On the Thomas Precession of Accelerated Axes

The Thomas precession theorem, which is of interest in the study of atomic spectra and also finds a new application in the preceding note, may be derived with less algebra or more elementary concepts than heretofore,[1] as follows: We consider a coordinate system P, in which the center of gravity of a "planetary" particle remains at rest. P has a velocity and an acceleration $\mathbf{a}$ in another system N, in which a nucleus or laboratory is at rest. Because of the acceleration, we cannot by means of special relativity transform from P to N, but the rotation (or "Thomas precession") of P in N may be demonstrated by considering two nonaccelerated coordinate systems P' and P'' with which P coincides at times t and $t+\delta t$, respectively.

We orient the axes in such a way that P' and N have their x axes (which we call x' and x, respectively) parallel to their relative velocity $\mathbf{v}$, and that x'' and x' make the same angle with the relative velocity $\mathbf{u}'$ of their systems P'' and P', $\mathbf{u}'$ being in the xy planes. Thus a Lorentz transformation without space rotation will serve between P'' and P', and between P' and N. We wish to show that the transformation between P'' and N does involve a rotation nevertheless (or, in other words, that observers in P'' and N would agree that their axes are not parallel, although each agrees with an observer in P' on parallelism with his axes). We do this by showing that the axes x'' and x make different angles ϕ'' and ϕ with the relative velocity $\mathbf{u}$ of P'' and N.

From the familiar Lorentz transformation

$$dx=(dx'-vdt')/(1-v^2/c^2)^{\frac{1}{2}}, \quad dy=dy',$$
$$dt=(dt'-vdx'/c^2)/(1-v^2/c^2)^{\frac{1}{2}},$$

is had by division the usual rule for relativistic velocity addition:

$$u_x=(u_x'+v)/(1-vu_x'/c^2),$$
$$u_y=u_y'(1-v^2/c^2)^{\frac{1}{2}}/(1-vu_x'/c^2).$$

For infinitesimal δt and finite acceleration, $\mathbf{u}'$ is infinitesimal, and we have for the angle between $\mathbf{u}$ and x

$$\phi=\tan\phi=u_y/u_x=u_y'(1-v^2/c^2)^{\frac{1}{2}}/v.$$

To get the components of $\mathbf{u}$ in P'', in terms of $\mathbf{v}$ and $\mathbf{u}'$, we have to transform $\mathbf{v}$, known in P', to P'', the relative velocity $\mathbf{u}'$ being infinitesimal. The infinitesimal relativistic correction to ordinary velocity addition may be neglected ($\mathbf{u}'$, although oblique, being analogous to v above), and we have

$$\phi''=\tan\phi''=u_y'/v.$$

We see that ϕ'' and ϕ are, indeed, not the same, their difference being the angle through which P has turned in N about the z axis during δt:

$$\delta_T\delta t=\phi-\phi''=u_y'\{(1-v^2/c^2)^{\frac{1}{2}}-1\}/v\approx -u_y'v/2c^2.$$

Substituting $u_y'=a_y'\delta t'\approx a_y\delta t$ (neglecting higher powers of v/c), we have the angular velocity of the Thomas precession of P in N:

$$\boldsymbol{\omega}_T=-\mathbf{v}\times\mathbf{a}/2c^2.$$

Any directed quantity, such as an angular momentum, which is constant in the coordinate system of an accelerated and moving particle, has this rotation in a "stationary" coordinate system. In the well-known atomic case, there is a magnetic field $\mathbf{H}$ in P corresponding to the electric field $\mathbf{E}$ in N, and a classical electron spin would be constant n a Larmor coordinate system having a precession $\boldsymbol{\omega}_L$ in P or $\boldsymbol{\omega}=\boldsymbol{\omega}_L+\boldsymbol{\omega}_T$ in N. (This summation assumes that P has a time scale differing from that of N only in higher order in v/c, as do the Lorentz frames P' and P''.) Using $\mathbf{a}=-\mathbf{E}e/m$, we have

$$\boldsymbol{\omega}_L=\mathbf{H}(\mu/S)=-[\mathbf{v}\times\mathbf{E}/c](e/mc)=\mathbf{v}\times\mathbf{a}/c^2=-2\boldsymbol{\omega}_T.$$

The term $\boldsymbol{\omega}\cdot\mathbf{S}$ (which we may ordinarily consider as a perturbation term) in the Hamiltonian of any classical model of an atomic electron with spin $\mathbf{S}$, consists of two terms, the magnetic term $\boldsymbol{\omega}_L\cdot\mathbf{S}$ being half canceled by the relativistic term $\boldsymbol{\omega}_T\cdot\mathbf{S}$, which thus introduces the familiar "Thomas factor" $\frac{1}{2}$ in the atomic spin-orbit energy.

