Collision of Neutron and Proton

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An exact solution is found for the bound and free states of a neutron-proton pair under the influence of a potential field $-D[2\exp(2/r_0)(r_1-r)-\exp(4/r_0)(r_1-r)]$, where the field can be of the Majorana type and may depend on the spin coupling. Values of binding energy (for the deuteron), of elastic cross section, of capture cross section and of cross section for the photoelectric disintegration of the bound state are computed for a wide range of the parameters used. Curves of the results are shown. In order to correspond to the experimental values of binding energy and of elastic cross section for slow neutrons, the value of D for the singlet state, D_s, must be made about one-half that for the triplet state, D_t. Separate calculations are made, assuming a bound singlet state, and assuming no bound singlet state. When this is done, the capture and photoelectric cross sections check the experimental values fairly well, the computed results being practically independent of r_0 and r_1 as long as these parameters are less than 6×10^{-13} cm. Thus a proper choice of two relations between the four parameters D_s, D_t, r_0, r_1, gives agreement with four different experimental values; the other two relations needed to determine all four parameters will require further experimental data for their determination.

INASMUCH as the fundamental theory of the interactions between the particles in the nucleus has not yet been developed, it is necessary to obtain knowledge of these interactions by empirical methods. Accordingly, it is of interest to assume a number of different types of force fields between the constituent nuclear particles, to work out as accurately as possible the behavior of these particles corresponding to the assumed forces, and to see which of the behaviors corresponds best to the experimental data. This program has already been initiated in a number of papers.[1, 2] The present article continues the work by studying in some detail the interaction between proton and neutron.

THE WAVE FUNCTIONS

A potential field which allows of exact solution and which permits a considerable variation of form, is given by the expression

$$V(r)=-2D\exp(2/r_0)(r_1-r)+D\exp(4/r_0)(r_1-r). \quad (1)$$

When $r_1=0$ the field is everywhere attractive, having a minimum at $r=0$. For r_1 larger than zero the field is attractive for large r and repulsive for small r, having a minimum at $r=r_1$. Fig. 1 shows typical shapes of the potential.

Writing energies and lengths in nuclear units: energy unit $=m_ec^2=506{,}000$ ev

length unit $=(h/2\pi c)(1/m_em_p)^{\frac{1}{2}}=8.97\times10^{-13}$ cm,

the wave equation for the motion of a neutron and proton relative to their center of mass is $[\nabla^2+(W-V)]\psi=0$. If the wave function has spherical symmetry, $\psi=R(r)/r$, this reduces to

$$(d^2R/dx^2)+[-m^2+2ke^{(x_1-x)}-ke^{2(x_1-x)}]R=0 \quad (2)$$

where $x=(2r/r_0)$, $x_1=(2r_1/r_0)$, $m^2=-(r_0/2)^2W$ and $k^2=(r_0/2)^2D$.

The solution of this equation corresponding to the bound state is the confluent hypergeometric function

$$R_0=z^{-\frac{1}{2}}M_{k,\,m}(z)=z^me^{-z/2}\left[1+\frac{\frac{1}{2}+m-k}{1!(2m+1)}z+\cdots\right] \quad (3)$$

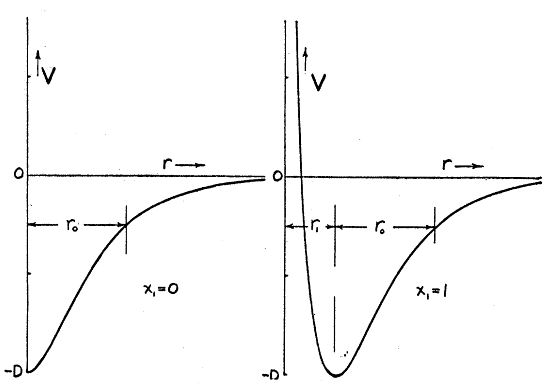

FIG. 1. Typical shapes of the assumed potential field.

