The Dependence of Nuclear Forces on Velocity

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It is shown that the Majorana force may be considered as a special case of a neutron-proton interaction dependent on velocity and angular momentum. Any interaction of this more general form, which is non-Wignerian, and represents attraction, accounts for the saturation properties of nuclear binding in the same way as do the Heisenberg and Majorana potentials. An analysis of the binding in heavy nuclei in terms of angular momentum is given on the basis of the Fermi-Thomas statistical model. An illustration is given showing that the assumption of velocity dependent forces is also consistent with the properties of light nuclei: a particular form of interaction, containing two parameters, is found to give results much the same as those of the ordinary Majorana theory. Finally, the most general dependence of two-particle forces on spin, separation, and velocity, consistent with the conservation laws, is determined, and it is pointed out that experiments on the scattering of fast neutrons in hydrogen are capable of deciding between Majorana forces and a more general form of velocity dependence.

WITH forces of an exchange nature acting between neutrons and protons, Heisenberg was able to account for the dominating feature of nuclear stability—the fact that for heavy nuclei the binding energy per particle is nearly a constant. Majorana's modification of the interaction to account for the closed shell nature of the alpha-particle gave support for a neutron-proton force which corresponds to an exchange of the coordinates but not the spins of the two particles. In contrast with Heisenberg's potential, however, just this last feature of the Majorana force takes away the possibility of a simple picture of the neutron-proton interaction in terms of the kind of exchange familiar in the theory of molecular structure.

It is therefore interesting to notice that Majorana's interaction may be described as an ordinary potential without exchange but with dependence on both the separation and the relative velocity of the neutron and proton. To bring this fact into evidence let us describe the motion of the two particles by a wave function ψ depending on the position of their center of gravity $\mathbf{A}=(A_1, A_2, A_3)$ and on their separation $\mathbf{X}=(X_1, X_2, X_3)$. Then, when we multiply the Majorana potential $V_{12}\cdot$ into $\psi(\mathbf{A}, \mathbf{X})$, we obtain a new function of $\mathbf{A}$ and $\mathbf{X}$, given by

$$V_{12}\cdot\psi = V(\mathbf{X})\psi(\mathbf{A}, -\mathbf{X}),$$

the change from $\mathbf{X}$ to $-\mathbf{X}$ bringing about the exchange of the neutron and proton. Taylor's theorem assures us now that the operator

$$V(\mathbf{X}) \sum_{n_1, n_2, n_3} \frac{(-2X_1)^{n_1}(-2X_2)^{n_2}(-2X_3)^{n_3}}{n_1!n_2!n_3!}\left(\frac{\partial}{\partial X_1}\right)^{n_1}\left(\frac{\partial}{\partial X_2}\right)^{n_2}\left(\frac{\partial}{\partial X_3}\right)^{n_3}$$

has all the properties of $V_{12}\cdot$, provided that ψ and its derivatives are continuous. But the operator $\partial/\partial\mathbf{X}$ represents just $i/\hbar$ times the relative momentum, $\mathbf{P}$, of the two particles. Consequently, we have in

$$V_{12}\cdot = V(\mathbf{X}) \sum_{n_1, n_2, n_3} (-2iX_1/\hbar)^{n_1}(-2iX_2/\hbar)^{n_2}(-2iX_3/\hbar)^{n_3}P_1^{n_1}P_2^{n_2}P_3^{n_3}/n_1!n_2!n_3! \quad (1)$$

the representation of the Majorana interaction in terms of relative momentum and separation of the neutron and proton.

The expression (1) shows the very special dependence on momentum of Majorana's force, and at the same time suggests the question: What are the consequences for nuclear structure of assuming between neutrons and protons a more general interaction. In the following we attempt partially to answer this question.

