of Tuve, Heydenburg and Hafstad are comparatively free of statistical effects and their comparison with theory shows that (a) There is an unmistakable difference between the observed scattering and that to be expected according to Mott's formula which uses the inverse square law. (b) This difference can be explained by using practically entirely effects of the phase shift in the partial wave having $L=0$ (head on collisions; s wave distortions). The distortion of p and d waves ($L=\hbar$, $2\hbar$) is secondary and the experimental accuracy does not yet suffice to enable their quantitative determination. (c) The variation of the scattering anomaly with proton energy is in approximate agreement with that to be expected from an interaction potential independent of the energy. (d) For a given range of nuclear forces the interaction potential is accurately determined by the data. The values obtained are in good agreement with those found by Feenberg and Knipp and by Bethe from the mass defects of H^2, H^3, He^4 provided the mass defect calculations are made on the basis of a proton-neutron interaction which depends on the relative orientation of the spins of proton and neutron in accordance with Wigner's explanation of the large scattering of slow neutrons by hydrogen. Mass defect calculations based on a proton-neutron interaction indicated by the binding energy of H^2 without dependence on the spin orientation give a much lower value for the interaction between like particles than that obtained from the proton-proton scattering experiments. The "like-particle" interaction for a Gauss error potential is $39mc^2e^{-17r^2}$ with 8.97×10^{-13} cm as the unit of length and the interaction energy is 11.1 mev for a potential which is constant (except for its Coulombian part) within a distance

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$e^2/mc^2=2.82\times10^{-13}$ cm. (e) The interaction between protons as derived from the scattering experiments is found to be very nearly equal to that between a proton and a neutron in the corresponding condition of relative spin orientation and angular momentum (^1S state). The proton-neutron values which come closest to being equal to the proton-proton values are those obtained by Fermi and Amaldi from the scattering and absorption of slow neutrons.

The close agreement between the empirical values of the proton-proton and proton-neutron interactions in ^1S states suggests that aside from Coulombian and spin effects the interactions between heavy particles are independent of their charge and that the apparent preference for equal numbers of protons and neutrons in the building up of nuclei is conditioned more by the operation of the exclusion principle than by the greater values of proton-neutron forces.

1. Introduction

OBSERVATIONS on the anomalous scattering of protons by protons have been made by Wells,[1] White[2] and Tuve, Heydenburg and Hafstad.[3] In the experiments of Wells a cloud chamber was used. On account of the small number of observed collisions the accuracy of his experiments was insufficient to make theoretical conclusions possible. White's experiments were made by the same method. A greater number of tracks was observed and White concluded that there is a large discrepancy between the actual scattering and that to be expected on the assumption of the inverse square law of force between protons. Such a discrepancy will be referred to below as a *scattering anomaly*. The theoretical interpretation of White's results has been considered in more detail by Present.[4] It was found impossible to account for White's angular distribution by any ordinary theory with a potential describing the interaction between protons. Nevertheless the order of magnitude of the scattering anomaly turned out to be in approximate agreement with the attractive potential which Feenberg and Knipp[5] found from the binding energies of H_1^2, He_2^3, He_2^4 by using a mixed Heisenberg-Majorana operator for the interaction between protons and neutrons. This operator gives an interaction between protons and neutrons in singlet states that is somewhat weaker than in triplet states, as has been suggested by Wigner from evidence on the scattering of slow neutrons by hydrogen.[6] The difficulties with the angular distribution and energy dependence in White's experiments make, however, conclusions about the interaction potential obtained from his data very uncertain.

The work of Tuve, Heydenburg, and Hafstad was done with electrical counters. Large numbers of scattered particles were observed and their data are comparatively free of statistical fluctuations due to an insufficiency of such particles. The scattering anomaly is smaller in their experiments than in those of White and it starts at a higher energy. It will be seen below that their final results are in reasonably good agreement with a simple form of scattering theory with respect to both the angular distribution and the energy dependence. There is no consistent evidence for the presence in the scattering of anything but an s wave (head-on collisions, $L=0$) which is in agreement with expectation for interaction forces confined to distances smaller than 10^{-12} cm. Although there were indications in the early experiments of THH of a too rapid variation of scattering anomaly with energy, later and more accurate data are in approximate agreement with theory also in this respect. The observations available at present are not yet precise enough to determine accurately the range of the proton-proton forces. Nevertheless, they appear to be good enough to eliminate strong long range forces. Thus interaction energies of constant magnitude through a distance of $3e^2/mc^2=8.5\times10^{-13}$ cm are in disagreement with the energy dependence of the

[1] W. H. Wells, Phys. Rev. **47**, 591 (1935).

[2] M. G. White, Phys. Rev. **47**, 573 (1935); **49**, 309 (1935). We are very grateful to Dr. White for communicating to us his complete data.

[3] M. A. Tuve, N. P. Heydenburg and L. R. Hafstad, Phys. Rev. **49**, 402 (1936); **50**, 806 (1936) (Preceding paper). We are very grateful to Messrs. Tuve, Heydenburg and Hafstad for making their results available to us before publication and for their wholehearted cooperation in answering by experiment the questions which came up in the interpretation of their earlier results.

[4] R. D. Present, Phys. Rev. **48**, 919 (1935).

[5] E. Feenberg and J. K. Knipp, Phys. Rev. **48**, 906 (1935).

[6] Dunning, Pegram, Fink and Mitchell, Phys. Rev. **47**, 970 (1935). Bjerge and Westcott, Proc. Roy. Soc. **A150**, 790 (1935). Fermi and Amaldi, La Ricerca Scientifica **1**, 1 (1936); Fermi, ibid., July, 1936. We are very grateful to Professor Fermi for informing us of his last results before publication. The value of 130 kv used for the position of the virtual level by us is too high. The effect of changing it to 110 kv is scarcely noticeable in Table XIV.

scattering anomaly while a constant interaction potential through $e^2/mc^2=2.8\times10^{-13}$ cm is in fair accord with observation. The magnitude of the interaction potential corresponds to $D=10.3$ mev through e^2/mc^2 if one uses square wells and to $A=39mc^2$ for the potential $Ae^{-\alpha r^2}$ with $\alpha=17$ and $\hbar(Mm)^{-\frac{1}{2}}c^{-1}=8.97\times10^{-13}$ cm as the unit of length. This value is in good agreement with the result of Feenberg and Knipp who obtain $A=41mc^2$ for the same α. The sensitivity of the expected scattering anomaly to the magnitude of the interaction potential is great and the above values of A and D are determinable to about $0.1mc^2$ from the scattering data at any given energy aside from uncertainties due to the possible effect of higher phase shifts.

In their present form the scattering experiments of THH give information about the force between two protons ($\pi-\pi$ force) when they collide head on (S state) and when they have antiparallel spins (^1S state). According to Wigner a similar state is of importance for the scattering of slow neutrons by hydrogen and the magnitude of the proton-neutron attraction in ^1S states ($\pi-\nu$ force) can be determined from scattering experiments combined with measurements of the absorption of slow neutrons in hydrogen. The $\pi-\pi$ and $\pi-\nu$ attractions will be compared for the ^1S states and it will be seen that the more careful experiments indicate a practically exact equality of the $\pi-\pi$ and $\pi-\nu$ forces. Although this comparison has been made only in the ^1S state the agreement is so striking as to suggest that the interactions between heavy particles are *universally equal*, i.e., that the only essential difference in the interactions between like and unlike particles is due to the exclusion principle. The magnetic moment of the proton is different from that of the neutron and, therefore, the force between protons cannot be expected to be exactly equal to that between neutrons or to that between protons and neutrons. The energy due to the magnetic interaction between two protons is, however, of the order $9(e\hbar/2Mc)^2/(e^2/mc^2)^3\sim0.012mc^2$ which is very much smaller than the main part of the interaction energy ($\sim20mc^2$). As a tentative hypothesis we may consider the interactions between heavy particles to be universally equal except for the Coulombian effects between protons and the spin effects between all the heavy particles. Whether this interaction has actually one of the forms used by Feenberg and Knipp and by Bethe still remains to be seen. Since the form

$$[(1-g)P^M+gP^H]J(r),$$

which has been used successfully by them (P^M = Majorana exchange operator, P^H = Heisenberg exchange operator) gives a value of the "like particle" interaction agreeing with that arrived at from the scattering experiments on protons, one may regard this form as the most likely. Since in H^3, He^3 and He^4 like particles can be considered as nearly in ^1S states the above operator would give for them the same effect as for the ^1S state of unlike particles. We should like to acknowledge our indebtedness to Dr. Feenberg who noticed independently that the above operator could be applied both to like and to unlike particles and kindly communicated his considerations to us. It is possible that the universal operator contains as a part of it a small Wignerian term.

The observations of Bjerge and Westcott and of Dunning[6] on the scattering of slow neutrons by hydrogen indicate that there is either a virtual or a stationary ^1S level of the deuteron at ±43 kv. If the level is stationary no agreement whatsoever is obtained between the $\pi-\pi$ and $\pi-\nu$ interactions. Assuming it to be virtual, the $\pi-\nu$ interaction as derived from these experiments corresponds to $A=42mc^2$ and is thus somewhat larger than the proton-proton interaction. The difference of $3mc^2$ is still too large to consider the interactions as the same because this difference will be seen to have significant and quite observable consequences for the scattering amounting to approximately a factor of 3 for neutrons. More recently improved measurements on neutrons were made by Fermi and Amaldi.[6] Their observations on the absorption of slow neutrons lead them to the conclusion that the level is virtual. Secondly the scattering cross section in hydrogen has been found by them to be 12×10^{-24} cm^2 instead of the previous larger values and the position of the virtual level has been raised to about 130 kv. *With these changes the difference between the interactions of like and unlike particles is insignificant and* *may be considered as lying within the limits of error of the experiments.*

Recent experiments of Goldhaber[7] raised doubts concerning Wigner's explanation of the scattering of neutrons in hydrogen. According to Goldhaber the mean free path of 200 kv neutrons is about three times greater than that to be expected theoretically. If Goldhaber's result were correct all of the conclusions arrived at in this paper would have to be changed because the $\pi - \nu$ interaction would be modified and because Feenberg and Knipp's value of the like-particle interaction would be altered. In view of these radical implications of Goldhaber's experiment it was repeated by Tuve and his colleagues using carbon bombarded with deuterons as a source. The energy of the deuterons is between 600 kv and 1200 kv. Its precise value does not matter for the interpretation of the experiments. They find a mean free path in paraffin of 2.2 cm which is to be compared with a theoretical mean free path of 2.4 cm using 30 kv for the position of the virtual level and 600 kv for the neutron energy and 2.2 cm using 140 kv for the position of the virtual level and again 600 kv for the neutron energy. Their mean free path is smaller than Goldhaber's even though the energy of the neutrons is higher. Goldhaber's result thus implies a minimum in the scattering at about 200 kv of neutron energy and is improbable. We are very grateful to Dr. Tuve and his colleagues for permission to quote their experiments in this connection.

2. Phase Shift Analysis

The scattering of charged particles by an inverse square field of force has been considered by Gordon and Mott[8] while the effect of symmetry due to the identity of particles has been worked out by Mott.[9] The scattering anomaly due to deviations from an inverse square field has been discussed by Taylor[10] who applied the theory to the scattering of alpha-particles in helium and in hydrogen. The general form of the theory is well established in these papers, and its systematic presentation is given by Mott and Massey.[11] It will suffice here to suppose the methods as known and it will not be necessary to give the derivations of the formulas.