[1] Feenberg and Knipp, Phys. Rev. **48**, 906 (1935); Heisenberg, Zeits. f. Physik **77**, 1 (1932); Majorana, Zeits. f. Physik **82** (1933); Massey and Mohr, Proc. Roy. Soc. **A152**, 693 (1935); Thomas, Phys. Rev. **47**, 903 (1935); Wigner, Phys. Rev. **43**, 252 (1933); Young, Phys. Rev. **48**, 913 (1935).

[2] Bethe and Peierls, Proc. Roy. Soc. **A148**, 146 (1935); Feenberg, Phys. Rev. **47**, 857 (1935); Fermi, Phys. Rev. **48**, 570 (1935); Hall, Phys. Rev. **49**, 401 (1936).

TABLE I. *Values of k for the bound state.*

x_1	$m=0$	$m=0.5$	$m=1.0$
0	0.900	1.480	2.032
0.5	.728	1.228	1.713
1.0	.608	1.070	1.540

where $z=2ke^{x_1-x}$, and where m is adjusted so that $R=0$ when $z=0(r=\infty)$ and when $z=z_0=2ke^{x_1}$ $(r=0)$. The first requirement is met when m is real and positive. The second is met by solving the transcendental equation $M_{k,\,m}(2ke^{x_1})=0$. Values of k satisfying this equation are given in Table I, for different values of m and x_1. Intermediate values can be obtained with considerable accuracy by second-difference interpolation.

The solution of Eq. (2) for the states of positive energy are obtained by setting $m=i\mu$, $\mu^2=(r_0/2)^2W$;

$$R=(i/2z^{\frac{1}{2}})[(z_0)^{-i\mu}e^{-i\delta_0}M_{k,\,i\mu}(z)-(z_0)^{i\mu}e^{i\delta_0}M_{k,\,-i\mu}(z)] \quad (4)$$

where the phase angle δ_0 is obtained by computing the real and imaginary parts of the series

$$\mathrm{Re}+i\,\mathrm{Im}=1+\frac{\frac{1}{2}-k+i\mu}{1!(1+2i\mu)}z_0+\frac{(\frac{1}{2}-k+i\mu)((3/2)-k+i\mu)}{2!(1+2i\mu)(2+2i\mu)}z_0^2+\cdots;$$

$$\tan(\delta_0)=(\mathrm{Im}/\mathrm{Re}). \quad (5)$$

This solution is zero at $r=0(z=z_0)$, and has the asymptotic form

$$R\rightarrow\sin(w^{\frac{1}{2}}r+\delta_0), \qquad (r\rightarrow\infty,\ z\rightarrow 0).$$

The important expression for the computation of elastic scattering is $\sin(\delta_0)/\mu$. The limiting value of this as μ goes to zero will be called Δ. The quantity $\pi\Delta^2$ is plotted in Fig. 2 as a function of k and x_1.

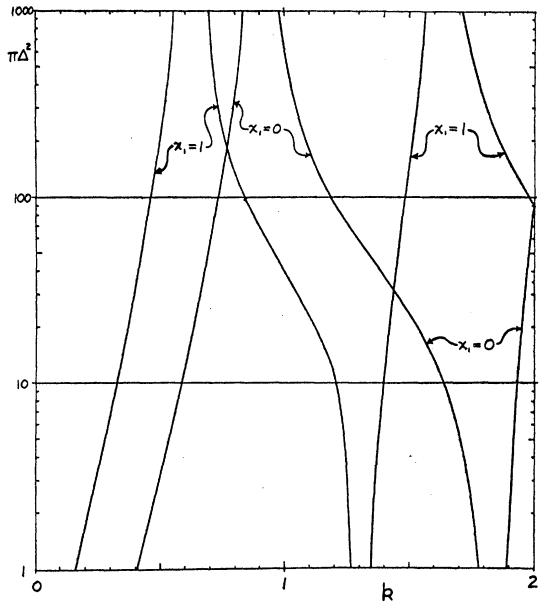

FIG. 2. Cross-section factor $\pi\Delta^2$ for slow neutrons.