If we assume an interaction, $V_{12}\cdot$, depending on momentum and separation in a general way, $V(\mathbf{X}, \mathbf{P})$, we will find it convenient to represent it not as a differential operator by replacing

$\mathbf{P}$ by $-i\hbar\partial/\partial\mathbf{X}$, but as an integral operator,[1] according to the equation

$$V_{12}\cdot\psi(x)=\int J(\mathbf{X},\,\xi)\psi(\xi)d\xi, \qquad (2)$$

where

$$J(\mathbf{X},\,\xi)=h^{-3}\int V(\mathbf{X},\,\mathbf{P})\exp\{i\mathbf{P}(\mathbf{X}-\xi)/\hbar\}d\mathbf{P}. \quad (3)$$

As is well known, the Majorana force in this representation is simply $J(\mathbf{X},\,\xi)=V(\mathbf{X})\delta(\mathbf{X}+\xi)$, and the Wigner force $V(\mathbf{X})\delta(\mathbf{X}-\xi)$.

The general interaction "kernel,"[2] $J(\mathbf{X},\,\xi)$, may be viewed as a matrix of (continuously) many rows and columns. The matrix element at the point $\mathbf{X}$, ξ determines that part of the time rate of change at ψ at $\mathbf{X}$ which is proportional to the probability amplitude for the two particles being at some other separation ξ. To be in accord with the law of conservation of energy, $J(\mathbf{X},\,\xi)$ must be self-adjoint: $J(\xi,\,\mathbf{X})=J^*(\mathbf{X},\,\xi)$. Because of the fact that J is independent of the position of the center of gravity, conservation of total linear momentum is automatically maintained. The requirement of constancy of angular momentum is equivalent to saying J must be independent of the orientation of the reference system in space; i.e., J depends on the magnitudes r and ρ of the vectors $\mathbf{X}$ and ξ and the angle θ_{12} between them, but not on the orientation of these vectors in space.

What conclusions of a general nature can we draw from the binding energies of heavy nuclei as to the interaction $J(\mathbf{X},\,\xi)$ between neutrons and protons? Following Majorana, let us apply to a heavy nucleus the Fermi-Thomas statistical method, in the form given it by Dirac.[3] In the first approximation, the wave function for a nucleus composed of P protons and N neutrons is given by

$$(P!)^{-\frac{1}{2}}(N!)^{-\frac{1}{2}}\begin{vmatrix}\Phi_1(1) & \cdots & \Phi_1(P)\\ \cdot & \cdot & \cdot\\ \Phi_P(1) & \cdots & \Phi_P(P)\end{vmatrix}\cdot\begin{vmatrix}\Psi_1(1) & \cdots & \Psi_1(N)\\ \cdot & \cdot & \cdot\\ \Psi_N(1) & \cdots & \Psi_N(N)\end{vmatrix}.$$

For the potential energy due to the neutron-proton interaction we obtain

$$V_{np}=\sum_{i,\,k}\int\Phi_i^*(p)\Psi_k^*(n)\,V_{ik}\cdot\Phi_i(p)\Psi_k(n)d\tau_pd\tau_n.$$

If we may neglect spin forces, we have $\Phi_i(p)=f_i(\sigma_p)\varphi(x_p)$, etc. On expressing the integral in terms of the position, $\mathbf{R}$, of the center of gravity of the typical neutron and proton, and the separation $\mathbf{X}$ of the two particles, we then have

$$V_{np}=\int\left\{\sum_{i=1}^{P}\varphi_i^*\left(\mathbf{R}+\frac{\mathbf{X}}{2}\right)\varphi_i\left(\mathbf{R}+\frac{\xi}{2}\right)\right\}J(\mathbf{X},\,\xi)\left\{\sum_{k=1}^{N}\psi_k^*\left(\mathbf{R}-\frac{\mathbf{X}}{2}\right)\psi_k\left(\mathbf{R}-\frac{\xi}{2}\right)\right\}d\mathbf{X}d\xi d\mathbf{R}. \qquad (4)$$

Applying the Fermi-Thomas statistical treatment, we express the density of particles at any point in terms of the maximum momentum of the particles there through the relations

$$\rho_p(\mathbf{x}_p)=2h^{-3}\int_0^{\mathbf{P}_p(\mathbf{x}_p)}d\mathbf{P}\quad\text{and}\quad\rho_n(\mathbf{x}_n)=2h^{-3}\int_0^{\mathbf{P}_n(\mathbf{x}_n)}d\mathbf{P}'.$$

Analogously, following Dirac, we express the mixed proton and neutron densities in the brackets in (4) by

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[1] The well-known equivalence of the two representations may be demonstrated by partial integrations, as shown for example by Dirac. (In the expansion of $V(\mathbf{X},\,\mathbf{P})$, the order of factors is taken to be $\mathbf{X}^m\mathbf{P}^n$.)