Notation

The following notation will be used:

M = mass of proton.
$\mu = M/2$ = reduced mass in the collision of two protons.
v = relative velocity of the two protons before the collision.
E = kinetic energy of incident protons $= \frac{1}{2}Mv^2$.
E' = energy in frame of center of gravity $= \frac{1}{2}E$
$\Lambda = h/\mu v$ = de Broglie wave-length.
$k = 2\pi/\Lambda$.
$a = \hbar^2/\mu e^2$.
$\eta = 1/ka = (e^2/\hbar c)ZZ'c/v$.
r = distance between proton and neutron or proton and proton.
$\rho = kr$, $y = \rho\eta = r/a$.
$L\hbar$ = angular momentum of proton and neutron or proton and proton around common center of gravity.
P_L = Legendre function.

[7] M. Goldhaber, Nature **137**, 824 (1936). Cf. reference to M. A. Tuve in text. M. A. Tuve and L. R. Hafstad, Phys. Rev. **50**, 490 (1936).

[8] W. Gordon, Zeits. f. Physik **48**, 180 (1928). N. F. Mott, Proc. Roy. Soc. **A118**, 542 (1928).

[9] N. F. Mott, Proc. Roy. Soc. **A126**, 259 (1930). Cf. J. R. Oppenheimer, Phys. Rev. **32**, 361 (1928).

In applications of the solutions of Gordon and Mott the meaning of the physical condition represented by the solution is usually not clearly stated. This solution is given by Eqs. (1) (1′) in the text. It represents the wave inside a very large screening sphere of radius R. For $r>R$ there is supposed to be no field of force while for $r<R$ the field is given by the Coulombian potential. For elastic collisions between a proton and a hydrogen atom or between two hydrogen atoms one is interested in the state of the system before and after the atoms have collided, i.e. in the state in a force free region. Thus rigorously it is the solution "outside" the screening sphere ($r>R$) that matters. Such a solution is complicated and for ordinary applications it is not considered explicitly for the following reasons. According to the asymptotic expansion given by Eq. (3) in the text the solution inside the screening sphere is the sum of two parts represented by the two exponentials. The second of these is of special interest because it represents the scattered wave. Over a small area of the screening sphere the "spherical wave" represented by this term may be approximated by a plane wave which may be regarded as subject to reflection and refraction at $r=R$. Actually the screening is taking place through a region large compared with the wave-length and therefore the reflection may be neglected. There will also be no refraction as long as the wave may be considered as plane. For very small scattering angles, however, the wave cannot be considered as plane because the second term in the curly brackets in the spherical wave part of Eq. (3) is not negligible in comparison with unity. Thus at small angles one may expect the solutions used here to give incorrect results, in agreement with the Born method calculation of Coulomb scattering due to Wentzel. In order that the second term in curly brackets should become ~ 1 for 1 Mev protons at 0.53×10^{-8} cm it is necessary to have $2\Theta=\theta\sim1°$. The magnitude of the term decreases with Θ^{-2} at small angles and therefore no serious effect due to this cause is expected in practical applications.

[10] H. M. Taylor, Proc. Roy. Soc. **A134**, 103 (1931); **A136**, 605 (1932).

[11] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press).

θ = scattering angle in the reference system of center of gravity.
$\Theta = \theta/2$ = scattering angle in the reference system of the laboratory.
$\mathbf{c} = \cos \Theta$. $\mathbf{s} = \sin \Theta$.
Γ = the gamma function.
$\sigma_L = \arg \Gamma(L+1+i\eta)$.
F_L = regular solution of the differential equation for r times the radial wave function in a Coulomb field normalized so as to be asymptotic to a sine wave of unit amplitude at ∞. The asymptotic form is given by $F_L \sim \sin (\rho - \frac{1}{2}L\pi - \eta \ln 2\rho + \sigma_L)$ and the diff. eq. is $[d^2/d\rho^2 + 1 - 2\eta/\rho - L(L+1)/\rho^2]F_L = 0$.
G_L = irregular solution of the same differential equation normalized so that its asymptotic form is given by $G_L \sim \cos (\rho - \frac{1}{2}L\pi - \eta \ln 2\rho + \sigma_L)$.
K_L = phase shift defined by the asymptotic form of $\mathfrak{F}_L = r$ times the radial wave function in the actual field. This form shall be $\mathfrak{F}_L \sim e^{iK_L} \sin (\rho - \frac{1}{2}L\pi - \eta \ln 2\rho + \sigma_L + K_L)$.

The above notation for the Coulomb wave functions is the same as that used by Yost, Wheeler Breit[12] in tabulations of these wave functions.

The plane wave e^{ikz} is changed by the Coulomb field into

$$\psi^c = \sum_0^\infty i^L (2L+1) P_L(\cos \theta) e^{i\sigma_L} F_L/\rho. \tag{1}$$

An alternative form is

$$\psi^c = e^{-\frac{1}{2}\pi\eta + ikz}\Gamma(1+i\eta)\mathbf{F}-i\eta;\,1;\,ik(r-z) \tag{1'}$$

where $\mathbf{F}$ is the confluent hypergeometric series. If the field is Coulombian at large distances but not at small ones then the same plane wave is changed into

$$\psi = \sum_0^\infty i^L (2L+1) P_L e^{i\sigma_L} \mathfrak{F}_L/\rho. \tag{2}$$

At large distances the asymptotic form of ψ^c is

$$\psi^c \sim \left\{1 + \frac{\eta^2}{ik(r-z)} + \cdots\right\} \exp\{i[kz + \eta \ln k(r-z)]\} - \frac{\eta}{k(r-z)}\left\{1 + \frac{(1+i\eta)^2}{ik(r-z)} + \cdots\right\} \times \exp\{i[kr - \eta \ln k(r-z) + 2\sigma_0]\}. \tag{3}$$

The collision cross section per unit solid angle of the laboratory reference system is, on the classical theory, for Coulombian fields:

$$\boldsymbol{\sigma}_{\mathrm{Cl}} = 4\mathbf{c}(\mathbf{s}^{-4} + \mathbf{c}^{-4})(e^2/2\mu v^2)^2. \tag{4}$$

The effect of taking into account the symmetry of the wave functions is according to Mott such as to change this into

$$\boldsymbol{\sigma}_{\mathrm{Mott}} = 4\mathbf{c}[\mathbf{s}^{-4} + \mathbf{c}^{-4} - \mathbf{s}^{-2}\mathbf{c}^{-2} \cos (\eta \ln \mathbf{s}^2\mathbf{c}^{-2})](e^2/2\mu v^2)^2. \tag{4'}$$

If the field deviates from the Coulombian the collision cross section per unit solid angle is

$$\boldsymbol{\sigma} = 4\mathbf{c}\mathbf{P}, \tag{4''}$$

where

$$\mathbf{P} = \tfrac{1}{4}|f(\theta) + f(\pi-\theta)|^2 + \tfrac{3}{4}|f(\theta) - f(\pi-\theta)|^2$$

in the notation of Mott and Massey.[11] The quantity $\mathbf{P}$ and hence $\boldsymbol{\sigma}$ can be computed using

[12] F. L. Yost, John A. Wheeler and G. Breit, Phys. Rev. **49**, 174 (1936); Journal of Terrestrial Magnetism and Atmospheric Electricity, December, 1935, p. 443. See also T. Sexl, Zeits. f. Physik **56**, 62 (1929) for discussion of $L=0$.

$$\mathbf{P}=(e^2/2\mu v^2)^2\Big\{\mathbf{s}^{-4}+\mathbf{c}^{-4}-\mathbf{s}^{-2}\mathbf{c}^{-2}\cos(\eta\ln\mathbf{s}^2\mathbf{c}^{-2})-\frac{2}{\eta}\sum_L g_L(2L+1)[\mathbf{s}^{-2}\cos\varphi_L{}^s+(-)^L\mathbf{c}^{-2}\cos\varphi_L{}^c]$$
$$+\frac{4}{\eta^2}\sum_L g_L(2L+1)^2P_L{}^2\sin^2 K_L+\frac{8}{\eta^2}[\sum_e+3\sum_o]_{L'>L}(2L+1)(2L'+1)P_LP_{L'}$$
$$\times\sin K_L\sin K_{L'}\cos(\varphi_L-\varphi_{L'})\Big\},\quad (5)$$

where g_L has the value 1 for even L and 3 for odd L. The last term in braces contains two sums, the first referring to even L, as indicated by suffix e, and the second referring to odd L as indicated by suffix o. Both of these sums are taken over pairs of unequal values of L. No cross products between even and odd L occur. As before, the argument of the Legendre functions P_L is $\cos\theta$. The other quantities needed in this formula are:

$$\varphi_L=K_L+2(\sigma_L-\sigma_0);\quad \varphi_L{}^s=\varphi_L+\eta\ln\mathbf{s}^2;\quad \varphi_L{}^c=\varphi_L+\eta\ln\mathbf{c}^2,\quad (5.1)$$

$$\sigma_1-\sigma_0=\tan^{-1}\eta;\quad \sigma_2-\sigma_1=\tan^{-1}(\eta/2);\quad \sigma_L-\sigma_{L-1}=\tan^{-1}(\eta/L).\quad (5.2)$$

If all K_L beyond K_2 vanish

$$\mathbf{P}=\mathbf{P}_M+(\Delta\mathbf{P})_0+(\Delta\mathbf{P})_1+(\Delta\mathbf{P})_2,\quad (6)$$

with

$$(2\mu v^2/e^2)^2\mathbf{P}_M=\mathbf{s}^{-4}+\mathbf{c}^{-4}-\mathbf{s}^{-2}\mathbf{c}^{-2}\cos(\eta\ln\mathbf{s}^2\mathbf{c}^{-2}),\quad (6.1)$$

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_0=-\frac{2}{\eta}(\mathbf{s}^{-2}\cos\varphi_0{}^s+\mathbf{c}^{-2}\cos\varphi_0{}^c)\sin K_0+\frac{4}{\eta^2}\sin^2 K_0,\quad (6.2)$$

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_1=-\frac{18}{\eta}(\mathbf{s}^{-2}\cos\varphi_1{}^s-\mathbf{c}^{-2}\cos\varphi_1{}^c)P_1\sin K_1+\frac{108}{\eta^2}P_1{}^2\sin^2 K_1,\quad (6.3)$$

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_2=-\frac{10}{\eta}(\mathbf{s}^{-2}\cos\varphi_2{}^s+\mathbf{c}^{-2}\cos\varphi_2{}^c)P_2\sin K_2+\frac{100}{\eta^2}P_2{}^2\sin^2 K_2$$
$$+\frac{40}{\eta^2}\sin K_0\sin K_2\cos(\varphi_2-\varphi_0)P_2.\quad (6.4)$$

By means of (4″) and (6.1) one obtains Mott's value as is seen from (4′). The additions to $\mathbf{P}$ due to K_0, K_1, K_2 are given by (6.2), (6.3), (6.4), respectively. It will be noted that the effect of K_2 depends on K_0. The above formulas are convienent for computation if one is not interested in many values of the phase shifts. For such cases it is more convenient to expand (6.2), (6.3), (6.4) as follows

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_0=-\frac{2}{\eta}\left(\frac{\cos\alpha_0}{\mathbf{s}^2}+\frac{\cos\beta_0}{\mathbf{c}^2}\right)\sin K_0\cos K_0+\left(\frac{4}{\eta^2}+\frac{2}{\eta}\frac{\sin\alpha_0}{\mathbf{s}^2}+\frac{2}{\eta}\frac{\sin\beta_0}{\mathbf{c}^2}\right)\sin^2 K_0,$$
$$\alpha_0=\eta\ln\mathbf{s}^2,\quad \beta_0=\eta\ln\mathbf{c}^2,\quad (6.5)$$

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_1=-\frac{18}{\eta}P_1\left(\frac{\cos\alpha_1}{\mathbf{s}^2}-\frac{\cos\beta_1}{\mathbf{c}^2}\right)\sin K_1\cos K_1+\left[\frac{108}{\eta^2}P_1{}^2+\frac{18}{\eta}\left(\frac{\sin\alpha_1}{\mathbf{s}^2}-\frac{\sin\beta_1}{\mathbf{c}^2}\right)P_1\right]\sin^2 K_1,$$
$$\alpha_1=\alpha_0+2(\sigma_1-\sigma_0),\quad \beta_1=\beta_0+2(\sigma_1-\sigma_0),\quad (6.6)$$

$$(2\mu v^2/e^2)^2(\Delta\mathbf{P})_2=-\frac{10}{\eta}P_2\left(\frac{\cos\alpha_2}{\mathbf{s}^2}+\frac{\cos\beta_2}{\mathbf{c}^2}\right)\sin K_2\cos K_2+\left[\frac{100}{\eta^2}P_2{}^2+\frac{10}{\eta}P_2\left(\frac{\sin\alpha_2}{\mathbf{s}^2}+\frac{\sin\beta_2}{\mathbf{c}^2}\right)\right]\sin^2 K_2$$
$$+\frac{40}{\eta^2}\sin K_0\sin K_2\cos[K_2-K_0+2\sigma_2-2\sigma_0],\quad \alpha_2=\alpha_0+2(\sigma_2-\sigma_0),\quad \beta_2=\beta_0+2(\sigma_2-\sigma_0).\quad (6.7)$$

TABLE I. *Values of* $(2\mu v^2/e^2)^2 \mathbf{P}_M$.

