An Attempt to Calculate the Number of Energy Levels of a Heavy Nucleus

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Experiments on slow neutrons, and theoretical considerations of Bohr have shown that heavy nuclei possess an enormous number of energy levels which are very closely spaced if the nucleus is highly excited. A crude method is suggested for calculating the spacing between these levels. The method is statistical: The individual nuclear particles are supposed to move in a simple potential hole, and the energy of the complete nucleus is supposed to be the sum of the energies of the individual particles. A critical discussion of these assumptions is given in section 5. The problem then reduces itself to the calculation of the "entropy" of a Fermi gas containing a given number of particles A and having a given excitation energy Q above the zero point energy of the Fermi gas (cf. section 2 and 3). This calculation gives the total number of levels of the complete nucleus in a given energy interval irrespective of the angular momentum, which will, for most of the levels, be very large. For the theory of neutron capture, it is necessary to calculate the density of nuclear levels with a *given angular momentum* I (section 4). The spacing of nuclear levels is found to depend on the product of the mass number A and the excitation energy Q of the nucleus, and to be roughly given by

$$\Delta = 4.1 \cdot 10^6 x^4 e^{-x}/(2I+1) \text{ volts}$$
$$x = (AQ)^{\frac{1}{2}}/2.20,$$

Q being expressed in MV and I being the nuclear spin. For the capture of slow neutrons by nuclei of medium weight (A around 100), Δ is of the order 50 to 500 volts. The spacing between adjacent levels decreases rapidly with increasing atomic weight. For given atomic weight, the spacing of the nuclear levels responsible for neutron capture is wider if the capture leads to the formation of a radioactive nucleus than if a stable nucleus is formed. This explains the experimental fact that only moderately large cross sections are found for the capture of thermal neutrons leading to radioactive nuclei while the very largest cross sections are all connected with the formation of stable nuclei. The dependence of the spacing on various factors is discussed (section 6); the results seem to be in qualitative agreement with experiment.

1. STATEMENT OF PROBLEM

BOHR[1] has given strong reasons for the existence of a very great number of closely spaced energy levels for a highly excited heavy nucleus. Breit and Wigner[2] and Bohr[1] have shown that the assumption of such levels leads automatically to a completely satisfactory explanation of all phenomena connected with slow neutrons, in particular the selective absorption, the high capture cross section and the large ratio of capture to scattering. Various investigators[3] have measured the position of the neutron resonance levels for several substances. The resonances are found to lie at neutron energies ranging from about 0.1 volt (Cd) to about 50 volts (I). These measurements indicate that the spacing between adjacent energy levels of the nuclei concerned in the energy region investigated is very small, maybe of the order of 100 volts or even less.

It is the purpose of this paper to give some fairly crude calculations leading to an estimate of this spacing. We consider a nucleus containing N neutrons and Z protons. The total number of particles (mass number) will be denoted by $A=N+Z$. The nucleus will have a certain ground state of energy U_0. We are interested in the energy levels of the nucleus which lie by a certain amount Q higher than the ground state, and we ask for the density of energy levels in this region, i.e., for the number of levels between Q and $Q+dQ$ which we may call $\rho(Q)dQ$. $1/\rho(Q)$ will then be the average spacing between neighboring levels.

We shall be particularly interested in such values of Q which are just sufficient to dissociate the given nucleus A into a neutron and a residual nucleus of atomic weight $A-1$. These energy levels will be important for the capture of slow neutrons by the nucleus $A-1$. In general, the "dissociation energy" Q, i.e., the energy set free when a neutron is captured by the nucleus $A-1$, will be of the order 8 MV. This figure applies if the packing fractions of the nuclei $A-1$ and A are about equal, and represents the excess of the

[1] Bohr, Nature **137** (1936).
[2] Breit and Wigner, Phys. Rev. **49**, 519 (1936).
[3] Frisch and Placzek, Nature **137**, 357 (1936); Weekes, Livingston and Bethe, Phys. Rev. **49**, 471 (1936); Rasetti, Fink, Goldsmith and Mitchell, Phys. Rev. **49**, 869 (1936); Collie, Nature **137**, 614 (1936); Fermi and Amaldi, Ricerca scient. **1**, No. 7–8 (1936).

neutron mass over one mass unit. In particular cases, Q will be lower or higher than 8 MV, but it will probably lie in the interval from 5 to 10 MV for most cases (see section 6).