[2] Following the language of integral equations.

[3] P. A. M. Dirac, Proc. Camb. Phil. Soc. **26**, 376 (1930). See also W. Heisenberg, *Report of 1933 Solvay Congress* (Paris, 1934).

$$2h^{-3}\int_0^{\mathbf{P}_p(\mathbf{R}+(\mathbf{X}+\xi)/4)} \exp\{i\mathbf{P}(\xi-\mathbf{X})/2\hbar\}d\mathbf{P} \tag{a}$$

and

$$2h^{-3}\int_0^{\mathbf{P}_n(\mathbf{R}-(\mathbf{X}+\xi)/4)} \exp\{i\mathbf{P}'(\mathbf{X}-\xi)/2\hbar\}d\mathbf{P}', \tag{b}$$

respectively. Now if the density of particles in the nucleus is very great, and if further $\rho_n > \rho_p$, as is in general the case, then the mixed densities (a) and (b) behave with respect to $\mathbf{X}-\xi$ very nearly as δ-functions, but (b) dominates, having a narrower width and higher peak. Thus the integral giving the potential energy of the nucleus,

$$V_{np}=\int d\mathbf{R}\int\int d\mathbf{X}d\xi J(\mathbf{X},\xi)\int_0^{\mathbf{P}_p(\mathbf{R}+(\mathbf{X}+\xi)/4)} 2h^{-3}d\mathbf{P}\int_0^{\mathbf{P}_n(\mathbf{R}-(\mathbf{X}+\xi)/4)} 2h^{-3}d\mathbf{P}' \exp\{i(\mathbf{P}'-\mathbf{P})(\mathbf{X}-\xi)/2\hbar\},$$

only depends on the value of $J(\mathbf{X}, \xi)$, in the neighborhood of $\xi \doteqdot \mathbf{X}$, and we have approximately

$$V_{np}=\int d\mathbf{R}2\rho_p\int 8d\mathbf{X}J(\mathbf{X},\mathbf{X}) \tag{5}$$

under two conditions:

(A) that the neutron and proton densities do not vary greatly within distances of the order of the range of action of J; and

(B) that $J(\mathbf{X}, \xi)$ shall not be too narrow a function in its dependence on $\mathbf{X}-\xi$; or more precisely, that for a fixed ξ, J shall not vary rapidly when $\mathbf{X}-\xi$ is varied in the range 0 to $\sim 2\hbar/P_n$ (width of the mixed density function (b)).

Condition (A) is the usual requirement for the accuracy of the statistical method. Condition (B), however, is definitely not satisfied by an interaction of the Wigner type, $J(\mathbf{X}, \xi) = V(\mathbf{X})\delta(\mathbf{X}-\xi)$. In this case, as is well known, we obtain instead of (5) the expression

$$V_{np}=\int\int d\mathbf{R}d\mathbf{X}\rho_p(\mathbf{R}+\mathbf{X}/2)V(\mathbf{X})\rho_n(\mathbf{R}-\mathbf{X}/2),$$

in contradiction with the saturation character of nuclear binding. On the other hand, the Majorana potential, $J(\mathbf{X}, \xi)=V(\mathbf{X})\delta(\mathbf{X}+\xi)$, is in agreement with (B), and (5) gives the often demonstrated result:

$$V_{np}=\int d\mathbf{R}2\rho_p V(0).$$

At the same time we see from (5) that the Majorana force is a very special one, and that *any interaction depending upon* $\mathbf{X}$ *and* ξ *in accordance with condition B will equally well account for the saturation effect in nuclear binding.*