η	E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
0.4775	108.8	219.1	68.84	27.74	13.15	7.19	4.694	4
.3821	170.0	215.4	67.42	27.14	12.90	7.10	4.673	4
.2867	302.4	212.3	66.25	26.65	12.70	7.02	4.657	4
.2069	580	210.3	65.52	26.35	12.58	6.98	4.648	4
.1751	810	209.7	65.27	26.25	12.53	6.97	4.644	4
.1591	981	209.4	65.18	26.21	12.52	6.96	4.644	4
.1017	2400	208.6	64.88	26.10	12.47	6.94	4.638	4

TABLE II. *Values of coefficients of* $-\sin K_0 \cos K_0$ *for* $\mathbf{P}/\mathbf{P}_M$.

η	E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
0.4775	108.8	0.0994	0.339	0.758	1.426	2.380	3.441	3.96
.3821	170.0	.212	.541	1.088	1.938	3.128	4.426	5.05
.2866	302.4	.385	.854	1.609	2.755	4.339	6.032	6.84
.1989	628	.665	1.376	2.484	4.15	6.42	8.84	9.95
.1830	741	.741	1.511	2.720	4.53	7.02	9.64	10.84
.1671	890	.830	1.679	3.004	4.99	7.69	10.53	11.88
.1512	1088	.936	1.873	3.350	5.54	8.54	11.69	13.16
.1432	1211	.995	1.990	3.54	5.86	9.06	12.34	13.88
.1332	1400	1.081	2.155	3.83	6.33	9.71	13.25	14.95
.1174	1800	1.244	2.468	4.37	7.21	11.04	15.06	16.98
.1017	2400	1.455	2.872	5.07	8.35	12.78	17.43	19.61

TABLE III. *Values of coefficients of* $\sin^2 K_0$ *for* $\mathbf{P}/\mathbf{P}_M$.

E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
108.8	−0.213	−0.194	−0.004(3)	0.493	1.380	2.469	3.025
170.0	−.185	−.0828	.332	1.244	2.769	4.579	5.48
302.4	−.117	.215	1.114	2.971	5.819	9.124	10.79
628	.114	1.000	3.11	7.12	13.37	20.5	23.9
741	.198	1.276	3.80	8.58	15.99	24.4	28.4
890	.309	1.643	4.71	10.49	19.42	29.6	34.4
1088	.460	2.130	5.94	13.07	24.02	36.4	42.4
1211	.565	2.438	6.71	14.68	26.96	40.6	47.4
1400	.700	2.907	7.87	17.11	31.3	47.3	55.0
1800	1.009	3.90	10.36	22.31	40.6	61.2	71.2
2400	1.471	5.40	14.07	30.07	54.6	82.1	95.3

TABLE IV. *Values of coefficients of* $-\sin K_1 \cos K_1$ *for* $\mathbf{P}/\mathbf{P}_M$.

η	E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
0.4775	108.8	1.946	3.23	4.14	4.19	3.06	1.100	0
.2866	302.4	3.50	5.48	6.85	6.84	4.92	1.739	0
.1989	628	5.14	7.92	9.85	9.75	6.99	2.471	0
.1671	890	6.15	9.44	11.69	11.60	8.32	2.91	0
.1432	1211	7.19	10.98	13.60	13.52	9.70	3.39	0
.1174	1800	8.80	13.45	16.61	16.48	11.78	4.13	0
.1017	2400	10.17	15.54	19.18	19.02	13.59	4.77	0

TABLE V. *Values of coefficients of* $\sin^2 K_1$ *for* $\mathbf{P}/\mathbf{P}_M$.

E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
108.8	0.635	3.21	6.63	9.00	7.51	2.927	0
302.4	3.68	10.88	20.03	25.94	22.15	8.63	0
628	8.78	23.94	42.6	54.4	46.1	17.89	0
890	12.91	34.1	60.5	77.3	65.1	25.22	0
1211	17.93	46.8	82.7	105.2	88.9	34.3	0
1800	27.21	70.1	123.7	157.1	132.2	51.1	0
2400	36.6	93.7	165.1	209.5	176.4	68.0	0

TABLE VI. *Values of coefficients of* $-\sin K_2 \cos K_2$ *for* $\mathbf{P}/\mathbf{P}_M$.

E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
628	2.36	2.74	1.48	−2.48	−9.68	−18.36	−22.62
890	2.82	3.29	1.79	−3.02	−11.81	−22.48	−27.68
1211	3.32	3.88	2.10	−3.58	−14.10	−26.80	−33.02

TABLE VII. *Values of coefficients of* $\sin^2 K_2$ *for* $\mathbf{P}/\mathbf{P}_M$.

E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
628	4.90	6.17	1.840	2.19	34.0	104.0	147.0
890	6.89	8.54	2.426	3.50	49.9	150.8	214.3
1211	9.34	11.40	3.14	5.10	69.5	208.6	293.4

TABLE VIII. *Values of* $(40/\eta^2)P_2/[s^{-4}+c^{-4}-s^{-2}c^{-2}\cos(\eta \ln s^2c^{-2})]$.

E in kv	$\Theta=15°$	20°	25°	30°	35°	40°	45°
628	3.00	5.88	4.60	−10.05	−47.0	−99.1	−126.3
890	4.27	8.35	6.54	−14.29	−66.7	−140.4	−179.0
1211	5.82	11.39	8.93	−19.48	−91.0	−191.2	−243.7

TABLE IX. *Values of* $2\sigma_2-2\sigma_0$.

E in kv =	108.8	170.0	302.4	580	1088	1800	2400
$2\sigma_2-2\sigma_0$ =	77.9°	63.5°	48.3°	35.4°	25.8°	20.1°	17.4°

In these equations, the coefficients of $\sin K_L \times \cos K_L$, $\sin^2 K_L$ are functions only of the energy and of the scattering angle. In formula (6.7) the cross product term in K_0, K_2 is conveniently computed directly. Values of the coefficients and other quantities for the computation of $\mathbf{P}$ are given in Tables I, II, $\cdots$ IX. In Table I are given values of $(2\mu v^2/e^2)^2\mathbf{P}_M$. In the first column are listed values of η and the second column gives the approximate value of the energy of the incident proton for this η. The succeeding columns give the values of $(2\mu v^2/e^2)^2\mathbf{P}_M$ for $\Theta=15°$, 20°, etc. as indicated at the top of each column. The values of $(2\mu v^2/e^2)^2\mathbf{P}_M$ are seen to vary slowly with E and interpolation can be easily made in Table I. In Tables II, III are given values of the coefficients of $-\sin K_0 \cos K_0$ and $\sin^2 K_0$ for the calculation of $(\Delta\mathbf{P})_0/\mathbf{P}_M$.

The values of η listed in Table II were used in the computations of the coefficients for Tables II and III. The values of E are not as accurate as those of η because their calculation involves the somewhat uncertain values of the fundamental physical constants. The same applies to Table I and to Tables III, IV, V, VI. The numbers tabulated above, where they are plotted against E, form smooth curves. The graphs for the coefficient of $\sin^2 K_0$ are practically straight lines. By means of such graphs the coefficients can be easily obtained for the needed values of E. The same applies to the coefficients for the effect of K_1 tabulated in Tables IV, V and the coefficients for the effect of K_2 tabulated in Tables VI, VII. In the computation of the cross product term in K_0, K_2 contained in (6.7) it is also convenient to have values of

$$(40/\eta^2)P_2/[\mathbf{s}^{-4}+\mathbf{c}^{-4}-\mathbf{s}^{-2}\mathbf{c}^{-2}\cos(\eta \ln \mathbf{s}^{-2}\mathbf{c}^{-2})]$$

which gives the coefficient of the trigonometric functions for $\mathbf{P}/\mathbf{P}_M$. These are given in Table VIII. Values of $2(\sigma_2-\sigma_0)$ for different E which are needed for the last trigonometric function in (6.7) can be computed by means of Eq. (5.2). Some values are given in Table IX. It will be noted that the coefficients of $\sin^2 K_1$, $\sin^2 K_2$ are practically linear functions of E for most of the energies covered here.

When the values of the energy and of the scattering angle are fixed $\mathbf{P}/\mathbf{P}_M$ turns out to be a smooth and almost linear function of K_0 and it is thus possible to interpolate for most values of K_0 having made calculations for a few of them.

Since it is probable that these scattering experiments will be repeated and since it is advisable for the experimentalists to have a ready means of testing their data for agreement with a possible analysis in terms of the phase shifts K_0, K_1, K_2 we summarize the procedure for using the above tables. From the experiments one obtains the collision cross section per unit solid angle. This is $\boldsymbol{\sigma}$ of Eq. (4″); it corresponds to the total number of protons observed and it thus includes the effect of recoil protons. Then $\mathbf{P}$ is obtained by means of Eq. (4″). By means of Table I one obtains $\mathbf{P}_M$ and hence $\mathbf{P}/\mathbf{P}_M$. Fixing the voltage the curve of $\mathbf{P}/\mathbf{P}_M$ against energy has to be fitted by means of K_0, K_1, K_2. The effect of K_0 is obtained by the following procedure. The coefficients given in Tables II, III are plotted for each scattering angle as a function of E. Using the graphs the coefficients for the needed voltage E are found. Usually no great accuracy is required in these coefficients. The coefficients of Table II are then multiplied by $-\sin K_0 \cos K_0$, those of Table III are multiplied by $\sin^2 K_0$ and the results are added. The result is the contribution to $\mathbf{P}/\mathbf{P}_M$ due to K_0. When added to unity it gives the expected $\mathbf{P}/\mathbf{P}_M$ for this K_0 if the other K are zero. These contributions are plotted as functions of K_0 keeping Θ fixed. The graphs give then the values of K_0 which are needed to fit the actual $\mathbf{P}/\mathbf{P}_M$ for any scattering angle as well as values of $\mathbf{P}/\mathbf{P}_M$ that correspond to this K_0 for other Θ. The same procedure may be used for the calculation of the effects of K_1 and K_2.

3. Calculation of Phase Shifts for Given Laws of Interaction

The nature of the forces between protons is not yet known. They may be partly describable by means of exchange potentials and partly by means of ordinary potentials. In spite of this apparent complication all kinds of interactions that have been seriously considered so far in nuclear theory give in effect a simple potential which is a function only of the distance r, at any fixed value of the relative angular momentum $L\hbar$. Thus exchange forces of the Majorana type give interaction potentials of the same absolute value but of opposite signs for states with odd and with even L. We will therefore suppose that there is some interaction potential for any L which may be different for different L. On account of the spins of the two protons they may be either in singlet or in triplet states. If the state is a singlet the orbital wave function is symmetric and therefore contains only terms with even L. If the state is a triplet the orbital wave function is antisymmetric and contains only odd L. Conversely the even L occur only in singlets and the odd only in triplets. For each L there is thus no necessity of considering interactions for singlets and triplets separately.