NORMAL STATE OF THE DEUTERON

The potential operator for the neutron-proton interaction may have a dependence on the coupling of the two spins and it may include a permutation operator of the Majorana type.[1] The observed value of the binding energy of the deuteron gives us a relationship between the width and depth of the interaction potential for the triplet state. We have assumed that the observed value of this binding energy W_t is -4 nuclear units. The results given in the rest of this paper will not be appreciably altered if the binding energy is chosen as large as -4.4, as recent work[3] seems to indicate.

The results given in Table I can be utilized to find the relation between D and r_0 which makes $W_t=-4$. The depth D_t is plotted (solid lines) in Fig. 3 as a function of r_0 for two different values of x_1. The corresponding values of the normalizing factor

$$N_t=\left[4\pi\int_0^\infty\psi^2r^2dr\right]^{-\frac{1}{2}}=\left[2\pi r_0\int_0^{z_0}M^2_{k,\,m}(z)dz/z^2\right]^{-\frac{1}{2}}$$

are given in Table II. The shapes of two normalized wave functions, with their corresponding potential energy curves, are given in Fig. 4.

ELASTIC SCATTERING

The cross section for elastic scattering of neutrons by protons is given by the formula[4]

$$Q=(\pi r_0^2/\mu^2)\sum_l(2l+1)\sin^2(\delta_l). \quad (6)$$

[3] Oliphant, Nature **137**, 396 (1936).
[4] Allis and Morse, Zeits. f. Physik **70**, 567 (1931).

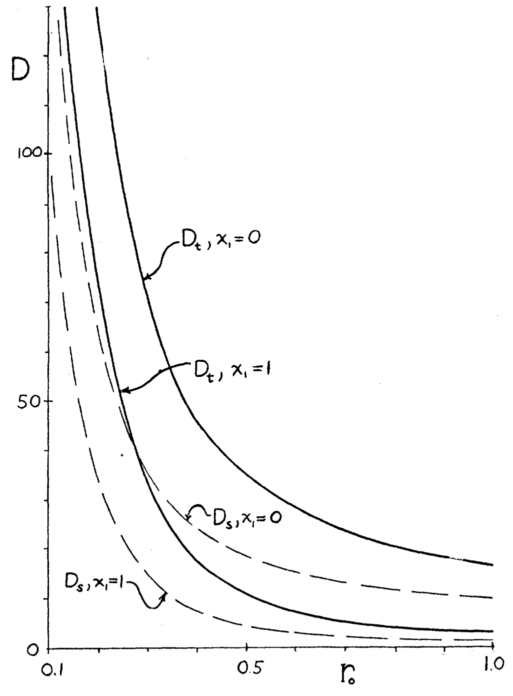

FIG. 3. Relation between depth and radius of triplet potential to give observed deuteron binding energy (solid lines), and for singlet potential to give observed elastic cross section for slow neutrons (dotted lines).

The phase angle δ_0 is obtained from Eq. (5) and the angles for larger l are obtained by numerical integration of the equation

$$R_l''+[\mu^2+2\epsilon_l k e^{x_1-x} -\epsilon_l k e^{2(x_1-x)}-l(l+1)/x^2]R_l=0, \quad (7)$$

and calculation of the phase in the usual manner.[4] The factor ϵ_l is equal to $(-1)^l$ if the potential is of the Majorana type; it is equal to unity if the potential is of the usual type.

For very slow neutrons only the term in $l=0$ is important, and the limiting value of the cross section is

$$Q_0=\pi r_0^2\Delta^2, \qquad \Delta=\lim\ (\sin\delta_0/\mu).$$

Fig. 5 shows a curve of Q_0 against r_0 for those values of D which give the right binding energy for H^2. The curve is for $x_1=0$, but the curves for $x_1=0.5$ and $x_1=1.0$ are practically identical, not differing by more than five percent at any part of the range shown. Since the experimental value[5] of Q_0 is about 45 square nuclear units, it is seen that the calculated cross section does not agree, except for improbably large values of r_0. Evidently the potential field must depend on spin orientation.