How this result comes about is most easily visualized by analyzing into groups of different relative momenta all the neutrons which interact with a given proton in a heavy nucleus. On the Fermi-Thomas statistical model, there are in the small element of volume $d\tau_p$ just $N_p=2h^{-3}(4\pi/3)P_p{}^3d\tau_p$ protons, distributed uniformly in momentum and direction of motion up to a certain maximum momentum, P_p, and similarly there are in $d\tau_n$, $N_n=2h^{-3}(4\pi/3)P_n{}^3d\tau_n$ neutrons. The number of protons in $d\tau_p$ which are in interaction with neutrons at a given distance r, having with respect to them relative momenta p_r parallel to r and p_a perpendicular to r, is determined by the volume common to two spheres of radii $P_p/2$ and $P_n/2$ separated by a distance $p=(p_r{}^2+p_a{}^2)^{\frac{1}{2}}$. On expressing p_a in terms of the quantum number L giving the mutual angular momentum of neutron and proton, we find (provided that the neutron density does not vary appreciably with r) that the average number of neutrons at a distance between r and $r+dr$ from a given proton, having with respect to it an angular momentum $L\hbar$ and a radial momentum in the range p_r to p_r+dp_r, is given by

$$\rho_L(p_r, r)dp_rdr=(2L+1)(dp_rdr/h)16f(P_p{}^{-1}[p_r{}^2+(L+\tfrac{1}{2})^2\hbar^2/r^2]^{\frac{1}{2}}), \tag{6}$$

where

$$f(x)=1 \qquad \text{if } x\leqslant P_n/2P_p-\tfrac{1}{2},$$

$$f(x)=\frac{x^3}{2}-\frac{3x}{4}\left(1+\frac{P_n^2}{P_p^2}\right)+\tfrac{1}{2}\left(1+\frac{P_n^3}{P_p^3}\right)-\frac{3}{32x}\left(\frac{P_n^2}{P_p^2}-1\right)^2 \qquad \text{if } x \text{ is between limits,}$$

$$f(x)=0 \qquad \text{if } x\geqslant P_n/2P_p+\tfrac{1}{2}.$$

A neutron-proton potential $V_L(r, p_r)$ dependent on velocity and angular momentum gives rise to an interaction energy

$$\sum_L \int V_L(r, p_r)\rho_L(p_r, r)dp_r dr$$

between the given proton and all neutrons. However, to avoid complications arising from the operator nature of a quantum-mechanical momentum, we transfer this expression to the form

$$\sum_L \int J_L(r, \rho)\sigma_L(\rho, r)drd\rho, \tag{7}$$

where $J_L(r, \rho)$ is the interaction kernel, related to $V_L(r, p_r)$ by an equation similar to (3), and $\sigma_L(\rho, r)$ is a mixed density matrix, connected with $\rho_L(p_r, r)$ by an analogous relation.[4] In particular, the diagonal values of the density matrix determine the number of neutrons of a given L in a range of distance, dr:

$$\sigma_L(r, r)dr=\int \rho_L(p_r, r)dp_r dr.$$

Fig. 1 shows the radial density, $\sigma_L(r, r)$, of neutrons of various angular momenta. (For definiteness, we have taken the density of neutrons equal to the density of protons and the volume of a nucleus of atomic weight A as $(4\pi/3)(1.48\times10^{-13})^3A$.) As a check, the density summed over all angular momenta (heavy curve) is compared with the expected value $4\pi r^2\rho_n$ (lower dotted curve).

Let us suppose we have a Wigner type of interaction between neutrons and protons; it has the same dependence on r for all values of the angular momentum, and expression (7) for the potential energy of the given protons becomes

$$\sum_L \int V(r)\sigma_L(r, r)dr.$$

If we increase the density of neutrons in the nucleus, all the curves in Fig. 1 are increased in height and move in to smaller values of r, so that the interaction energy is increased in proportion to the neutron density. On the other hand, a Majorana interaction gives attraction for the neutrons of angular momentum $L=0$, repulsion for $L=1$, attraction for $L=2$, etc. Thus the total potential energy per proton is an alternating sum,

$$\sum_L (-1)^L \int V(r)\sigma_L(r, r)dr,$$

whose value is not changed by an increase in the neutron density, essentially because the greater magnitude of the densities, σ_L, is balanced by the increased cancelation of terms of opposite sign which occurs as the curves move in to smaller values of r. This is, of course, only another way of putting Eq. (5), which states that in the general non-Wignerian case the interaction energy per proton is

$$16\int J(\mathbf{X}, \mathbf{X})d\mathbf{X}=16\sum_L(2L+1)\int J_L(r, r)dr, \tag{8}$$

a result independent of the neutron density.