2. Method of Calculation

In order to estimate the "density" $\rho(Q)$ of the levels of the nucleus as a whole, we shall start from the statistical model of the nucleus; in other words, from the individual-particle picture. We are fully aware of the crudeness of this assumption, but reasons will be given below (section 5) for the belief that the density of levels will come out fairly correctly from this picture, although the wave function of a particular state of the nucleus will differ greatly from that obtained by this picture.

In this model, we shall obtain, first of all, a certain set of energy levels for the individual neutrons and protons in the nucleus. The positions of these levels will depend on the potential which we assume to act on the particles. The ground level of the *nucleus as a whole* is then obtained by filling all the lowest "individual" states with particles and leaving all the higher individual states unoccupied. An excited level of the nucleus will be obtained by taking one particle out of one of the low individual states and putting it into one of the higher individual states, or else by leaving *two* of the "low" individual states empty and having two higher individual states occupied instead, etc. In fact, for excited levels about 8 MV above the ground level of the nucleus, we shall in general have a fairly large number (of the order $\sqrt{A}$) of the individual particles "excited." Since there is a great variety of "low" individual states which may be left empty, and an equally great variety of "high" states which may be occupied by a particle, it is obvious that there will be a very large number of different ways in which a certain excitation energy of the nucleus as a whole may be realized, particularly if the energy is sufficient to have many particles excited.

The problem of finding the number of these different modes of realization is identical with the problem of finding the "probability number" (entropy) of a Fermi gas whose energy is given and has a value larger than the minimum possible energy for the gas. The solution of this problem is well-known in Fermi statistics, and it is only necessary to go one step farther in the accuracy than is usually done because usually only the logarithm of $\rho(Q)$, i.e., the entropy, is wanted while we want to calculate $\rho(Q)$ itself correctly to quantities of the relative order $1/A$ or $1/\sqrt{A}$ where A is the total number of particles.

Since we know that Fermi statistics is applicable to our problem, the probability that a given *individual-particle* state of energy ϵ is occupied, will be

$$f(\epsilon)=1/(e^{\beta(\epsilon-\zeta)}+1) \tag{1}$$

where β and ζ are two constants which have to be determined from the total number of particles N and the total energy U of the nucleus. As regards the number of particles, we have to consider neutrons and protons separately because both these numbers are given for a given nucleus. We shall, for the present, refer to neutrons alone and let U denote the total energy of the neutrons alone; U is then, of course, *not* given, but we must, later on, integrate over all possible distributions of the total nuclear energy between neutrons and protons.

The conditions determining β and ζ are then

$$N=\sum_i f(\epsilon_i), \quad U=\sum_i \epsilon_i f(\epsilon_i), \tag{2}$$

the sums extending over all possible states of an individual neutron. We now assume that the neutrons and protons are contained in a box of volume

$$\Omega=(4\pi/3)R^3, \tag{3}$$

where R is the nuclear radius. Then the number of neutron levels of energy between ϵ and $\epsilon+d\epsilon$ is given by the well-known formula

$$\varphi(\epsilon)d\epsilon=(2^{5/2}/3\pi)(MR^2/\hbar^2)^{\frac{3}{2}}\epsilon^{\frac{1}{2}}d\epsilon=(3/2)C\epsilon^{\frac{1}{2}}d\epsilon \tag{4}$$

with

$$C=(2^{7/2}/9\pi)(MR^2/\hbar^2)^{\frac{3}{2}}. \tag{4a}$$

Then (2) reduces to

$$N=(3/2)C\int f(\epsilon)\epsilon^{\frac{1}{2}}d\epsilon, \tag{5}$$

$$U=(3/2)C\int f(\epsilon)\epsilon^{\frac{3}{2}}d\epsilon. \tag{5a}$$

The number $\rho(U)dU$ of energy levels of the nucleus as a whole, between U and $U+dU$, is

found very easily from Sommerfeld's[4] derivation of the Fermi statistics. We have

$$\log \sum_{N'} \int dU' \rho(U') e^{\beta\zeta N' - \beta U'} = \sum_i \log (1+e^{\beta(\zeta-\epsilon_i)})$$

$$= (3/2) C \int \epsilon^{\frac{1}{2}} d\epsilon \log (1+e^{\beta(\zeta-\epsilon)}). \quad (6)$$

The notation is the same as in Sommerfeld's paper (Eqs. (3), (11)), except that his α has been denoted by $-\beta\zeta$. The left hand side of (6) is known to have a sharp maximum for $N'=N$ and $U'=U$. This fact makes it easy to determine $\rho(U)$ once the right hand side of (6) has been evaluated. $1/\rho(U)$ gives directly the desired spacing between the nuclear levels.