To study this effect we assume a singlet potential energy having the same value of r_0 as for the triplet, but having a smaller depth, D_s. The elastic cross section will then be

$$Q=(1/4)(Q_s+3Q_t). \quad (8)$$

The problem can then be worked backward, to find a value of D_s such that Q_0, computed from Eq. (8) and using the value of D_t which gives the

[5] Dunning, Pegram, Fink and Mitchell, Phys. Rev. **48**, 265 (1935). Other work, Fermi and Amaldi, Ricerca Scient. **1**, 56 (1936), indicates the possibility of somewhat smaller values for Q_0, but this will not appreciably alter the general results of this paper.

TABLE II. *Values of the constants for the deuteron.*

		$x_1=0$				$x_1=1$			
	r_0	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
	k_t	1.135	1.366	1.593	1.814	0.792	0.977	1.163	1.351
	m_t	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
Triplet	D_t	128.8	46.64	28.20	20.57	62.73	23.86	15.03	11.41
State	N_t	0.6737	0.7217	0.7103	0.6843	0.6203	0.6150	0.5660	0.4984
	Δ_t	−6.64	−3.77	−2.13	−0.01	−6.88	−3.90	−2.22	+0.79
	W_t	−4	−4	−4	−4	−4	−4	−4	−4
	k_s	0.865	0.834	0.809	0.791	0.580	0.552	0.528	0.508
Singlet	D_s	74.8	17.4	7.27	3.91	33.7	7.61	3.10	1.59
Unstable	Δ_s	36.1	17.8	12.1	9.46	35.9	17.7	12.0	9.36
	k_s	0.937	0.978	1.017	1.053	0.637	0.671	0.705	0.736
	m_s	0.031	0.066	0.099	0.130	0.031	0.069	0.105	0.139
Singlet	D_s	87.8	23.9	11.5	6.93	40.6	11.3	5.52	3.39
Stable	N_s	0.2310	0.2513	0.2634	0.2680	0.2206	0.2504	0.2544	0.2576
	Δ_s	−36.1	−17.8	−12.1	−9.46	−35.9	−17.7	−12.0	−9.36
	W_s	−0.096	−0.109	−0.109	−0.106	−0.096	−0.119	−0.123	−0.121

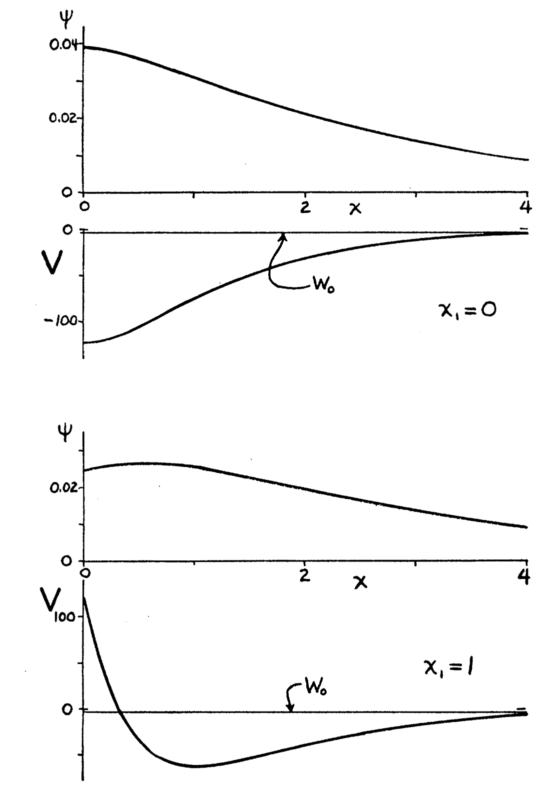

FIG. 4. Possible wave functions and corresponding potential energy curves for the deuteron. Horizontal lines on the energy plots show the deuteron energy level.

correct binding energy for H^2, comes out equal to 45. There are several values of D_s which satisfy this requirement, one having a stable singlet level very close to the top of the potential hole, and one having no stable singlet state. The constants for these two possibilities are given in Table II for different values of r_0 and x_1. In the case of the stable state the value of the allowed energy W_s is also given. It is interesting to notice that this value is practically independent of r_0 and of x_1 for a wide range of values of these parameters. The level, if it exists, is only about 60,000 ev below the top of the potential hole. The dotted curves in Fig. 3 show D_s for the case of no stable singlet state.