It will be supposed that for any L the potential is practically Coulombian beyond a certain distance r_0. The radial wave equation determines the function $\mathfrak{F}_L$ for $r<r_0$ to within a constant factor (see list of notation). In order to indicate that this solution involves only calculation inside r_0 it will be written as F_i. The suffix L will be omitted for the present so as not to complicate the formulas. The derivatives of F_i, F, G with respect to ρ will be written as F_i', F', G'. The relations between F'/F and K are

$$\tan K=(F'F_i-FF_i')/(GF_i'-F_iG') = (F^2\delta)/(1-FG\delta), \quad (7.1)$$

$$\delta=F'/F-F_i'/F_i, \quad (7.2)$$

$$(F_i'/F_i)=(F'+G'\tan K)/(F+G\tan K). \quad (7.3)$$

Here all quantities are supposed to be taken at $r=r_0$. The calculation of F, G, F', G' can be made by means of the formulas given by Yost, Wheeler, Breit[12] in terms of the series Φ, Φ^*, Ψ, Ψ^*. The necessary relations are

$$F_L=C_L\rho^{L+1}\Phi_L;\quad \rho F_L'/F_L=\Phi_L^*/\Phi_L;$$
$$G_L=D_L\rho^{-L}\Theta_L;$$
$$\Theta_L=\Psi_L+\rho^{2L+1}(p_L\ln 2\rho+q_L)\Phi_L. \quad (7.4)$$

Using the last form in (7.1) one does not need G'. The tabulations of Yost, Wheeler, Breit do not cover the range of values needed here. The necessary numbers are given in Tables X, XI, XII, XIII. From the phase shifts determined by means of the angular distribution $\rho F_i'/F_i$ can be determined by means of (7.1) or (7.3). This is necessary for example if one wishes to determine the magnitude of the interaction when its shape as a function of distance is known. For small radii and $L=0$ the use of (7.1) is not advisable because the numerical accuracy is then poor. In such a case it is better to use

$$\frac{\rho F_i'}{F_i}=y\frac{X+(2\ln 2y+f)\Phi_0^*}{\Psi_0+y(2\ln 2y+f)\Phi_0}, \quad (7.5)$$

where $y=\rho\eta=r/a$;

$$f=-2\ln\eta+q_0/\eta+(C_0^2/\eta)\cot K_0, \quad (7.6)$$

$$X=(\Psi_0^*+2y\Phi_0)/y=2-\left(4+\frac{1}{\eta^2}\right)y-4y^2+\left(\frac{1}{6\eta^4}+\frac{37}{27\eta^2}-\frac{32}{27}\right)y^3+\cdots,$$

$$\Phi_0=1+y+\left(\frac{1}{3}-\frac{1}{6\eta^2}\right)y^2+\left(\frac{1}{18}-\frac{1}{9\eta^2}\right)y^3+\cdots,$$

$$\Phi_0^*=1+2y+\left(1-\frac{1}{2\eta^2}\right)y^2+\left(\frac{2}{9}-\frac{4}{9\eta^2}\right)y^3+\cdots, \quad (7.7)$$

$$\Psi_0=1-\left(3+\frac{1}{2\eta^2}\right)y^2+\left(\frac{1}{9\eta^2}-\frac{14}{9}\right)y^3+\left(\frac{1}{24\eta^4}+\frac{43}{108\eta^2}-\frac{35}{108}\right)y^4+\cdots,$$

$$q_0=2\eta\left[\gamma-\frac{1}{1+\eta^2}+(s_3-1)\eta^2-(s_5-1)\eta^4+\cdots\right],$$

$s_3=1.2021$; $s_5=1.0369$; $s_7=1.00835$.

If r_0 is not very small it is more convenient to use

$$\frac{\rho F_i'}{F_i}=\frac{\Phi_0^*}{\Phi_0}-\frac{1}{\Phi_0\Theta_0+C_0^2\rho\Phi_0^2\cot K_0}. \quad (7.8)$$

For $r=0$ it is found from Eqs. (7.1) and (7.5) that

$$\tan K_0=\frac{C_0^2/\eta}{(\bar{C}_0^2/\bar{\eta})\cot\bar{K}_0+2\ln\eta/\bar{\eta}+\bar{q}/\bar{\eta}-q/\eta}, \quad (7.9)$$

where the barred quantities refer to that energy at which $K_0=\bar{K}_0$. Some useful values are given in Tables X, XI. The values of η in Tables X, XI, XII, XIII correspond to those in Tables I, $\cdots$ VIII wherever the listed E are the same. As before η is the actual quantity used in the calculations while E was computed from η using values of the fundamental constants. Similarly the values of y given in the headings of the tables are accurate while the values of r_0 are approximate. The values tabulated are those needed for Eq. (7.8) in order to calculate $\rho F_i'/F_i$ from K_0 as well as for the calculation of K_0 from $\rho F_i'/F_i$ by means of the second form of Eq. (7.1).

The calculation of K_0 is now reduced to finding $\rho F_i'/F_i$. If the interaction potential is constant within r_0 the expression for $L=0$ is

$$\rho F_i'/F_i=z\cot z;\quad z=[2\mu\hbar^{-2}(D+E')]^{\frac{1}{2}}r_0$$
$$=0.439(rmc^2/e^2)(D+E')^{\frac{1}{2}}\text{ mv}. \quad (8)$$

Here D is the negative of the potential energy in $0<r<r_0$. For attractive forces D is positive. Estimates show that K_1, K_2, are probably very small if the range of the nuclear forces is of the order of magnitude arrived at from nuclear mass defects. However, nuclear mass defect calculations are not sensitive to small interactions at large distances. The magnitude of the expected phase shifts can be estimated using

$$K_L\cong-\int(V/E')F_L^2d\rho. \quad (8.1)$$

Graphs of F_L for $L=0$, 1, 2 for energies $E=0.4$, 0.6, 0.8, 1.0 Mev are given in Figs. 1, 2. For 1 Mev these functions are compared with corresponding functions in the absence of a Coulomb field. The validity of the approximation implied in Eq. (8.1) was tested by an explicit numerical integration

TABLE X. *Coulomb functions for* $y=0.02445$; $r_0 \cong e^2/2mc^2$.

$2\pi\eta$	$E\cong$	Φ_0^*/Φ_0	$\Phi_0\Theta_0$	$C_0^2\rho\Phi_0^2$
0.95	1088	1.0156	0.9276	0.1008
1.00	981	1.0165	.9268	.0931
1.05	890	1.0172	.9258	.0862
1.10	810	1.0178	.9248	.0800
1.15	741	1.0184	.9237	.0743
1.20	681	1.0188	.9227	.0692
1.30	580	1.0197	.9206	.0602
1.50	436	1.0208	.9163	.0461

TABLE XI. *Coulomb functions for* $y=0.0489$; $r_0 \cong e^2/mc^2$.

$2\pi\eta$	Φ_0^*/Φ_0	$\Phi_0\Theta_0$	$C_0^2\rho\Phi_0^2$
0.95	1.014	0.8847	0.2062
1.00	1.017	.8861	.1910
1.05	1.020	.8869	.1772
1.10	1.022	.8873	.1646
1.15	1.025	.8872	.1533
1.20	1.027	.8866	.1427
1.30	1.030	.8851	.1258
1.50	1.034	.8804	.0959

for $L=1$ using the Gauss error potential $Ae^{-\alpha r^2}$ with values of A and α that correspond to those found by Feenberg and Knipp. The agreement is good and the use of Eq. (8.1) appears to be justified for such estimates. If resonance is approached on account of a sufficiently large V the equation becomes unreliable. Direct calculation for K_1 using accurate formulas with a square well having a radius $2e^2/mc^2$ and a depth of 2 Mev gives $K_1 \cong 0.3°$. Since this radius is too great one may expect the principal part of the interaction potential to give rise only to negligible higher phase shifts.

4. Discussion of Experiments

In Fig. 3 are given values of $\sigma \sin \Theta$ from White's experiments. Crosses mark the experimental points observed for energies of the incident protons ranging from 600–750 kv. The dotted curve was calculated for a pure Coulomb field from Mott's formula. The full curve was obtained by adjusting the phase shift K_0 for the distorted s wave so as to give agreement with experiment in the neighborhood of 45°. It is to be noticed that if the two points at 37.5° and 42.5° are rejected, there is surprisingly good agreement with exact Coulomb scattering. However, there is apparently no reason except for statistical fluctuations specially to doubt the reliability of these two points. If they are correct it becomes necessary to explain the strange deficiency at 30°. The effect of K_1 cannot reasonably explain this condition because it increases relatively to the effect of K_0 at smaller angles. Since this point is in contradiction with the observations of Tuve, Heydenburg, Hafstad it appears simplest to attribute it to statistical fluctuations in White's experiments. Conclusions about the magnitude of the interaction potential from his work appear to be somewhat unsafe in view of this erratic angular dependence. Nevertheless his values at 45° for the 600–750 kv range give an interaction potential which is in approximate agreement with that obtained from the experiments of THH. This happens essentially because the observed scattering anomaly is so large that its explanation calls for approximate *resonance*. In more detail the situation is as follows.

There are in general two values of K_0 that will account for the scattering at 45°. The smallest positive K_0 turns out to be the most probable. It can be explained in terms of an attractive potential agreeing closely with the calculation of Feenberg and Knipp. The other value would either require repulsive forces or a definitely stronger attraction than that obtained by Feenberg and Knipp with the mixed operator for neutron-proton interactions. Additional strong

TABLE XII. *Coulomb functions for* $y=0.0978$; $r_0 \cong 2e^2/mc^2$.

$2\pi\eta$	Φ_0^*/Φ_0	$\Phi_0\Theta_0$	$C_0^2\rho\Phi_0^2$
0.95	0.9525	0.7663	0.4084
1.00	.9674	.7802	.3823
1.05	.9799	.7918	.3580
1.10	.9904	.8008	.3353
1.15	.9998	.8085	.3143
1.20	1.0079	.8139	.2945
1.30	1.0213	.8218	.2593
1.50	1.0400	.8277	.2025

TABLE XIII. *Coulomb functions for* $y=0.1467$; $r_0 \cong 3e^2/mc^2$.

$2\pi\eta$	Φ_0^*/Φ_0	$\Phi_0\Theta_0$	$C_0^2\rho\Phi_0^2$
0.95	0.798	0.586	0.560
1.00	.837	.620	.535
1.05	.869	.649	.509
1.10	.896	.673	.484
1.15	.919	.694	.459
1.20	.939	.711	.435
1.30	.971	.738	.390
1.50	1.016	.769	.312

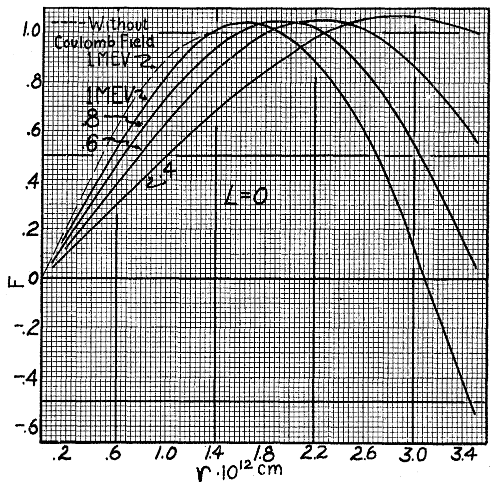

FIG. 1. Coulomb wave functions for $L=0$. Normalization is such as to give unit amplitude of sine wave at ∞.