3. Evaluation of the Number of Levels

Since the excitation Q of the nucleus is small compared to the total kinetic energy U of all the nuclear particles, the "Fermi gas" is highly degenerate and the formulae known from the theory of metals may be applied. We have

$$N = C\zeta^{\frac{3}{2}}(1+(\pi^2/8)(\beta\zeta)^{-2}+\cdots), \quad (7)$$

$$U = \tfrac{3}{5} C\zeta^{5/2}(1+(5\pi^2/8)(\beta\zeta)^{-2}+\cdots). \quad (7a)$$

C may be regarded as a given constant, essentially determined by the nuclear radius. Therefor ζ is practically determined by the number of neutrons:

$$\zeta = (N/C)^{2/3}(1-(\pi^2/12)(\beta\zeta)^{-2}+\cdots). \quad (8)$$

ζ, in its turn, determines U except for a small term of the relative order $(\beta\zeta)^{-2}$. There will, therefore, be a large zero-point energy of the nucleus, plus a small additional (excitation) energy. The latter determines the constant β.

If the nucleus is in its ground state, β will be infinity. In this case, the "Fermi energy" becomes (cf. (8))

$$\zeta_0 = (N/C)^{2/3} \quad (9)$$

and the total kinetic energy of all neutrons (cf. (7a), "zero-point energy")

$$U_0 = \tfrac{3}{5} C\zeta_0^{5/2} = \tfrac{3}{5} N\zeta_0. \quad (9a)$$

From (8) and (9), we have

$$\zeta = \zeta_0(1-(\pi^2/12)(\beta\zeta_0)^{-2}+\cdots). \quad (10)$$

Inserting this into (7a), we find

$$U = \tfrac{3}{5} C\zeta_0^{5/2} + \tfrac{1}{4}\pi^2 C\beta^{-2}\zeta_0^{1/2}. \quad (10a)$$

Therefore the excitation energy is

$$Q = U - U_0 = \tfrac{1}{4}\pi^2 C\zeta_0^{1/2}\beta^{-2}, \quad (11)$$

wherefrom

$$\beta^{-2} = 4Q/\pi^2 C\zeta_0^{\frac{1}{2}} \quad (11a)$$

or, by inserting (7)

$$\beta^{-1} = (2/\pi)(\zeta_0 Q/N)^{\frac{1}{2}}. \quad (12)$$

β^{-1} is the average excitation energy of the individual particles (width of the "tail" of the Fermi distribution).

Numerically, ζ_0 turns out to be somewhat, but not very much, larger than Q. Therefore β^{-1} is of the order $QN^{-\frac{1}{2}}$. In other words, the excitation energy is, in our model, shared between $N^{\frac{1}{2}}$ particles.

The right-hand side of (6) can be transformed by partial integration; we obtain exactly (i.e., not only for large β)

$$(3/2) C \int \epsilon^{\frac{1}{2}} d\epsilon \log (1+e^{\beta(\zeta-\epsilon)}) = \beta C \int \epsilon^{\frac{3}{2}} d\epsilon/(e^{\beta(\zeta-\epsilon)}+1) = \tfrac{2}{3}\beta U. \quad (13)$$

Therefore

$$\Phi = \log \sum_{N'} \int dU' \rho(U') e^{\beta\zeta(N'-N)-\beta(U'-U)} = (5/3)\beta U - \beta\zeta N. \quad (14)$$

Inserting (7) (7a) we get

$$\Phi = \beta C\zeta^{5/2}\tfrac{1}{2}\pi^2(\beta\zeta)^{-2} = \tfrac{1}{2}\pi^2 C\beta^{-1}\zeta^{\frac{1}{2}} \quad (14a)$$

and with (11a) (9)

$$\Phi = \pi(NQ/\zeta_0)^{\frac{1}{2}}. \quad (15)$$

This formula contains the fundamental result of our calculations. The following calculations, down to the end of section 4, represent only refinements.