The variation of cross section with neutron velocity can then be computed. Fig. 6 shows curves of Q for $x_1=0$, plotted as a function of $W^{\frac{1}{2}}$, assuming a nonpermuting potential function; i.e., one where $\epsilon_l=1$. The differences between the curves are due to the increasing importance of the $l=1$ term in the series for Q as the value of r_0 is

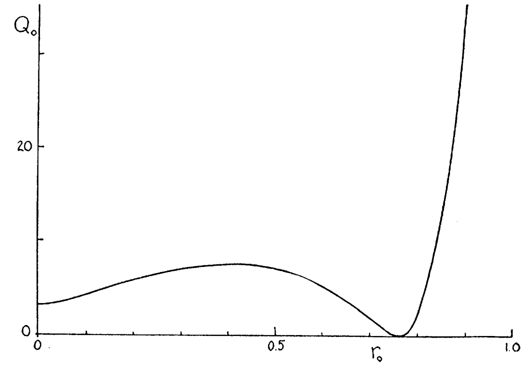

FIG. 5. Cross section for slow neutrons using the same potential as for bound state.

increased. The terms for $l>1$ are negligible for the range of r_0 studied. Fig. 7 shows corresponding curves for $x_1=1$. The chief difference is that at $x_1=1$ the term in $l=1$ becomes important for a smaller value of r_0.

When a potential of the Majorana type, $\epsilon_l=(-1)^l$, is assumed, all the terms in the series for Q are negligible except the first, and the cross-section curves for $x_1<1$ and $r_0<1$ are all practically identical with the curves shown for $r_0=0.2$.

Thus the variation of elastic cross section with neutron velocity is not a critical test for the value of r_0. If the experimental curve is similar to

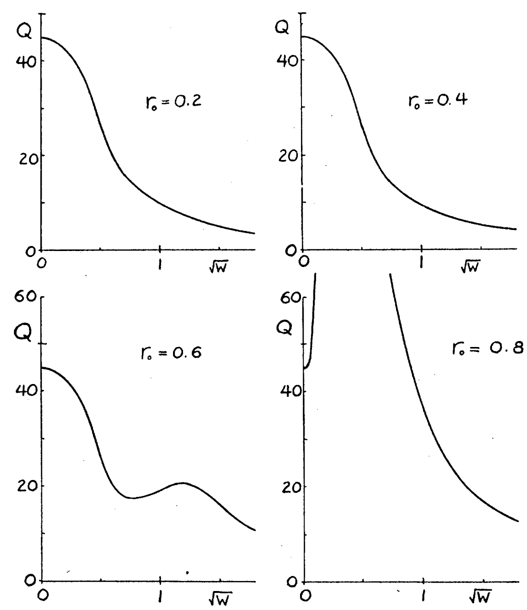

FIG. 6. Variation of cross section with neutron velocity for $x_1=0$, for different values of r_0, assuming a nonpermuting type of potential. For a Majorana potential all curves would be similar to that shown for $r_0=0.2$.

the first one shown in Fig. 6, the potential can be nonpermuting with $r_0<0.4$, or it can be of the Majorana type with $r_0<1$. If, on the other hand, the curve shows a "Ramsauer effect" as in the curves for $r_0=0.8$, the potential is probably of the nonpermuting type with $r_0=0.8$ or larger.[6]

Capture Cross Section

During the collision of neutron and proton there is a finite probability that the neutron will be captured by the proton, forming a deuteron, radiating away the excess energy. The effective cross section for such a transition is given by the usual expression

$$q=(16\pi^4\nu^3/3c^3hv)M^2 \tag{9}$$

where v is the incoming neutron velocity and M is the effective moment,[7] either electric or magnetic, due to the transition. The wave function for the incoming neutron is normalized to unit amplitude at large distances. This means that the radial parts of the terms in the series for plane-plus-scattered wave must have the asymptotic form $(1/W^{\frac{1}{2}}r)\sin(W^{\frac{1}{2}}r-\delta_l-\frac{1}{2}l\pi)$.