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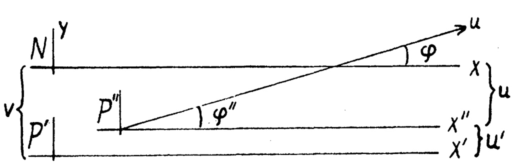

FIG. 1. The brackets indicate relative velocities.

[1] L. H. Thomas, Nature **514** (1926), Phil. Mag. **3**, 1 (1926); Frenkel, Zeits. f. Physik **37**, 243 (1926); Goudsmit and Uhlenbeck, Physica **6**, 276 (1926); Ruark and Urey, *Atoms, Molecules, and Quanta*, p. 162; Sommerfeld, *Convegno di Fisica Nucleare* (Rome, 1932); Kramers, Physica **1**, 825 (1934); see also Zeeman Verhandelingen, 405 (1935).

On the Introduction of Nonelectric Forces into Dirac's Equations

The doublet separations of nuclear energy levels have been discussed by Inglis,[1] who assumed that on account of the nonelectric character of the forces the main contribution to the splittings comes from the Thomas relativistic effect. In this note we shall see how such a result follows from Dirac's equations. Although it is not generally supposed that these equations actually hold for a heavy particle, they still represent our only way of treating the relativistic quantum mechanics of a particle with spin, so that it is interesting to see what their application gives. The addition of extra terms[2] in order to obtain agreement with observed values of the magnetic moments would not affect these results.

The Dirac equations for a particle subject to an electrostatic force are

$$c\{p_0-(H_0+V_e)/c\}\psi=c\{p_0-V_e/c+(\boldsymbol{\alpha}\cdot\mathbf{p})+\beta mc\}\psi=0, \quad (1)$$

where $V_e=$(charge)$\cdot$(electrostatic potential) = potential energy due to the electrostatic force. It is evident that if we try to introduce a nonelectric potential energy V_n by adding it to the Hamiltonian as V_e is here added, the new term will be indistinguishable from V_e in all its effects on the behavior of the wave function. Adding a term in this way really corresponds to introducing a four-vector whose space components vanish in the particular Lorentz frame used. On the other hand if we introduce a term proportional to V_n in such a way that it transforms like the Lagrangian, which is a scalar invariant, it can just as reasonably be called a potential energy as can a term added to the Hamiltonian; and the scalar so introduced will differ in its effects from a nonscalar such as the term in V_e. Accordingly, for a particle subject to both an electrostatic force and a nonelectric force we write

$$c\{p_0 - V_e/c + (\boldsymbol{\alpha}\cdot\mathbf{p}) + \beta(mc + V_n/c)\}\psi = 0. \qquad (2)$$

The various relativistic quantities[3] which can be formed from the Dirac matrices offer only these two possibilities for introducing quantities which may reasonably be regarded as potential energies, and which have a suitable effect on the behavior of ψ in the lowest order in v/c.

If in (2) we introduce the usual convenient notations,

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}, \qquad (3)$$

where the σ's and 1 are two-rowed matrices and χ, φ are two-component functions, we get

$$c\{(p_0 + mc - V_e/c + V_n/c)\chi + (\boldsymbol{\sigma}\cdot\mathbf{p})\varphi\} = 0 \qquad (4a)$$

$$c\{(p_0 - mc - V_e/c - V_n/c)\varphi + (\boldsymbol{\sigma}\cdot\mathbf{p})\chi\} = 0. \qquad (4b)$$