In order to obtain ρ from (14), (15) we remark that the argument of the logarithm on the left-hand side in (14) is essentially $\rho(U)$ because the integrand has a sharp maximum for $U'=U$. Therefore we put

$$\rho(Q) = \lambda(N, Q) e^{\Phi(Q)}, \quad (16)$$

where λ is a slowly varying function of Q. We shall determine λ by carrying out the summation

[4] Sommerfeld, Zeits. f. Physik **47**, 1 (1927).

in (14), regarding λ as constant over the range of N' and U' involved. We find then from (14)

$$\lambda(N, Q)\int dQ' \exp\left[\beta\zeta(N'-N) - \beta(U'-U)+\Phi(Q')-\Phi(Q)\right]=1, \quad (17)$$

having replaced the integration variable U' by Q'.

In order to evaluate (17), we consider the exponent

$$f(\beta', \zeta')=\beta\zeta(N'-N)-\beta(U'-U)+(5/3)\beta' U' - \beta'\zeta' N'-(5/3)\beta U+\beta\zeta N=((5/3)U'-N'\zeta')(\beta'-\beta) + \tfrac{2}{3}\beta(U'-U)-N'\beta(\zeta'-\zeta), \quad (18)$$

where β' and ζ' are the parameters corresponding to $N'U'$. Using the formulae (7), (7a) repeatedly, we find by an elementary calculation

$$(5/3)U'-N'\zeta'=\tfrac{1}{2}\pi^2 C\beta'^{-2}\zeta'^{\frac{1}{2}}, \quad (18a)$$

$$\tfrac{2}{3}(U'-U)-N'(\zeta'-\zeta)=-\tfrac{3}{4}C\zeta'^{\frac{1}{3}}(\zeta'-\zeta)^2 + \tfrac{1}{4}\pi^2 C\zeta'^{\frac{1}{3}}(\beta'^{-2}-\beta^{-2}), \quad (18b)$$

leaving out some terms of the relative order $(\beta\zeta)^{-2}$. Inserting into (18), we have

$$f=-C\beta\zeta^{\frac{1}{3}}\left[\tfrac{3}{4}(\zeta'-\zeta)^2+\tfrac{1}{4}\pi^2(\beta'-\beta)^2(\beta\beta')^{-2}\right]. \quad (19)$$

The exponential in (17) behaves thus as $e^{-a(\zeta'-\zeta)^2-b(\beta'-\beta)^2}$ as should be expected. The differences $\zeta'-\zeta$ and $\beta'-\beta$ may be expressed in terms of $N'-N$ and $Q'-Q$, respectively, using (9) and (12). Neglecting again some small terms, we get

$$f=-\frac{\pi}{2}\left(\frac{N}{\zeta_0 Q}\right)^{\frac{1}{3}}\left[\tfrac{1}{3}(N'-N)^2\frac{\zeta}{N}+\frac{1}{4}\frac{(Q'-Q)^2}{Q}\right]. \quad (19a)$$

This may be inserted into (17) and the integrations carried out. Then

$$\frac{1}{\lambda(N, Q)}=\int_{-\infty}^{+\infty}dQ'\int_{-\infty}^{\infty}dN' \exp\left[-\frac{\pi}{6}\left(\frac{N}{\zeta Q}\right)^{\frac{1}{2}}\frac{\zeta}{N}(N'-N)^2 - \frac{\pi}{8}\left(\frac{N}{\zeta Q}\right)^{\frac{1}{3}}\frac{(Q'-Q)^2}{Q}\right]=\frac{\pi}{48^{-\frac{1}{3}}\pi}\left(\frac{\zeta Q}{N}\right)^{\frac{1}{3}}\left(\frac{N}{\zeta}\right)^{\frac{1}{3}}Q^{\frac{1}{2}}=48^{\frac{1}{3}}Q. \quad (20)$$

Therefore we find finally (cf. (16), (15))

$$\rho(Q)dQ=48^{-\frac{1}{2}}e^{\pi(NQ/\zeta)^{\frac{1}{2}}}dQ/Q \quad (21)$$

for the number of states of the system composed of all the neutrons, having a total excitation energy between Q and $Q+dQ$. A similar expression holds for the protons. Therefore the total number of levels of the nucleus as a whole per dQ is

$$\rho(Q)=\int dQ_1 \exp\left(\pi(NQ_1/\zeta_1)^{\frac{1}{2}} + \pi(ZQ_2/\zeta_2)^{\frac{1}{2}}\right)/48Q_1Q_2, \quad (22)$$

where $Q_2=Q-Q_1$ is the excitation energy of the protons.