From Eq. (9) can be derived the expression for the reciprocal of the mean life, before capture, of a neutron in paraffin, $(1/\tau)=nqv$, which can be measured experimentally.[8] The quantity n, the number of protons per cc in paraffin, is taken to be 0.72×10^{23}.

The expression for that part of $(1/\tau)$ due to electric dipole radiation is

$$(1/\tau)_e=\tfrac{3}{4}n\left(\frac{e^2h}{6\pi c^2m_p^2}\right)(m_e/m_p)^{\frac{1}{2}}W_r^3\left[\int\psi_0\Phi_t x\,dV\right]^2$$

where all lengths in the integral are in nuclear units. The constant W_r is the energy given off during the transition: it equals $W-W_t$. The function ψ_0 is the normalized wave function for the bound state and Φ_t is the unit amplitude wave for the triplet state. The factor (3/4) comes because only the triplet state takes part in this transition.

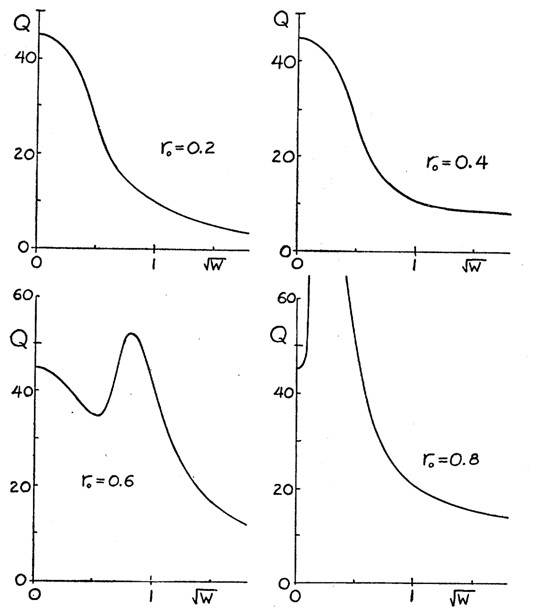

Fig. 7. Curves similar to those of Fig. 6 for $x_1=1$. Again the curves for a Majorana potential would all be similar to the first curve shown.

Values of $(1/\tau)_e$ have been computed by Breit and Condon[7] for a "square" potential hole, for a wide range of sizes. Inasmuch as our results for this quantity agree on the whole with theirs, it is not considered necessary to reproduce our curves in detail. We need only mention that the electric dipole cross section is more sensitive to changes in x_1 (i.e. in the shape of the potential curve) than any of the other quantities discussed in this paper.

This part of $(1/\tau)$ is very small for small values of $W(W<1)$, which is not in accord with experiment.[8] Therefore the part due to magnetic dipole radiation must predominate at low neutron velocities. The expression for the effective magnetic dipole for the allowed transition (singlet to triplet) is[8]

$$M_m=3^{\frac{1}{2}}(eh/4\pi m_pc)(g_p-g_n)\int\psi_0\Phi_s\,dV. \tag{10}$$

where g_p is the proton g factor, g_n that for the neutron, and c.g.s. units are used in the integral. We have used the following values[9] for the g's: $g_p=5.7$; $g_n=-4.0$.

[6] None of these curves are in agreement with the results of Goldhaber, Nature **137**, 824 (1936), who reports a cross section of 3.5 at $W=0.4$ nuclear unit. In order to check both Dunning's and Goldhaber's results, we should have to use a value of r_0 much larger than unity; in which case none of the results on inelastic collision, discussed later in this paper, would agree with experiment. This difficulty occurs whether Majorana forces are assumed or not.

[7] For a discussion of the validity of this formula, see Breit and Condon, Phys. Rev. **49**, 904 (1936).

[8] Fermi, Phys. Rev. **48**, 570 (1935).

[9] We are indebted to Professors O. Stern and I. I. Rabi on this point. Probable values of magnetic moment for the proton and the deuteron are 2.85 and 0.85 nuclear magnetons, respectively.