[4] The value of the mixed density matrix $\sigma_L(\rho, r)$ may also be obtained from the equation just before (5) by expressing J in terms of its components J_L and carrying out the integration over θ_{12}. Otherwise the mixed density matrix is not uniquely defined in the text (unless the neutron and proton densities are everywhere constant). In the general case, where these densities depend on distance, $\sigma_L(r, \rho)$ should be regarded as giving, not the density of neutrons referred to a given point in proton space, but the density of neutron-proton pairs of a given mutual angular momentum referred to a given position, $\mathbf{R}$, of the center of gravity of the pairs. Then the σ_L's may be consistently defined. The total mixed density matrix $\sigma(\mathbf{X}, \boldsymbol{\xi}; \mathbf{R})$ is given by

$$\{\sum_i^P \varphi_i^*(\mathbf{R}+\mathbf{X}/2)\varphi_i(\mathbf{R}+\boldsymbol{\xi}/2)\} \times\{\sum_{k=1}^N \psi_k^*(\mathbf{R}-\mathbf{X}/2)\psi_k(\mathbf{R}-\boldsymbol{\xi}/2)\},$$

and the σ_L's may be found from this by inverting an equation very similar to Eq. (9) below.

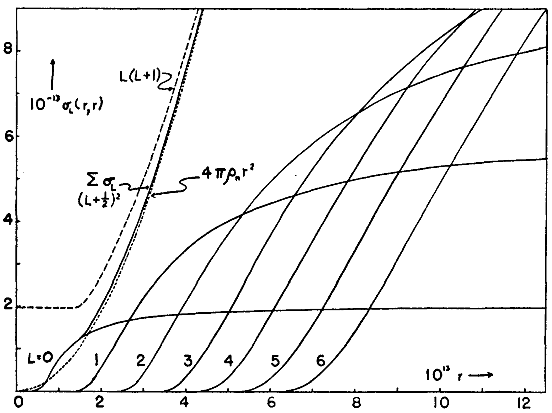

FIG. 1. Density of neutrons of various angular momenta around a given proton, using $(L+1/2)^2\hbar^2$ for the square of the angular momentum. Use of $L(L+1)\hbar^2$ gives the upper dotted curve, in disagreement with the expected value $4\pi\rho_n r^2$ (lower dotted curve) for the total density.

The Majorana force, which is independent of the relative radial momentum p_r of the two interacting particles, since it depends on ρ as $\delta(r-\rho)$, is nevertheless a very singular type of interaction, for it relies entirely upon the $(-1)^L$ dependence of potential upon angular momentum to bring about the saturation effect in neutron-proton binding. This is seen in the fact that it gives a divergent result for the right-hand side of (8) although the left-hand side converges. Indeed, any dependence of interaction upon angular momentum alone, other than the Majorana type, must make the energy per proton dependent on the density of neutrons.

In other words, any interaction which is to give saturation binding, and is not to be of the Majorana type, must depend on radial momentum; only then does the right-hand side of Eq. (8) exist. The criterion which we have now developed for a satisfactory interaction, that the right-hand side of (8) shall converge, may seem to add restrictions to the earlier condition B (following Eq. (5)), but in fact it causes difficulties only for the Majorana force, which is a special singular case, and would have been realized as such if we had not interpreted condition B liberally above. The dependence of the general interaction on radial momentum makes it clear that if the nuclear density is increased, resulting in greater mean velocities of neutrons and protons, constancy of the energy per proton is ensured by the change of potential with velocity. Indeed, if the interaction kernel, $J(\mathbf{X}, \xi)$, is not of the singular Majorana type, we can conclude that the potential $V(\mathbf{X}, \mathbf{P})$ must decrease in magnitude for sufficiently high velocities, for it represents the Fourier transform of a regular function.