The integration is facilitated by the fact that, for all existing nuclei, practically $N/\zeta_1=Z/\zeta_2$. The Fermi energies $\zeta_1\zeta_2$ are given by (cf. 9)

$$\zeta_1=(N/C)^{\frac{2}{3}}, \qquad \zeta_2=(Z/C)^{\frac{2}{3}}, \quad (23)$$

so that

$$a_1=\pi(N/\zeta_1)^{\frac{1}{2}}=\pi C^{\frac{1}{3}}N^{1/6}, \quad a_2=\pi(Z/\zeta_2)^{\frac{1}{2}}=\pi C^{\frac{1}{3}}Z^{1/6}. \quad (24a)$$

The ratio a_2/a_1 is, even for uranium, only $(146/92)^{1/6}=1.08$ and for other nuclei even closer to unity. We put therefore

$$a_1=a_2=a=\pi C^{\frac{1}{3}}(\tfrac{1}{2}A)^{1/6} \quad (24)$$

and have

$$\rho(Q)=\int dQ_1 e^{a(Q_1^{\frac{1}{2}}+(Q-Q_1)^{\frac{1}{2}})}/48Q_1(Q-Q_1). \quad (25)$$

The exponential has a sharp maximum for $Q_1=\frac{1}{2}Q$. We may write

$$Q_1^{\frac{1}{2}}+(Q-Q_1)^{\frac{1}{2}}=\sqrt{2}Q^{\frac{1}{2}}(1-\tfrac{1}{8}(2Q_1-Q/Q)^2+\cdots), \quad (25a)$$

while the denominator may simply be replaced by $12Q^2$. The integration gives then

$$\rho(Q)=\tfrac{1}{12}(\pi/a)^{\frac{1}{2}}2^{\frac{1}{4}}Q^{-5/4}e^{\sqrt{2}aQ^{\frac{1}{2}}}. \quad (26)$$

From its definition (24a), the constant a may be written

$$a=\pi(A/2\zeta_0)^{\frac{1}{2}}, \quad (27a)$$

where ζ_0 is the average Fermi energy for protons and neutrons. Then (26) reduces to

$$\rho(Q)=\tfrac{1}{12}\sqrt{2}\zeta_0^{\frac{1}{4}}Q^{-5/4}A^{-\frac{1}{4}}e^{\pi(AQ/\zeta_0)^{\frac{1}{2}}}. \quad (27)$$

The value of ζ_0 follows from (9) and (4a):

$$\zeta_0=(A/2C)^{\frac{2}{3}}=(3^{4/3}\pi^{\frac{2}{3}}/8)(\hbar^2A^{\frac{2}{3}}/MR^2). \quad (28)$$

Now the nuclear volume is proportional to the atomic weight A, so that

$$R=r_0A^{1/3}, \quad (29)$$

where r_0 may be calculated from the experimental data on α-radioactivity. Assuming $R=9\cdot10^{-13}$ cm for the average radius of radioactive nuclei, corresponding to A about 222, we have

$$r_0=1.48\cdot10^{-13}\text{ cm} \quad (29a)$$

and $$\zeta_0=(3^{4/3}\pi^{\frac{2}{3}}/8)(\hbar^2/Mr_0^2)=21.5\text{ MV}. \quad (30)$$

Putting now

$$x=\pi(AQ/\zeta_0)^{\frac{1}{2}}=(AQ/2.20)^{\frac{1}{2}} \quad (31)$$

(Q in MV), we have for the spacing between neighboring levels

$$1/\rho(Q)=12\cdot(2\pi)^{-\frac{1}{2}}Qx^{\frac{1}{2}}e^{-x}. \quad (32)$$

For medium atomic weight, let us say $A=110$, and for $Q=8$ MV, we have $x=20$ and the spacing (32) becomes

$$5\cdot8\cdot10^6\cdot4.5\cdot2\cdot10^{-9}=0.4\text{ volt}. \quad (32a)$$

This is obviously too small compared to the experimental spacing between neutron resonance levels. The reasons will be explained in the following section.