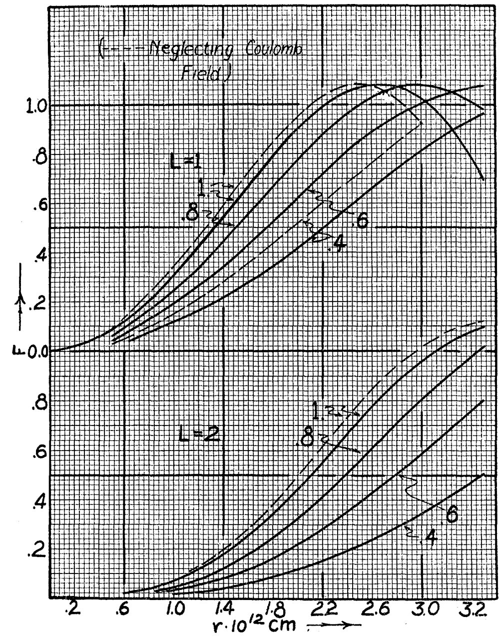

FIG. 2. Coulomb wave functions for $L=1, 2$. Normalization is such as to give unit amplitude of sine wave at ∞.

evidence against this K_0 will be seen to be contained in the experiments of THH. Combined evidence from all sources thus indicates only one of the two essentially different values of K_0 to be probable. In order to account for this value it becomes necessary to use approximate resonance of the s wave with the potential hole because the scattering anomaly is great. The expected scattering is then sensitive to the depth of the hole and nearly the same value of the depth is obtained for different values of scattering. By using the interaction energy $Ae^{-(r/a)^2}$ White's values are fitted with $A=45mc^2$ and $a=2.2\times10^{-13}$ cm. Between 450 and 600 kv his experiments show agreement with Coulomb scattering. The potential determined from the 600–750 kv range was used to calculate the scattering for energies between 450 and 600 kv. A value roughly six times Mott's was obtained. The disagreement between theory and experiment is here very definite and it is hard to account for it by any simple modification of the theory. Briefly White's angular distribution and voltage dependence of scattering do not allow of a simple theoretical explanation. The effects observed at a scattering angle of 45° for energies between 600 and 750 kv give nevertheless an interaction energy which is in approximate agreement with that found by THH, as will be seen presently.

The observations of THH will now be discussed and it will be seen that most of the scattering anomaly observed by them can be accounted for by the distortion of the s wave. At small scattering angles there are effects calling for p and perhaps d wave distortions. These effects are at present not decided enough to be regarded as definitely real. Nevertheless, they will be considered so as to give an idea of the reliability of the conclusions regarding the magnitude of the interaction potential. It will be seen that, aside from uncertainties having to do with the higher phases (p and d wave distortions), one can obtain very accurate values of the interaction energy from the experiments on the scattering of protons by hydrogen; the uncertainties due to higher phase shifts will be seen to be relatively small. The interaction energy derived from scattering experiments is in good agreement with that obtained from mass defect calculations.

The results of THH for $\mathbf{P}/\mathbf{P}_M$ at 900 kv are plotted in Fig. 4. The broken curve is drawn smoothly through their points. The full curves give the theoretical dependence of the scattering anomaly on the scattering angle when 30° and 31° are used for the phase shift K_0. Experiment and theory are seen to agree nicely at this energy.

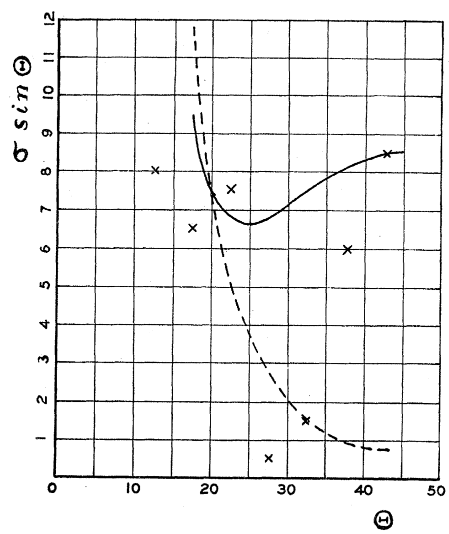

FIG. 3. White's data. Number of collisions of 600–750 kv protons within a 5° range plotted against angle. The experimental points are as communicated to us by Dr. White from his more complete data.

In Fig. 5 observations at 800 kv are compared with theory. It will be noted that here the agreement is less satisfactory if one uses only the s wave phase-shift K_0. At small angles there is relatively too much scattering. Thus if the experimental data are fitted in the neighborhood of 45° then one expects about 15 percent less scattering at 25° than is actually observed. It will be noted that this discrepancy is independent of the precise value which is used for $\mathbf{P}/\mathbf{P}_M$ at $\Theta = 45°$. Thus at $\Theta = 20°$ and 15° the curves for $K_0 = 26°$ and 28° give nearly the same $\mathbf{P}/\mathbf{P}_M$ while at $\Theta = 45°$ the values of $\mathbf{P}/\mathbf{P}_M$ which correspond to these K_0 differ by about 0.6 in a total of 3. The estimated error of the observations is shown on the same graph for some of the points. It is at the most 0.1 in the neighborhood of $\Theta = 45°$. There is also an uncertain error in the measurement of the scattering angle which is important for small Θ. It is not clear, however, why this error should matter at 800 kv and not at 900 kv. Curves for $K_0 = 30°$ and $K_0 = 35°$ are drawn in in order to show what happens when one attempts to fit the data at small Θ. There is then no indication of agreement between theory and experiment from $\Theta = 30°$ on to higher values. At the bottom of the graph is shown the contribution to $\mathbf{P}/\mathbf{P}_M$ which may be expected on account of a distortion of the p wave by a phase shift $K_1 = -1°$. This has a relatively insignificant effect for values of Θ between 35° and 45° and it raises the theoretically expected values by the necessary amount to agree with experiment from $\Theta = 15°$ to 30°. The data in their present form thus indicate $K_0 = 26°$ and $K_1 = -1°$ at 800 kv. Similarly in Fig. 6 comparisons between theory and experiment are made for 700 kv and 600 kv. At these energies and at high scattering angles the observations are supposedly more difficult on account of the increased importance of the stopping power of the window in the electrical counter. Thus at 600 kv and $\Theta = 40°$ the experimental point is known to be definitely too low and for this reason the experimental curve is drawn in by THH somewhat higher than the number of observed particles at this angle would indicate. It is difficult to be sure of the angular distribution curves sufficiently to make a definite phase angle analysis possible. The difference between the experimental and theoretical $\mathbf{P}/\mathbf{P}_M$ at 800 kv amounts to roughly 0.1 at $\Theta = 20°$ which when attributed to the distortion of the p wave gives roughly $-\frac{1}{2}°$ for K_1. There is thus an indication that the phase shift K_1 is present from 600 kv to 800 kv and that it is negligible at 900 kv. Such a variation of K_1 is contrary to all expectation for forces of such spatial extension as is usually assumed in theories of nuclear structure. Thus according to Fig. 2 and Eq. (8.1) the distortion of the p wave would have to be attributed to potentials extending to 3×10^{-12} cm. Otherwise Fig. 2 shows that F_1^2 will increase with E much too rapidly to make such a behavior of K_1 possible. In order to account for $K_1 = -1°$ at 1 mev one would need roughly an interaction energy of 10 kv extending through a distance of 10^{-12} cm. The Coulomb energy at 3×10^{-12} cm is about 50 kv. There appears to be at present no other evidence of such long range forces that is at all definite.

Quantitative conclusions about K_0 and K_1 are sensitive to possible effects of K_2. If K_1 is due to an interaction extending as far out as the above estimates would indicate then appreciable values of K_2 would also be expected. This is seen again from Fig. 2 by comparing the wave functions for $L = 2$ with those for $L = 1$. Fig. 7 shows qualitatively the effect of combining the effects of $K_0 = 35°$ and $K_2 = 2.2°$. This curve should be compared with that representing the effect of $K_0 = 29°$ and $K_2 = 0$ shown in the same figure. The difference in shape is seen to be relatively slight and it will be noted that it corresponds to the difference in shape between the experimental curves for 900 kv and 800 kv and the corresponding theoretical curves in Fig. 4 and Fig. 5 in that slightly higher values of $\mathbf{P}/\mathbf{P}_M$ are obtained in the region of 30–35°. The same applies to Fig. 6 for 700 kv while at 600 kv the angular distribution is well represented by K_0 alone. Less weight should be given to 600 kv because the number of observations is smaller and none were possible at 45°. The present data are seen to agree better with a combination of K_0, K_1, K_2 than with K_0 alone. To some extent this is doubtless due to the larger number of available parameters. The signs of K_1, K_2 suggested by the above discussion of the experiments are such as to correspond to attractive forces for $L=2$ and to repulsive forces for $L=1$. These are the signs which would be expected if the interaction were representable by a pure Majorana exchange operator or by the linear combination of Wigner and Majorana potentials which is expressible as a spin-spin interaction.[5, 13] Agreement in sign between the empirical and the expected phase shifts is an argument in favor of their reality. This argument is not very strong because the interaction potential for $L=0$ may change sign between the long range region of 3×10^{-12} cm and the short range region 3×10^{-13} cm. The value of K_0 derived from scattering at 45° does not depend on K_1 but is quite sensitive

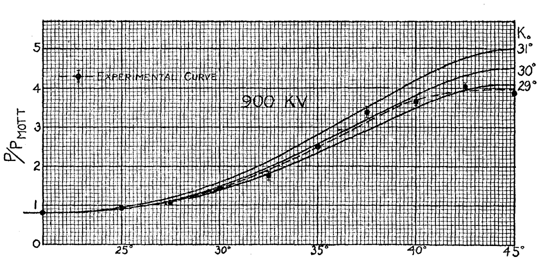

FIG. 4. Angular distribution of scattering anomaly at 900 kv according to THH. Statistical error as estimated in the observations is indicated where it exceeds size of dot. Points at small angles may be in error on account of difficult angle measurement. Some points in this and in Figs. 5 and 6 do not correspond to latest revision of data from which they differ by amounts insignificant for interpretation.

[13] J. H. Van Vleck, Phys. Rev. **48**, 367 (1935).

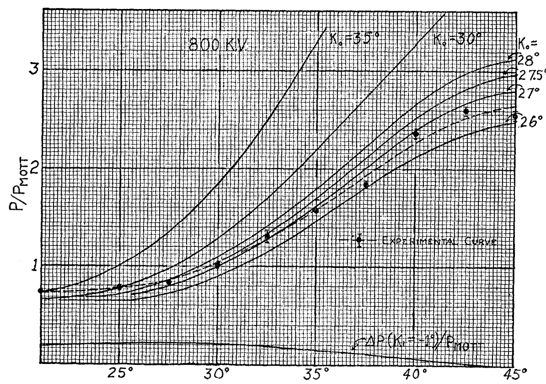

FIG. 5. Same as Fig. 4 but for 800 kv. Curve at bottom gives effect of K_1.

to the presence of small amounts of K_2. It is seen from Fig. 7 that inclusion of $K_2 = 2.2°$ in the theoretical analysis makes it necessary to change K_0 from 29° to about 36°. Such a change in K_0 will be seen to have serious consequences on the possible interpretation of the change of K_0 with energy as well as on the comparison of the proton-proton with proton-neutron interactions.