We obtain more detailed information as to the neutron-proton interaction by considering the simple two-body problem involved in the stability of the deuteron and the scattering of neutrons by protons. Referred to a frame of reference in which the center of gravity of the two particles is at rest, the wave equation is

$$(\hbar^2/2\mu)\nabla^2\psi+E\psi-\int J(\mathbf{X}, \xi)\psi(\xi)d\xi=0.$$

This equation may be separated in polar coordinates, r, θ, φ, by writing ψ as a spherical harmonic $Y_L(\theta, \varphi)$ of order L ($L\hbar$ being the angular momentum of the system) times an undetermined function, $f_L(r)/r$, and by analyzing the total interaction $J(\mathbf{X}, \xi)=J(r, \rho, \theta_{12})$ into parts referring to the interaction of two particles of definite angular momentum, thus:

$$J(r, \rho, \theta_{12}) =\sum_L(2L+1)P_L(\cos\theta_{12})J_L(r, \rho)/4\pi r\rho. \quad (9)$$

The result is that $f_L(r)$ satisfies the equation

$$(\hbar^2/2\mu)f''(r)+[E-L(L+1)\hbar^2/2\mu r^2]f(r) =\int J_L(r, \rho)f(\rho)d\rho. \quad (10)$$

One fact at once emerges of interest in connection with the analysis of scattering experiments: From the behavior of two particles with angular momentum $L\hbar$ we cannot in general, without further information, draw conclusions as to the behavior of the same two particles moving with some other angular momentum.

Now we know from the binding energies of H^2 and He^4 that neutrons and protons of zero angular momentum form stable configurations. Furthermore we have indications that the nuclear forces fall off rather rapidly with distance. Consequently, the first component, $J_0(r, \rho)$, of the neutron-proton interaction must be negative, and a reasonable assumption is that it varies with distance as e^{-br}. To give J the proper symmetry with respect to interchange of r and ρ, we might now take $J_0(r, \rho)$ equal to $-Ae^{-br}\delta(r-\rho)$, which represents a force independent of radial velocity. We have seen, however, that such a force is only consistent with the binding energies of heavy nuclei if it is of the Majorana type. If we do not wish to restrict ourselves to this form of interaction, we must bring in dependence on velocity, keeping J_0 symmetrical; our simplest possibility is to assume that J_0 has the form $-ae^{-b(r+\rho)}$.

For a velocity dependent interaction of the form just mentioned, the wave equation has an extremely simple solution, and there is found to be never more than one stable S state of the deuteron, whatever the values of a and b. Identifying the energy of this state with the experimental binding energy, D, of H^2, we obtain an equation giving the magnitude of the interaction in terms of its narrowness, b:

$$a/2b=(\hbar^2/M)(\kappa+b)^2, \qquad (11)$$

where $$(\hbar^2/M)\kappa^2=D.$$

The cross section, σ_0, for the collision of neutrons and protons of zero angular momentum may also be calculated exactly:

$$\sigma_0=16\pi b^2\{4k^2b^2+[b^2-k^2-(b^2+k^2)^2/(b+\kappa)^2]^2\}^{-1}, \qquad (12)$$

where $$(\hbar^2/M)k^2=E=E_0/2,$$

E_0 being the energy of the incident neutron. To calculate the total cross section σ, we must make some assumption as to the forces acting when $L>0$; the simplest possible assumption is $J_L(r, \rho)=0$ for $L>0$ (i.e., $J(r, \rho, \theta_{12})$ independent of θ_{12}). σ is then exactly equal to σ_0; however, even if J_1, J_2, etc., do not vanish, σ will be closely equal to σ_0 for neutron energies of several million volts and lower. Fig. 2 shows the dependence of cross section on neutron velocity for several values of b. The recent experiments of Fermi (capture of slow neutrons) give support for a ^3S level at $\sim+130{,}000$ ev; there is therefore little doubt that we can only obtain a complete account of neutron scattering by bringing in a spin dependence of J as is done in the Majorana theory. From binding energies of other nuclei, also, there seems to be evidence[5] for the dependence of nuclear forces on spin.

[5] E. Feenberg and J. K. Knipp, Phys. Rev. **48**, 906 (1935).