4. States with Given Angular Momentum

Most of the nuclear energy levels calculated in the preceding section will have very large angular momenta. Already the angular momentum of an individual particle in a heavy nucleus is apt to run up to about 6 (see Eq. (44a), below), and if the momenta of a fairly large number of particles are added, extremely large momenta for the nucleus as a whole may result. On the other hand, only small momenta of the nucleus A are of any importance for the capture of slow neutrons by the nucleus $A-1$. The nucleus $A-1$ will, in its ground state, have a certain total angular momentum ("nuclear spin"), say I_0. The neutron must have orbital momentum $l=0$ in order to be captured by the nucleus $A-1$, and therefore total momentum $j=\frac{1}{2}$, considering its spin. The resulting state of the "compound nucleus" A must therefore have an angular momentum $I_0\pm\frac{1}{2}$. Of all the levels of the nucleus A, we should therefore consider only those with angular momenta $I_0\pm\frac{1}{2}$; and I_0 will be small compared to the "average angular momentum" of all possible states of the nucleus A. Only a small fraction of the levels considered in the preceding section will fulfill this condition.[5]

We want to calculate the probability that a nuclear level has a given angular momentum I. In order to do this, we assume that each individual particle in the nucleus has the same angular momentum j. (The value of j will be calculated later; also we shall then consider the variation of j among the particles.) The resultant of all individual particle momenta will be the moment of the nucleus, I. Following the usual procedure, we consider the *components* of the angular momenta in a given direction z. Let m_i be the z-component of the momentum of particle i, and M the z-component of the total momentum. Then

$$M=\sum_i m_i, \tag{33}$$

We want to know the probability of a given resultant M, each value of m_i from $-j$ to $+j$ being equally probable. Provided the number n of particles is large, the probability for a given M is given by the "Gauss formula"

$$p(M)=(\tfrac{2}{3}\pi nj(j+1))^{-\frac{1}{2}}e^{-3M^2/2nj(j+1)}. \tag{34}$$

To prove this, we make use of three facts:

(1) If $p_n(M)$ is the probability that n momenta have the resultant M in the z direction, we must have the "addition theorem"

$$p_{r+s}(M)=\sum_{M_1}p_r(M_1)p_s(M-M_1)$$
$$=\int_{-\infty}^{\infty}p_r(M_1)p_s(M-M_1)dM_1. \tag{34a}$$

If this is to be generally true for arbitrary values of r, s and M, then p must be of the form

$$p_n(M)=c_ne^{-\alpha M^2/n}, \tag{35a}$$

where α is a constant independent of n while c_n depends on n.

(2) The average of M^2, *viz.*

$$\overline{M^2}=\sum_M M^2p_n(M), \tag{36}$$

must be given by

$$\overline{M^2}=\sum_{ik}\overline{m_im_k}=\sum_i\overline{m_i^2}+\sum_{i\neq k}\overline{m_im_k}=n\overline{m^2}, \tag{36a}$$

since two different m's (m_i and m_k) are independent of each other, and the average value of an individual m is zero. Now

$$\overline{m^2}=\sum_{m=-j}^{j}m^2/(2j+1)=\tfrac{1}{3}j(j+1). \tag{36b}$$

On the other hand, we have from (35a)

$$\overline{M^2}=\int p_n(M)M^2dM/\int p_ndM=n/2\alpha. \tag{37}$$

Comparing (36a), (36b), (37), we find

$$\alpha=3/2j(j+1). \tag{37a}$$

(3) The total probability must be unity:

$$\int_{-\infty}^{\infty}p_n(M)dM=1. \tag{38}$$

This fixes the constant c_n in (35a).

The number of states of given total angular momentum I is, as is well known, equal to the number of states with $M=I$, minus the number of states with $M=I+1$. Since $\rho(Q)dQ$ is the total number of nuclear states in the energy interval dQ, the number of states with a given M is $\rho(Q)p(M)dQ$ and the number of energy levels with a given I therefore

$$\rho(Q)[p(I)-p(I+1)]dQ. \tag{39}$$

[5] The importance of the angular momentum for the number of nuclear states was first pointed out to me by Dr. Placzek, to whom I am indebted for this suggestion.