In the above discussion it was supposed that the main effect is due to K_0 which is a reasonable hypothesis on present views regarding the range of nuclear forces. As has been already noted in connection with White's experiments there are for any scattering angle essentially two values of K_0 which account for a given experimental value of $\mathbf{P}$. In the discussion of angular distributions due to K_0 alone the effect of an addition of π to K_0 cannot be noticed. To every K_0 in the first quadrant there corresponds another possible K_0 in the third and to every K_0 in the second quadrant there corresponds another possible K_0 in the fourth. For the present purpose one can consider values of K_0 differing by π as equivalent. They are also equivalent for the purpose of drawing conclusions about the interaction potential for $L=0$ using K_0, because only tan K_0 enters into the expression for $\rho F_i'/F_i$. Aside from this duplicity it is possible to fit the experimental values by means of K_0 lying either in the first or in the fourth quadrant. The possibilities in the fourth quadrant were not considered above. For such K_0 the values of $\mathbf{P}/\mathbf{P}_M$ remain consistently above unity while according to the experimental points presented in Figs. 4, 5, 6 the actual $\mathbf{P}/\mathbf{P}_M$ drop below unity for small scattering angles in

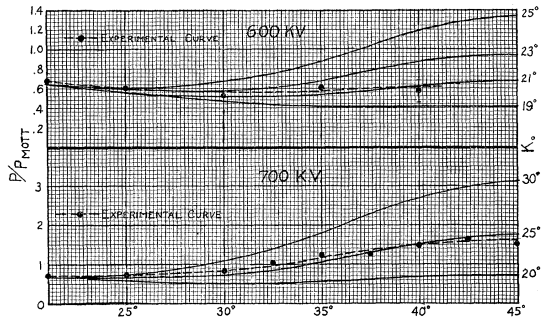

FIG. 6. Same as Fig. 4 for 600 kv and 700 kv.

all cases. At 600 kv $\mathbf{P}/\mathbf{P}_M$ remains in fact below 1 for all angles at which observations were made. In addition the dependence of $\mathbf{P}/\mathbf{P}_M$ on the scattering angle for all voltages experimentally examined is represented poorly by means of such K_0 even for those angles for which $\mathbf{P}/\mathbf{P}_M<1$. If values of K_0 in the fourth quadrant were to be seriously considered one would need to use large values of the phase shifts for higher L in order to bring about agreement with the observed angular distributions. The difference between the first and fourth quadrant for K_0 can be understood qualitatively as follows. The interaction potential which exists in addition to that representing the inverse square law may be imagined to be either increased or decreased by small amounts starting with zero. If it is increased one gets repulsive forces and values of K_0 in the fourth quadrant; if it is decreased the forces are attractive and K_0 is positive. In the first case the repulsive forces reinforce the Coulombian effect and a larger scattering is to be expected. For attractive forces the Coulombian effect is partly counteracted and a smaller scattering should be found. If, however, the attractive force is made sufficiently great then the Coulombian effect may be practically entirely overcome and the scattering will become nearly zero. As the attractive force is increased further the scattering becomes due primarily to the attraction and may exceed that which would exist if only the Coulombian force were acting. Qualitatively this corresponds to the condition of the theoretical curves shown for 900 kv in Fig. 4 for most of the scattering angles. At small scattering angles the effect of the inverse square field on the wave function becomes great and is sufficient to partly neutralize the effect of attraction so that $\mathbf{P}/\mathbf{P}_M$ again becomes <1. As the energy of the incident protons decreases the effect of the attraction becomes less pronounced since the proton penetrates into the attractive region with greater difficulty. The region of $\mathbf{P}/\mathbf{P}_M<1$ thus moves towards higher scattering angles. These qualitative features of the attractive potentials which correspond to the values of K_0 in the first quadrant are in good agreement with the data which appear to give very *direct evidence against repulsions and for attraction inside the nucleus*. Further evidence in favor of this view will be found in a quantitative discussion of the variation of the scattering anomaly with energy.

In Figs. 8, 9, 10, 11 calculations with "square wells" are compared with experiment. The interaction potential is here constant for $0<r<r_0$ and its value will be referred to as $-D$. For $r>r_0$ the potential is supposed to be Coulombian. The curves represent the theoretical dependence of K on the energy. In Fig. 8, r_0 was taken to be $e^2/2mc^2\cong 1.4\times 10^{-13}$ cm. The three curves correspond to $D=47.9$, 47.0, 46.3 mev. The ovals mark the values of K_0 derived from the data of Figs. 4, 5, 6 using scattering close to 45° and neglecting possible effects of K_2. The dotted curve gives the theoretical dependence of K_0 on the energy when $r_0=0$. This curve may be raised or lowered by approaching the limit of $r_0=0$ in different ways. Its shape does not vary greatly when this is done. The same curve is reproduced in Figs. 9, 10, 11 so as to give a standard of comparison. No relativistic corrections and no spin forces were taken into account in the calculation of the curves. The experimental

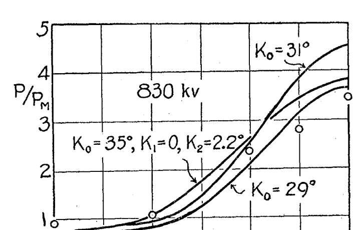

FIG. 7. Effect of K_2 on angular distribution. Circles represent experimental points as obtained by THH in their first set of experiments. These points should be lowered by 13 percent on account of a geometrical correction.

points are seen to be in good agreement with $D=47.0$ Mev. It will be noted from the figure that D is determined by this fit with an apparent accuracy of about 0.2 Mev. In order to illustrate this sensitivity the differences in the values of D were used accurately even though the absolute values are perhaps not quite accurate for each r_0. These depend on the correctness of numerical conversion factors such as that occurring in Eq. (8). (Comparisons with neutron-proton forces will be made without demanding great accuracy of the conversion factors and the absolute values of D.) The heavy straight line cutting obliquely across the three curves gives an average of the dependence of K_0 on E in an earlier set of data taken by THH. This is drawn in because it illustrates how hard it would be to fit these earlier data by means of a simple theory. It should be noted that with the plausible potentials used here such steepness cannot be attained and that, therefore, the rate at which K_0 varies with the energy can be used to rule out some kinds of interactions. Fortunately the newer data represented by ovals are free of this troublesome feature. In Figs. 9, 10, 11 similar comparisons of empirical and theoretical calculations for K_0 are made for $r_0=e^2/mc^2$, $2e^2/mc^2$, $3e^2/mc^2$, respectively. The theoretical dependence for $r_0=3e^2/mc^2$ is seen to fit experiment poorly.

The theoretical curves for $r_0=0$, $e^2/2mc^2$, e^2/mc^2, $2e^2/mc^2$ are seen to be in fair agreement with observation. The interval from $E=700$ kv to 900 kv appears to be too small to make it possible to determine r_0 to a higher accuracy. Inspection of the curves shows that *in a larger energy interval more precise information about r_0 should be obtainable.* It should be noted that the interaction energy $-D$ is determinable with great accuracy in all the cases considered as is obvious from the graphs. It would nevertheless be premature to claim at present an absolutely precise determination of the depth of the potential well because the question of the possible presence of the higher phase shifts has not been settled. Although it appears probable that K_2 is not the largest phase shift it is seen that a small positive K_2 of less than 2° would necessitate using a K_0 larger than what has been used by about 5°. The whole difference between the curves corresponding to $D=-47.9$ and 47.0

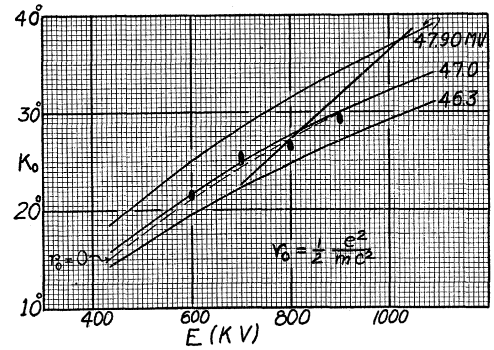

FIG. 8. Theoretical variation of K_0 with proton energy for an interaction energy constant within $r_0=e^2/2mc^2$ $=1.4\times10^{-13}$ cm. Coulombian potential is supposed to be inoperative for $r<r_0$. Ovals represent values of K_0 derived from the newer experiments of THH using data near $\Theta=45°$. Straight line gives average of the first set of data of THH. Dashed line represents theoretical behavior for $r_0=0$. Numbers like 47.0 refer to negative of interaction energy in mev.

Mev amounts to roughly 4° in K_0. The graphs of Figs. 8, 9, 10, 11 thus give an exaggerated impression of accuracy if K_2 is of importance, i.e., if the forces extend to large distances, even to a small degree.

In Fig. 12 are given theoretical and experimental values of $\mathbf{P}/\mathbf{P}_M$ as a function of E for scattering at 45°. The experimental values are indicated by ovals. The full lines refer to $r_0=0$ and $r_0=e^2/2mc^2$. All of the theoretical curves are labeled by means of two numbers such as 1, 10.03. The first of these gives the radius r_0 in units of e^2/mc^2. The second gives the depth in mev. For $r_0=0$ the graph is extended to low energies. It will be noted that around 400 kv the scattering at 45° should become very small as a consequence of dealing with an attractive potential. As the energy is decreased towards 100 kv the penetration through the Coulombian barrier becomes small and the attractive potential ceases to be effective. From 100 kv down one may expect the scattering to obey Mott's formula closely, in agreement with the experiments of Gerthsen.[14] However, even the relatively low energy of 200 kv is definitely of interest in drawing conclusions about forces between protons, since the scattering can be expected to be roughly $\frac{1}{2}$ of Mott's value in this region. The sensitivity of $\mathbf{P}/\mathbf{P}_M$ to the magnitude of the interaction energy is even more striking than the

[14] C. Gerthsen, Ann. d. Physik 9, 769 (1931).

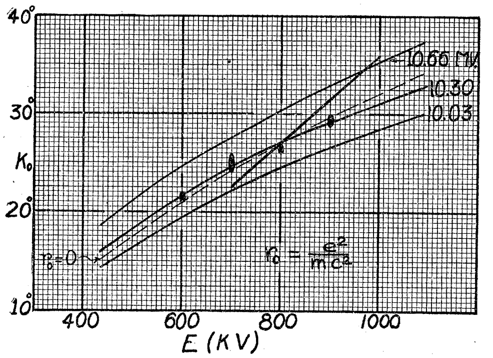

FIG. 9. Same as Fig. 8 but for $r_0=e^2/mc^2=2.8\times10^{-13}$ cm.

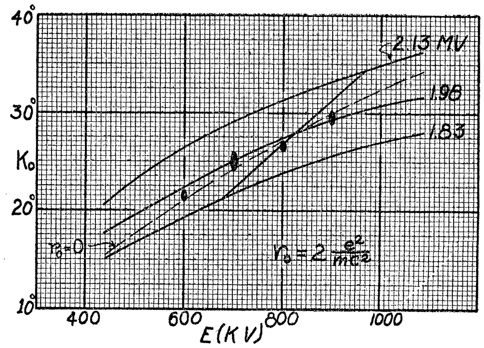

FIG. 10. Same as Fig. 8 but for $r_0=2e^2/mc^2$.

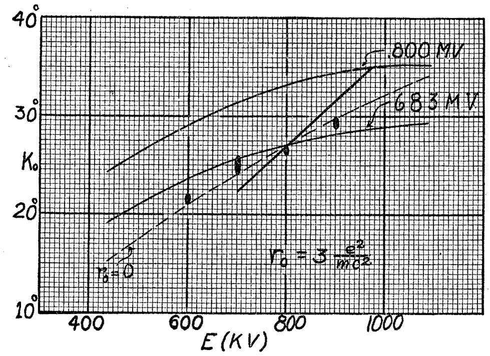

FIG. 11. Same as Fig. 8 but for $r_0=3e^2/mc^2$. Note insufficient slope of theoretical curve.

sensitivity of K_0. Thus the difference between $r_0=e^2/mc^2$, $D=10.30$ Mev and $r_0=e^2/mc^2$, $D=10.03$ Mev is quite unmistakable, and the latter value is seen to be definitely excluded by comparison with experiment. Comparison of observation at 800 kv with that at 900 kv favors values of $r_0<e^2/mc^2$ while comparison of 700 kv with 800 kv is in better agreement with somewhat larger ranges. This comparison is of course theoretically equivalent to that made in Figs. 8, 9, 10, 11, but it puts relatively more emphasis on the 900 kv point on account of the sensitivity of $\mathbf{P}/\mathbf{P}_M$ to K_0 in this region. As before, the apparent precision in the determination of the depth of interaction may be deceptive on account of the possible presence of higher phase shifts.