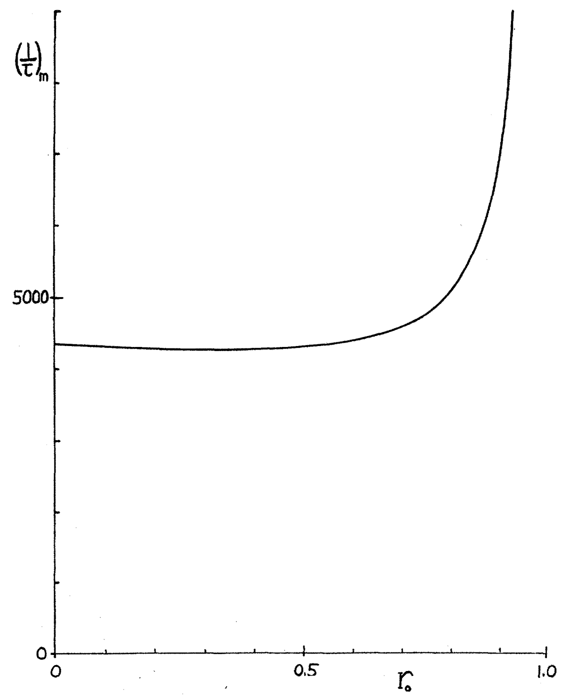

FIG. 8. Reciprocal mean life time of a neutron in paraffin. The observed value is of the order of 10^4 sec^{-1}.

If the singlet potential is equal to that of the triplet this moment will be zero, due to the orthogonality of ψ_0 and Φ_s. If the singlet field is used which gives the correct value of the elastic cross section, by Eq. (8), then M_m will not be zero. Transforming the integration into nuclear units and combining terms, the expression for that part of $(1/\tau)$ which is due to magnetic dipole radiation becomes

$$(1/\tau)_m = \tfrac{1}{4}n(e^2h/8\pi c^2 m_p^2)(m_e/m_p)^{\frac{3}{2}} \times W_r^3(g_p-g_n)^2[\int\psi_0\Phi_s dV]^2 \quad (11)$$

where ψ_0 and Φ_s are given in terms of the functions defined in Eqs. (3) and (4) for the proper values of k.

A curve of $(1/\tau)_m$ for very slow neutrons is shown in Fig. 8 as a function of r_0 for $x_1=0$, for the case where there is no stable singlet state. In contrast to $(1/\tau)_e$, the quantity $(1/\tau)_m$ is practically independent of r_0 for r_0 less than 0.8, and of W when W is less than unity; having a value of 4300 sec.$^{-1}$ within this range. The curve for $x_1=1$ is practically identical with the one shown. The corresponding curves for the case of a stable singlet state are also similar, the value of $(1/\tau)_m$ for the range ($W<1$, $r_0<0.8$, $x_1<1$) being 3600 sec.$^{-1}$. In this latter case one must include the probability of transition from the triplet incoming state to the stable singlet state; but this quantity is negligible due to the small value of the factor W_r^3 in the expression corresponding to Eq. (11) for such a transition.

Therefore the magnetic dipole type of capture predominates for neutron energies below about 500,000 ev. The results obtained here indicate that measurements of slow neutron capture cross section will have to be much more accurate than is possible at present before this experiment, by itself, can decide whether the deuteron has a stable singlet state or not. The two values obtained, 4300 and 3600 sec.$^{-1}$, both check the experimental value within its probable error. Presumably other experiments can be devised which are more sensitive to the presence or absence of a stable singlet state.

PHOTOELECTRIC DISINTEGRATION OF THE DEUTERON

It can be shown[10] that the cross section for the disintegration of the deuteron by photons of frequency ν (where $h\nu=W_r=-W_t+W$ in nuclear units) is related to the cross section for the inverse process, the capture of neutrons, discussed in the previous section. In fact the disintegration cross section is

$$\sigma = 15.2\times 10^{-6}(W_r+W_t)^{\frac{1}{2}} (1/W_r)^2[(1/\tau)_m+(1/\tau)_e],$$

in nuclear units, and where $(1/\tau)_e$ and $(1/\tau)_m$ are the quantities considered in the previous section, considered as functions of $W=W_r+W_t$.