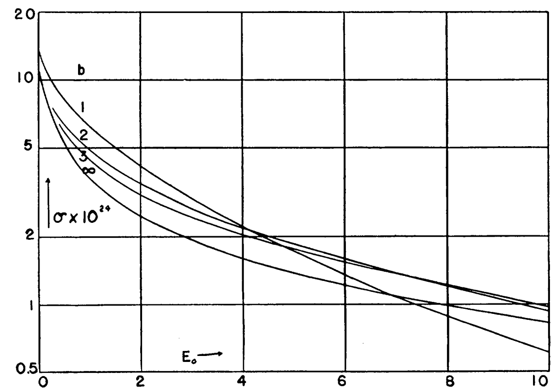

FIG. 2. Cross section for elastic scattering of neutrons in hydrogen as a function of neutron energy. Calculated on the basis of spin-dependent force of the type $-ae^{-b(r+\rho)}$ with constants adjusted to give singlet level of deuteron at 0.13 MEV, triplet level at -2.15 MEV. The curve $b=\infty$ is the same as that given on the Majorana-Heisenberg or Wigner theory.

Application of the interaction $J_0=-ae^{-b(r+\rho)}$ to the treatment of the alpha-particle also gives results not essentially different from those of the corresponding Majorana theory; we have therefore in this interaction a simple illustration of a velocity dependent force which gives quite reasonable physical results.

We bring this discussion to a close by investigating what is the most general dependence of the neutron-proton interaction kernel, $J(\mathbf{X}, \boldsymbol{\xi})$ on spin, position, and velocity consistent with the conservation laws. We denote by 1_p and $\mathfrak{n}_p=(\xi_p, \eta_p, \zeta_p)$ the unit matrix and the three anticommuting spin matrices of the proton and use 1_n and $\mathfrak{n}_n=(\xi_n, \eta_r, \zeta_n)$ for the corresponding neutron operators. Any function of the spins of the two particles is a matrix representable as a sum of the sixteen product matrices of the type $1_n\zeta_p$, with suitably chosen coefficients. In the case of the interaction operator, these coefficients must be such as to give invariance with respect to any rotation of the axes of reference; in other words, J must have the form

$$1_p1_n\cdot(\text{scalar functions, } s, \text{ of } \mathbf{X} \text{ and } \boldsymbol{\xi})$$
$$+1_p\mathfrak{n}_n\cdot(\text{vector function, } t, \text{ or } \mathbf{X} \text{ and } \boldsymbol{\xi})$$
$$+\mathfrak{n}_p1_n\cdot(\text{another vector function, } u)$$
$$+\mathfrak{n}_p\cdot(\text{a tensor function, } w, \text{ of } \mathbf{X} \text{ and } \boldsymbol{\xi})\cdot\mathfrak{n}_n.$$

We have already in Eq. (9) expressed the fact that the most general scalar invariant depends only on $|\mathbf{X}|$, $|\boldsymbol{\xi}|$, and $\mathbf{X}\cdot\boldsymbol{\xi}$, or r, ρ, and θ_{12};

similarly, the most arbitrary vector function may be written

$$t_x\mathbf{X}+it_0[\mathbf{X},\ \xi]+t_\xi\xi,$$

and the general tensor is

$$w_{xx}\mathbf{XX}+iw_{x0}\mathbf{X}[\mathbf{X},\ \xi]+w_{x\xi}\mathbf{X}\xi +iw_{0x}[\mathbf{X},\ \xi]\mathbf{X}-w_{00}[\mathbf{X},\ \xi][\mathbf{X},\ \xi]+iw_{0\xi}[\mathbf{X},\ \xi]\xi +w_{\xi x}\xi\mathbf{X}+iw_{\xi 0}\xi[\mathbf{X},\ \xi]+w_{\xi\xi}\xi\xi,$$

where the t's and w's are functions only of r, ρ and θ_{12}, and $\mathbf{X}\xi$ is a tensor in the sense that it gives the scalar $(\mathbf{X}\cdot\mathbf{n}_p)(\xi\cdot\mathbf{n}_n)$ when operating on the two vectors $\mathbf{n}_p$ and $\mathbf{n}_n$. The rotational invariance of the interaction obtained in this way insures constancy of angular momentum. The conservation of total linear momentum is assured because J depends on the separation of the particles but not the position of their center of gravity; and energy is conserved if (and only if) the following conditions for the self-adjoint nature of J are satisfied:

(a), $s(\rho,\ r,\ \theta_{12})=s^*(r,\ \rho,\ \theta_{12})$, with similar relations on t_0, u_0, w_{xx}, w_{00}, and $w_{\xi\xi}$; and

(b), $t_x(\rho,\ r,\ \theta_{12})=t_\xi^*(r,\ \rho,\ \theta_{12})$, and corresponding connections between u_x and u_ξ, w_{x0} and $w_{\xi 0}$, $w_{x\xi}$ and $w_{\xi x}$; and w_{0x} and $w_{0\xi}$.