Dr. J. A. Wheeler in his work on the scattering of alpha-particles has developed a criterion which makes it possible to eliminate certain kinds of potentials. This criterion will now be used to give an additional argument against the second possibility for K_0 which is due to the fact that it is related to $\mathbf{P}$ by an equation having two roots. The point of Wheeler's criterion is that the quantity $\rho F_i'/F_i$ must decrease with energy as a consequence of Green's theorem. According to Figs. 8, 9, 10, 11, 12, the K_0 which was used varies approximately as would be expected. For the other possible K_0 the following numbers are obtained at $r_0=2e^2/mc^2$ using Eq. (7.8). At 700 kv, $K_0=-4.1°$, $\rho F_i'/F_i=1.30$; at 800 kv, $K_0=-7.3°$, $\rho F_i'/F_i=1.54$; at 900 kv, $K_0=-11.3°$, $\rho F_i'/F_i=1.97$. These numbers show that $\rho F_i'/F_i$ would have to increase with energy if these K_0's were true. In the light of this, combined with the angular dependence as well as the presence in the experimental data of regions in which $\mathbf{P}/\mathbf{P}_M<1$ *one must consider the choice of* K_0 *made here as correct* provided higher phase shifts do not interfere with the analysis. If there were even an infinite repulsive interaction through e^2/mc^2 one would expect K_0 to be approximately $-11.3°$ and it is thus *impossible to use repulsive interactions* in s states for the explanation of the data both on account of the wrong energy dependence and on account of the difficulty of obtaining a sufficiently large scattering anomaly. In addition the angular dependence of the scattering anomaly would be wrong. It is satisfactory to have the data point so definitely to one rather than two possibilities of interpretation.

In Fig. 13 graphical comparisons are made between the observed scattering and the Gauss error potential $-Ae^{-\alpha r^2}$. This potential is supposed to be present in addition to the potential e^2/r which represents the Coulombian interaction. With mc^2 as the unit of energy and $\hbar(Mm)^{-\frac{1}{2}}c^{-1}=8.97\times10^{-13}$ cm as the unit of length the equation for $\mathfrak{F}$ is

$$\left[\frac{d^2}{dr^2}+E'-\frac{0.312}{r}+Ae^{-\alpha r^2}\right]\mathfrak{F}=0.$$

By introducing $x=\alpha^{\frac{1}{2}}r$ and letting α, A have the probable values 17, 40 this becomes

$$\left[\frac{d^2}{dx^2}+\frac{E'}{17}-\frac{0.075(7)}{x}+2.35(4)e^{-x^2}\right]\mathfrak{F}=0. \quad (9)$$

The quantity $-0.0757/x+2.354e^{-x^2}$ can be regarded as the negative of the potential energy in units of 17 mc^2. It is plotted against x in the figure. The quantity $-0.0757/x$ corresponds to the Coulombian potential and is also plotted for comparison in the same figure. The potential is seen to be very nearly Coulombian at $x=2.6$ which corresponds to $r_0 \cong 2e^2/mc^2$. There are three curves in the figure going through the origin which represent wave functions. Two of them are marked $E'=0$, $2E'=800$ kv. They are regular solutions of Eq. (9) corresponding to these values of E'. The third is marked f and is a regular solution of

$$d^2f/dx^2+2.66e^{-x^2}f=0. \quad (9.1)$$

This equation corresponds approximately to the condition of having $df/dx=0$ at large distances which makes the stationary level of the two particles in each other's field fall at $E'=0$. The function f will be used later in order to obtain the value of A for the proton-neutron interaction as well as in order to check on calculations using Eq. (9). The value of the constant multiplying factor for $\mathfrak{F}$ was not determined here by joining to the Coulomb wave functions. This factor cancels out in the applications. Corresponding to these three curves there are three other graphs starting from the point 2.0 on the axis of ordinates which represent $xd\mathfrak{F}/\mathfrak{F}dx$ and xdf/fdx. The uppermost of these carries the label $E=0$ at its right-hand end. It corresponds to the solution of Eq. (9) for $E=0$. The lowest gives values of xdf/fdx and is labeled by Eq. (9.1) at its lowest right-hand end. The intermediate curve of the set corresponds to $E=2E'=800$ kv. It should be compared with the curve marked 800 kv among the three pointed out as "experimental values of $rd\mathfrak{F}/\mathfrak{F}dr$" on the figure. These "experimental values of $rd\mathfrak{F}/\mathfrak{F}dr$" are obtained from empirical values of K_0 by means of Eq. (7.8), neglecting higher phase shifts, for the energies of 680 kv, 800 kv, 980 kv, respectively. For any r these curves give the value of $rd\mathfrak{F}/\mathfrak{F}dr$ which corresponds to an experimental K_0 provided the field is Coulombian at all distances greater than this r.

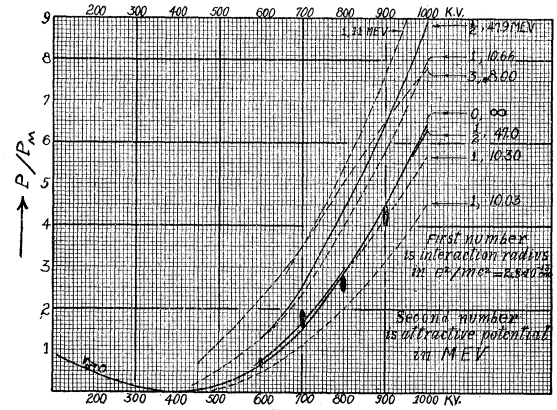

FIG. 12. Theoretical variation of $\mathbf{P}/\mathbf{P}_M$ with energy at a scattering angle of 45°. Curves are labeled by two numbers. First number gives the interaction radius of square well (r_0) in units of $e^2/mc^2=2.82\times10^{-13}$ cm. Second number gives the negative of interaction potential in mv. The ovals are obtained from the second set of data of THH using theoretical angular distribution curves fitted to experiment at higher scattering angles.

Comparison of the theoretical and experimental curves for $rd\mathfrak{F}/\mathfrak{F}dr=xd\mathfrak{F}/\mathfrak{F}dx$ at 800 kv shows, as is clear from the graph, that the two curves are very nearly the same for $x=2.6$ on towards larger r and that therefore the values of A and α used here are nearly right. The values of Feenberg and Knipp are $\alpha=17$, $A=41$ and the agreement is seen to be satisfactory. It is to be noted that Feenberg and Knipp considered two possibilities. In one of these the neutron-proton interaction was taken to be represented by a Majorana exchange operator. It gave $A\sim 26$ for the proton-proton and neutron-neutron interactions using the same α. In the other possibility Wigner's suggestion of using different interactions in the singlet and triplet states of the deuteron was used by regarding the proton-neutron interaction as a linear combination of a Majorana and a Heisenberg exchange operator. This view gave rise to the proton-proton and neutron-neutron interactions being represented by the Gauss error potential with $A=41$ and $\alpha=17$. There is on the whole a remarkable qualitative consistency in the way in which the proton-neutron scattering, the mass defects of H^2, He^3, H^3, He^4 and the proton-proton scattering fit in with each other. *The data on the scattering of protons in hydrogen are seen to speak definitely in favor of using different neutron-proton interactions in singlet and triplet states and make a pure Majorana force between neutrons and protons improbable.*

5. More Accurate Determinations and Comparisons with the Neutron-Proton Potential

It has been pointed out that, once definite values are assigned to the higher phases and to the range of the forces then accurate information can be obtained about the magnitude of the interaction potential. This is due to the fact that the experimental scattering anomaly is rather large and requires for its explanation interaction energies giving rise to approximate resonance. Thus the depth of the "square well" which was used to fit the data for $r_0 = e^2/mc^2$ is 10.3 Mev while the depth of the "square well" required to give a virtual level at $E=0$ is 12.8 Mev. Qualitatively this condition is similar to that in the interaction of a proton and neutron in their singlet S state. In order to account for the large scattering of slow neutrons in hydrogen it is necessary to suppose that there is either a virtual or a stationary ^1S level of the proton and neutron in their mutual field. Evidence as to whether the level is stationary or virtual is very scant. It appears of interest to see whether it is possible to consider the proton-proton and proton-neutron interactions to be identical in the ^1S states. We are indebted to Dr. L. A. Young, who made estimates analogous to and qualitatively agreeing with those presented here, for pointing out to us that one should not neglect to correct the depth of the "well" for the position of the stationary or virtual level in the neutron-proton potential.

In addition one must consider with greater care the possible effect of a Coulomb potential within the "square well." If it is supposed that the Coulombian force acts everywhere, a closer agreement is obtained between proton-proton and proton-neutron potentials than on the assumption that it acts only outside the square

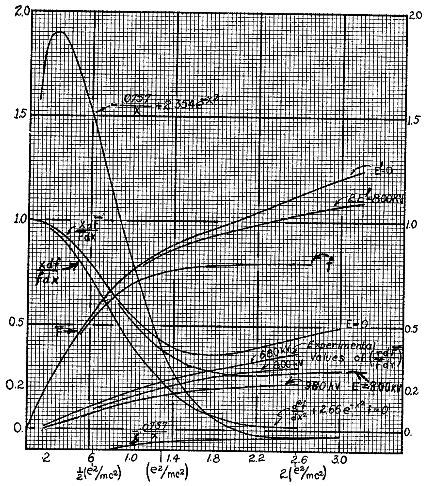

FIG. 13. Comparison of observation with the potential $Ae^{-\alpha r^2}$. Curve marked $-0.0757/x+2.354e^{-x^2}$ represents negative of potential energy in units of $17mc^2$. Coulombian part is shown by $-0.0757/x$. Note approximate agreement of experimental value of $rd\mathfrak{F}/\mathfrak{F}dr$ for 800 kv with corresponding theoretical curve for $xd\mathfrak{F}/\mathfrak{F}dx$. In the figure $\mathfrak{F}$ is written as F with a bar over it.

well ($r>r_0$). Since the range of interaction is not definitely determined by either the proton-proton or the proton-neutron scattering experiments, comparisons are listed below in Table XIV, for three values of the interaction radius r_0. The depth of the square well is given in Mev. The first row in the table gives the radius in units of e^2/mc^2. The second row $D_{\pi\pi}$ gives the depth that is obtained if the Coulomb potential does not act inside the well. The third row gives similarly the depth, if the potential inside the well is taken to be $-D_{\pi\pi}{}^c+e^2/r$. In order to counterbalance the effect of the repulsion the depth has to be increased by roughly mc^2. The fourth row gives the depth required to give a virtual or stationary level at $E=0$ taking the potential energy to be $-D$ inside the well and zero outside. In the fifth row $D_{\pi\nu}$ gives the depth for the proton-neutron interaction required to give a virtual level at 43 kv

TABLE XIV. *Comparison of proton-proton and proton-neutron singlet S interactions.*

$r_0mc^2/e^2=$	1/2	1	2
$D_{\pi\pi}$ =	47.0	10.3	1.98
$D_{\pi\pi}{}^c$ =	48.7	11.1	2.42
$(D)_{E=0}$ =	51.2	12.8	3.20
$(D_{\pi\nu})_{43}$ =	49.3	11.9	2.73
$(D_{\pi\nu})_{130}$ =	48.0	11.2	2.38

which corresponds to a collision cross section of 30×10^{-24} cm^2 for slow neutrons scattered by protons. The sixth row gives $D_{\pi\nu}$ using 130 kv for the position of the virtual level. The third and fifth rows agree fairly well indicating that the proton-proton force acting in addition to the Coulombian may be equal to the proton-neutron force in states with antiparallel spins. The third and sixth rows agree still better.