On carrying out the elimination of the "small components" χ in the usual way,[4] we obtain an equation for φ of the form $(cp_0 - mc^2 - \bar{H})\varphi = 0$, valid to order $(v/c)^2$, with

$$\begin{aligned} \bar{H} = p^2/2m + V_e + V_n \\ + \{(\hbar/4mc)(\boldsymbol{\sigma}\cdot[(\mathbf{p}/mc)\times \mathrm{grad}\,(V_e - V_n)])\} \\ + \{(i\hbar/4mc)((\mathbf{p}/mc)\cdot \mathrm{grad}\,(V_e - V_n))\}. \qquad (5) \end{aligned}$$

Thus, although V_n and V_e appear in the same way in the lowest order in v/c, they appear with opposite signs in the spin-orbit terms of the first bracket, in agreement with our expectations from Inglis' argument about the Thomas effect. They also oppose each other in the second bracket, in terms of a type first discussed by Darwin.[5]

The introduction of a scalar potential energy added to the proper energy corresponds closely to Nordström's special-relativistic theory of gravitation.[6] Since the forces in nuclei are surely not gravitational, the fact that astronomical observation decides for general relativity and against Nordström's gravitational theory does not impair the interest of this way of discussing the action of nonelectric forces.

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[1] D. R. Inglis, this issue, p. 783.
[2] W. Pauli, *Handbuch der Physik*, XXIV/1, p. 233.
[3] W. Pauli, reference 2, p. 221.
[4] Originally done by Darwin, reference 5; for brief treatment in a notation more like that used here, cf. reference 2, p. 237.
[5] C. G. Darwin, Proc. Roy. Soc. **A118**, 662 (1928).
[6] G. Nordström, Ann. d. Physik **42**, 540 (1913).

Mutual Reactions of Metals and Salts

When a metal is heated in contact with a salt one or more of the following phenomena may take place: (1) solution of the metal in the salt; (2) dissociation of the molecules of the salt; (3) diffusion of the dissociated metallic atoms of the salt into the metal; (4) formation of chemical compounds of liberated metalloids with the metal.

We shall describe an experiment made with a copper bar heated in cadmium chloride. At a temperature below the melting point of $CdCl_2$ the copper dissolves in the salt, which becomes colored (reddish) after about one hour of heating. At a temperature above the melting point of $CdCl_2$ copper dissolves rapidly in the liquid $CdCl_2$ and after cooling the faces of the crystals of the salt are covered with a thin sheet of metallic copper alloyed with cadmium (thickness of a few microns).

In this experiment dissociation of $CdCl_2$ and formation of chemical compounds with copper take place. The reaction is:

$$\alpha_1 Cu + \alpha_2 CdCl_2 \rightleftarrows \beta_1 CuCl + \beta_2 CuCl_2 + \beta_3 Cd.$$

The values of α and β depend upon the temperature. The heat evolved in the reaction (U_v) has a negative value. Consequently from the van't Hoff law $(\partial lgR_v/\partial T = U_v/RT^2)$ the value of the equilibrium constant (R_v) of the mass action law decreases when the temperature rises

$$R_v = \frac{C^{\alpha_1}(Cu)\cdot C^{\alpha_2}(CdCl_2)}{C^{\beta_1}(CuCl)\cdot C^{\beta_2}(CuCl_2)\cdot C^{\beta_3}(Cd)},$$

where the C's are the concentrations of the reacting substances and their products. This law shows that the concentrations of the products of the reaction increase with the temperature. Hence a rapid lowering of the temperature of the reacting bodies causes metallic copper to be separated out of the salt. This phenomenon is only possible in the exothermic reactions (thermit). The liberated metals: cadmium and copper are dissolved in the melted salt. The concentration of dissolved metals is greater in the liquid than in the solid state of $CdCl_2$ and so on crystallization of that salt the metal excluded from the salt during the solidification is deposited on the faces of the crystals.

Analogous phenomena occur also in the case of metals (copper) heated in an atmosphere of vapors of subliming chlorides such as $NiCl_2$ and $CdCl_3$.[1] The metals of the dissociated chlorides diffuse into the copper. This phenomenon is analogous to the phenomenon of diffusion of two metals in contact. (A theory of this phenomenon by Mr. J. Cichocki is to be published in the Journal de physique.)

Copper heated in $MgCl_2$ for 24 hours at a temperature of 640°C contains after the heating 3 atoms per hundred of magnesium. The metal liberated in these reactions is deposited on the copper and on the walls of the vessel and form porous deposits or metallic crystals.

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[1] Comptes rendus **181**, 463 (1925).