We have seen that when $W\ll 1$ and when $r_0<0.8$ and $x_1<1$, the magnetic dipole capture predominates. Therefore for photons of energy just above the photoelectric threshold, the disintegration cross section is

$$\sigma \simeq \begin{Bmatrix} 0.053 \\ 0.064 \end{Bmatrix} (W_r+W_t)^{\frac{1}{2}}(1/W_r)^2 \quad \text{nuclear units,} \quad (12)$$

where the two constants are for the presence or the absence of a stable singlet state, respectively.

The only experimental data available[11] leads

[10] Bethe and Peierls, Proc. Roy. Soc. **A148**, 146 (1935).
[11] Chadwick and Goldhaber, Proc. Roy. Soc. **A151**, 479 (1935).

to a value for σ of the order of 10^{-3} nuclear units for $W_r=5.2$, an energy outside the range of validity of Eq. (12). The results of Breit and Condon[7] show that at this energy, the part of σ due to electric dipole transition is as large as that due to magnetic dipole, or perhaps somewhat larger. Utilizing this, the computed value of σ comes out to be about 0.005 nuclear units, in fair agreement with the experimental result.

Conclusions

It has often been stated that the general characteristics of the interaction between neutron and proton are more or less independent of the shape of the field and of the size of the potential hole. We have shown, in a specific case which can be calculated by analytic means, that this is indeed the case. Once the constants are chosen so as to give the correct binding energy and elastic cross section for slow neutrons, the computed values of $(1/\tau)$ and σ (for low energy photons) are insensitive to variation of the remaining parameters r_0 and x_1 within quite wide ranges of values of these parameters. Nor are the results particularly sensitive to the presence of a Majorana exchange operator in the potential, or to whether there is a stable singlet state or not. Empirical determination of these properties will require the experimental measurement of quantities which depend more markedly on the angular momentum than do those discussed in the present paper.

The above result is a negative one, and as such is not particularly satisfying. However several positive conclusions can be drawn from our results. One is that the neutron-proton interaction must contain a term dependent on the coupling between the spins of proton and of neutron. Only in this way can be explained the large elastic cross section for slow neutrons, and the fact that the mean life of a slow neutron in the presence of protons is independent of its velocity, a property of magnetic dipole capture.

It is also apparent that the results are internally consistent: i.e., if one assumes that r_0 is less than 0.8, that x_1 is less than unity and that V_s is not equal to V_t, then a choice of parameters to fit two of the four bits of experimental data discussed above gives results which fit the other two experimental values.

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Electronic Energy Bands in Sodium Chloride

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The Wigner and Seitz method of cellular potentials has been applied to the calculation of wave functions in NaCl. A renormalized Hartree field has been used around the Cl and the Prokofjew field around the Na. The relative heights of the potentials are determined by use of Madelung's number. The problem of joining the functions at the cell boundaries has been treated by the Slater method of fitting ψ and ψ' at midpoints. For the outer Cl electrons a reasonable approximation is to join at Cl—Cl midpoints only. This gives rise to a face-centered lattice for which solutions of the Slater conditions have been found by Krutter. Several new solutions have been derived which allow fairly accurate energy contours in momentum space to be drawn for the Cl $3p$ band. If the joining is made at Cl—Na midpoints alone, a large number of unsatisfactory zero-width bands arise. When both Cl—Cl and Cl—Na midpoints are used, the boundary conditions can be treated only for special cases. For these they are consistent with the Cl—Cl solutions. Several attempts to calculate the ultraviolet absorption frequency are described and the difficulties involved are discussed.

I. Introduction*

There has been a great advance in the calculation of wave functions in solids in the last four years. The initial impetus was derived from the contributions of Wigner and Seitz.[1] From a consideration of the Pauli principle they concluded that an electron in a monovalent metal

* The writer is indebted to Dr. Seitz for discussions of this paper and that by Douglas H. Ewing and Frederick Seitz. The viewpoints of the two papers differ in that the ionic picture of the lattice has been adhered to in this paper and no attempt to obtain a self consistent field has been made.

[1] Wigner and Seitz, Phys. Rev. **43**, 804 (1933).