Eleven functions of r, ρ, and θ_{12} are therefore required to describe the spin interaction in the general case. In contrast with this is the situation when the interaction is restricted to depend on $\mathbf{X}$ and ξ as $\delta(\mathbf{X}+\epsilon\xi)$. $J(\mathbf{X},\ \xi)$ reduces to

$$\{s(r)+(\mathbf{X}\cdot\mathbf{n}_n)t(r)+(\mathbf{X}\cdot\mathbf{n}_p)u(r) +(\mathbf{X}\cdot\mathbf{n}_p)(\mathbf{X}\cdot\mathbf{n}_n)w_{xx}(r)+(\mathbf{X}\cdot[\mathbf{n}_p,\ \mathbf{n}_n])w_A(r) +(\mathbf{n}_p\cdot\mathbf{n}_n)w_S(r)\}\,\delta(\mathbf{X}+\epsilon\xi)$$

because we have only one vector instead of three with which to form invariants. Various special cases are summarized as follows:

(1) $\epsilon=-1$; only $s(r)$ different from zero (Wigner);
(2) $\epsilon=\ \ 1$; all functions vanish except $s(r)=w_S(r)$ (Heisenberg);
(3) $\epsilon=\ \ 1$; only $s(r)$ nonvanishing (Majorana);
(4) $\epsilon=-1$; all functions zero except $s(r)=w_S(r)$ (Bartlett);
(5) $\epsilon=-1$; $w_{xx}(r)=-3\mu_p\mu_n r^{-5}$; $w_S(r)=\mu_p\mu_n r^{-3}$; all other functions zero (magnetic spin forces).

The magnetic interaction between proton velocity and neutron spin requires for its representation a more general dependence of $J(\mathbf{X},\ \xi)$ on ξ than that given by $\delta(\mathbf{X}+\epsilon\xi)$.

The proper value of the kernel, $J(\mathbf{X},\ \xi)$, required to account for the interaction between neutrons and protons (or between protons and protons, and neutrons and neutrons) is of course not arbitrary, but fixed by general physical principles of which we have not as yet a proper understanding. The fact that present treatments of nuclear binding point to neutron and proton velocities in the nucleus of the order of $c/4$ suggest a question as to whether the simple mathematical forms of the kernel given by (2) and (3) really give correctly the interaction between two nuclear particles, at distances smaller than the classical electron radius, moving with high relative velocities. In this connection it may be mentioned that, in a paper now in preparation for publication, it is shown on the basis of present theory that the interaction between two normal alpha-particles of not too high energy is representable in terms of an equivalent potential $J(\mathbf{X},\ \xi)$. This kernel, whether derived on the assumption of Wigner or Majorana forces between neutrons and protons or the more general type of interaction we have considered in this paper, is itself definitely not representable in its dependence on ξ by $\delta(\mathbf{X}+\epsilon\xi)$, and consequently is not a force of the Wigner or Majorana type.

In summary, we have found:

(1) that the most general interaction between two particles consistent with the conservation laws requires for its specification ∞^3 numbers (dependence of force on distance, relative velocity, and angular momentum) in contrast with a Wigner or Majorana potential, which is described by only ∞ numbers (dependence on distance);

(2) that a simple form of the interaction, dependent on velocity, and acting only between neutrons and protons of zero mutual angular momentum, is well adapted to the treatment of the binding energies of light nuclei; and

(3) that this particular interaction is only one instance of a whole group of non-Wignerian potentials, any one of which will account for the saturation effect in nuclear binding.

A decision between the Majorana force and a force of the more general nature considered above would appear possible as follows:

(1) from the dependence of the neutron-proton collision cross section upon velocity for high energy neutrons. A swifter falling off is to be expected for velocity dependent forces.

(2) the consequences of the Majorana interaction for the detailed structure of heavy nuclei are somewhat different from those given by the general non-singular type of force. Detailed calculations are needed to show which features of nuclear structure are most sensitive to the characteristic differences between the two kinds of interaction.