Although the numbers in the above comparison suggest very strongly that they are actually the same, caution in drawing this conclusion should be exercised for the following reasons: (a) It was not known until recently whether the neutron-proton interaction should be taken so as to give a virtual or a stationary level. If the level were stationary rather than virtual the depth for $r_0=e^2/mc^2$ would be 13.7 Mev and there would be then no agreement between $D_{\pi\nu}$ and $D_{\pi\pi}^c$ for this r_0. However, the recent experiments of Fermi and Amaldi[6] show that the level is virtual and indicate that $D_{\pi\nu}=D_{\pi\pi}^c$. (b) The theoretical value of the neutron-proton scattering cross section is in disagreement with the experiments of Goldhaber.[7] Goldhaber's experiments are, however, contradicted by those of Tuve and may be considered as probably incorrect. (c) The values of $D_{\pi\pi}$ and $D_{\pi\pi}^c$ are sensitive to the value of K_2 which is used in the interpretation of proton-proton scattering experiments. (d) The numbers given in the third row of Table XIV for $D_{\pi\pi}^c$ would give a smaller scattering cross section of slow neutrons in hydrogen than 30×10^{-24} cm^2 by roughly a factor of four and the values of $D_{\pi\nu}$ given in the fifth row would give larger cross sections for proton-proton scattering at 45° by roughly a factor of 1.3. The sixth row in Table XIV corresponds to the measurements of Fermi and Amaldi and indicates a practically perfect agreement with $D_{\pi\nu}$ for reasonably large values of r_0. The agreement is poorer for $rmc^2/e^2=\frac{1}{2}$ but this is a smaller range of force than is usually considered probable.

In order to be sure of the differences $\delta D=D_{\pi\pi}^c-D_{\pi\pi}$ the calculations were carried out by two methods one of which is a direct calculation using Coulomb wave functions inside the "well." The other is a perturbation calculation using $D_{\pi\pi}$ as a starting point. For the direct calculation only the regular wave function inside the "well" need be known. The main part of this function is the same as though there were no Coulomb field and it is convenient to arrange the series so as to take this into account. The functions Φ_0, Φ_0^* of Eq. (7.4) can be expressed as

$$\Phi_0=\sin z/z+c_1y+c_2y^2+\cdots;\qquad \Phi_0^*=\cos z+2c_1y+3c_2y^2+\cdots, \tag{10}$$

where z is given by Eq. (8) and the coefficients are

$$c_1=1,\quad c_2=\frac{1}{3},\quad c_3=\frac{1}{18}-\frac{1}{9\eta^2},\quad c_4=\frac{1}{180}-\frac{1}{36\eta^2},$$

$$c_5=\frac{1}{2700}-\frac{1}{270\eta^2}+\frac{23}{5400\eta^4},$$

$$c_6=\frac{1}{56700}-\frac{1}{3240\eta^2}+\frac{7}{8100\eta^4}.$$

The perturbation calculations were made using the formula

$$\delta D=\frac{e^2}{r_0}\frac{z\int_0^z(\sin^2 z/z)dz}{\int_0^z \sin^2 zdz}=\frac{e^2z}{r_0}\frac{\ln 2z+0.5772-Ci(2z)}{z-\sin z\cos z} \tag{10.1}$$

with $$Cix=-\int_x^\infty \frac{\cos u}{u}du.$$

The two methods of calculation agree to the desired accuracy. This fact is of practical interest if exact calculations are more difficult to perform than those with "square wells." Use of the validity of the perturbation method will now be made for an improvement on calculations with the Gauss error potential. The general relations needed for extensions of the perturbation method are as follows. A given differential equation

$$[(d^2/dr^2)-E'+\lambda\chi(r)+A\varphi(r)]\mathfrak{F}=0 \tag{10.2}$$

of the type considered here has, to within an arbitrary constant factor, one and only one solution which is regular at $r=0$. For such a solution $rd\mathfrak{F}/\mathfrak{F}dr$ is, therefore, uniquely defined. It may be considered as a function of E', A and λ.

Using Green's theorem one obtains

$$\mathfrak{F}_1\mathfrak{F}_2\left[\frac{1}{\mathfrak{F}_1}\frac{d\mathfrak{F}_1}{dr}-\frac{1}{\mathfrak{F}_2}\frac{d\mathfrak{F}_2}{dr}\right]+(A_1-A_2)\int_0^r \varphi\mathfrak{F}_1\mathfrak{F}_2 dr=0$$

and similar equations with E' and λ. In the limiting cases $A_1=A_2$, $\lambda_1=\lambda_2$, $E_1'=E_2'$ they become:

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial\mathfrak{F}}{\mathfrak{F}\partial r}\right)=-\frac{1}{\mathfrak{F}^2}\int_0^r\mathfrak{F}^2\chi dr;$$
$$\frac{\partial}{\partial A}\left(\frac{\partial\mathfrak{F}}{\mathfrak{F}\partial r}\right)=-\frac{1}{\mathfrak{F}^2}\int_0^r\mathfrak{F}^2\varphi dr; \qquad (10.3)$$
$$\frac{\partial}{\partial E'}\left(\frac{\partial\mathfrak{F}}{\mathfrak{F}\partial r}\right)=\frac{1}{\mathfrak{F}^2}\int_0^r\mathfrak{F}^2 dr.$$

These formulas are useful for the following applications: (a) The determination of the change which must be made in A in order to compensate for a given change in λ so as to leave the phase shift unaltered. This is possible by means of Eqs. (10.3) as long as the rates of change of $\partial\mathfrak{F}/\mathfrak{F}\partial r$ with λ and A are nearly constant. For all that is required is to leave $\partial\mathfrak{F}/\mathfrak{F}\partial r$ unchanged at the boundary of the "well." Eq. (10.1) can be obtained by this procedure by regarding $\chi(r)$ as arising from the Coulombian energy. (b) From the values of the phase shift K_0 one can obtain $\partial\mathfrak{F}/\mathfrak{F}\partial r$ for definite energies and hence from the last Eq. (10.3) the quantity $(1/r\mathfrak{F}^2)\int_0^r\mathfrak{F}^2dr$ can be determined. It may be considered as a rough form factor of the wave function. By means of the form factor information can also be obtained about the character of the interaction potential. Thus the first set of data of THH indicated a variation of K_0 such as is shown by the straight line in Figs. 8, 9, 10, 11. Computing $(1/r\mathfrak{F}^2)\int_0^r\mathfrak{F}^2dr$ from the variation of $\partial\mathfrak{F}/\mathfrak{F}\partial r$ with energy gives then values >1 for the form factor which cannot be explained by the simple potentials used here but would have required other less probable possibilities. This is in agreement with the fact that the older data of THH and the data of White gave a more rapid variation of K_0 with energy than would be expected for $r_0=0$. The same effect is shown by the two curves for "experimental values of $r\partial\mathfrak{F}/\mathfrak{F}\partial r$" of Fig. 13 which are marked by 680 kv and 980 kv. These were computed for the old data of THH which were represented by straight lines in Figs. 8, 9, 10, 11. Comparing them with the theoretical curves for $E=0$ and $E=800$ kv the experimental variation of $r\partial\mathfrak{F}/\mathfrak{F}\partial r$ is found to be too great. It will be seen presently, however, that the newer data which were taken at 900, 800, 700 and 600 kv give an energy dependence of K_0 and of $r\partial\mathfrak{F}/\mathfrak{F}\partial r$ which is in approximate agreement with expectation for the Gauss error potential used for Fig. 13.

In order to improve the comparison of the Gauss error potential obtainable from proton-proton scattering with those derivable from mass defects and from neutron-proton scattering, calculations were made in which the field was supposed to become Coulombian for $r>2e^2/mc^2$ while for $r<2e^2/mc^2$ the potential was taken as in Eq. (9). By means of Eq. (10.3) the value of A used in Eq. (9) was then corrected so as to give the experimental value of $r\partial\mathfrak{F}/\mathfrak{F}\partial r$ at $r=2e^2/mc^2$ for $E=800$ kv. This calculation gave $A=38.5$ in units of mc^2 which is slightly lower than Feenberg and Knipp's value of $A=41$. It should be noted that if a small positive K_2 is present this value of A should be raised and that therefore the agreement with Feenberg and Knipp may be better than $A=38.5$ would indicate. For comparison with neutron-proton interactions it is desirable to eliminate cumulative errors which might be present in the numerical integration. This was done by using the solution for a stationary level at $E=0$ in the absence of a Coulomb field as a starting point for the determination of both the $\pi\pi$ and the $\pi\nu$ potentials. For $A_{\pi\pi}$ this method of calculation is not very accurate because the shape of the wave function changes appreciably between the initial condition and the final one. The value of A obtained for a stationary level at $E=0$ is 45.7. The result of applying the perturbation method using the wave function in this state (see curve marked f in Fig. 13) is to give $A_{\pi\pi}=40.9$. Using the wave function in the final state (marked 2 $E'=800$ kv in Fig. 13) we get $A_{\pi\pi}=37.8$. The mean from the two perturbation calculations is 39.3 which is somewhat higher than that obtained by direct calculation. The mean of the direct determination and the perturbation method is $A_{\pi\pi}=38.9$.

For the $\pi\nu$ interaction the perturbation calculation should be much more accurate. With the solution f as a starting point and by solving for $A_{\pi\nu}$ so as to have a virtual level at 43 kv. $A_{\pi\nu} = 42.0$. The difference $A_{\pi\nu} - A_{\pi\pi}$ is seen to be positive just as $D_{\pi\nu} - D_{\pi\pi}$ and it is of the same order of magnitude. Use of $A_{\pi\nu}$ for calculations on $\pi\pi$ scattering gives too high values and use of $A_{\pi\pi}$ for $\pi\nu$ scattering gives values too small by approximately the same amounts as for square wells with $r_0 = e^2/mc^2$. As expected, this feature of the comparison does not depend critically on the shape of the "well." Use of Fermi and Amaldi's position of the virtual level (130 kv), lowers the value of $A_{\pi\nu}$ to 39.2 which is in practically perfect agreement with $A_{\pi\pi} = 38.9$.

The rate of change of $\partial\mathfrak{F}/\mathfrak{F}\partial r$ with energy was calculated for $\alpha = 17$ by means of Eq. (10.3) and hence also the rate of change with energy of K_0. It was found that K_0 varies approximately in the same way at 800 kv as for a square well with $D = 10.3$ Mev and $r_0 = e^2/mc^2$. It definitely varies more rapidly than for $D = 1.98$ Mev and $r_0 = 2e^2/mc^2$.

Measurements at voltages below and above those used so far will be valuable in determining the range of nuclear forces as is clear from Figs. 8, 9, 10, 11 and they should be helpful in establishing the effects due to higher angular momenta. For an attractive interaction energy of 10 Mev through 2.8×10^{-13} cm the phase shift K_1 should be 0.2° and 1.5° at incident energies of 2 and 9 Mev, respectively; K_1 is roughly twice as great for $D = 2$ Mev, $r_0 = 5.6 \times 10^{-13}$ cm as for $D = 10$ Mev, $r_0 = 2.8 \times 10^{-13}$ cm at 2 Mev of incident proton energy. Attractive and repulsive potentials for $L = 1$ can be distinguished by the sign of K_1.

Summary. The experiments of THH indicate an interaction potential between protons equivalent to -11.1 Mev in a distance of 2.82×10^{-13} cm acting in addition to the Coulombian repulsion. The potential agrees closely with that obtained from mass defect calculations which use a neutron-proton interaction depending on spin orientation. Higher phase shifts than those for $L = 0$ are not called for sufficiently definitely to make their existence certain.

The magnitude of the interaction between like particles in ^1S states is arrived at here with a relatively high precision. It is compared with the proton-neutron interaction in the corresponding state as derived from the experiments of Fermi and Amaldi. The proton-proton and proton-neutron interactions in ^1S states are found to be equal within the experimental error. This suggests that interactions between heavy particles are equal also in other states.

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