Since p varies slowly with its argument M, this gives

$$\rho(Q, I)=\rho(Q)(dp/dM)_{I+\frac{1}{2}}. \quad (40)$$

Here we insert (34), carry out the differentiation and then put the exponential equal to unity, since for all cases of interest $I^2\ll nj(j+1)$. We obtain thus

$$\rho(Q, I)=\rho(Q)(2I+1)(8\pi)^{-\frac{1}{2}}(3/nj(j+1))^{3/2}. \quad (41)$$

We have now to compute $nj(j+1)\approx n(j+\frac{1}{2})^2$. We have therefore to know the average angular momentum j of the individual particles, and the number n of the particles which contribute to the resultant angular momentum of the nucleus. We know that in the ground state of a nucleus the resultant momentum is almost zero, because the momenta of the various individual particles nearly cancel each other ("closed shells"). Therefore the angular momentum of an excited state of a nucleus comes from the "tail of the Fermi distribution," i.e., from those particles whose energy is larger than the Fermi energy ζ, and from the empty states of energy smaller than ζ.

For j we have thus to take the average angular momentum of an individual particle whose energy is near ζ. We therefore calculate the number of quantum states (per unit energy) of given orbital momentum l for an individual particle of kinetic energy ζ which is enclosed in a deep spherical potential well of radius R. This problem is similar to the problem of the first appearance of an electron of orbital momentum l in the periodic system which was treated by Fermi.[6] The problem can be solved by separating the wave equation in polar coordinates, and then using the WKB (Wentzel Kramers Brillouin) method for treating the radial wave equation. This leads to the well known "quantum condition"

$$\int^R dr[(2M/\hbar^2)\epsilon-(l+\tfrac{1}{2})^2/r^2]^{\frac{1}{2}}=(n+\tfrac{1}{2})\pi, \quad (42)$$

where n is an integer and ϵ the energy of the particle. Since there is one quantum state for each integral value of n, the number of states of orbital momentum l in the energy interval $d\epsilon$ is

$$w(\epsilon)d\epsilon=\frac{d\epsilon}{\pi}\frac{d}{d\epsilon}\int^R dr\left[\frac{2M}{\hbar^2}\epsilon-\frac{(l+\frac{1}{2})^2}{r^2}\right]^{\frac{1}{2}}$$
$$=\frac{Md\epsilon}{\pi\hbar^2}\int^R\frac{rdr}{[2M\epsilon\hbar^{-2}r^2-(l+\frac{1}{2})^2]^{\frac{1}{2}}}$$
$$=\frac{d\epsilon}{2\pi\epsilon}[2M\epsilon\hbar^{-2}R^2-(l+\tfrac{1}{2})^2]^{\frac{1}{2}}. \quad (43)$$

The number of states with a given j is equal to the number of states with $l=j-\frac{1}{2}$, plus the number of states with $l=j+\frac{1}{2}$. Therefore it will be approximately proportional to

$$w'(j)=[2M\zeta R^2\hbar^{-2}-(j+\tfrac{1}{2})]^{\frac{1}{2}}, \quad (43a)$$

[6] Fermi, Zeits. f. Physik **48**, 73 (1928).

leaving out factors independent of j and putting the particle energy equal to ζ. The number of *particles* of angular momentum j will be proportional to $(2j+1)w'(j)$ because each level j has the statistical weight $2j+1$. Therefore the average value of $(j+\frac{1}{2})^2$ is

$$\overline{(j+\tfrac{1}{2})^2}=\int(2j+1)dj(j+\tfrac{1}{2})^2w'(j)/\int(2j+1)djw'(j)$$
$$=(2/5)\cdot 2MR^2\hbar^{-2}\zeta. \quad (44)$$

Inserting the value of ζ from (30) and R from (29), we find

$$\overline{(j+\tfrac{1}{2})^2}=(3^{4/3}\pi^{\frac{2}{3}}/10)A^{\frac{2}{3}}=0.93A^{\frac{2}{3}}. \quad (44a)$$

There remains the computation of n, which is the number of neutrons and protons having energies larger than ζ, plus the number of unoccupied states of energy smaller than ζ. Obviously, n is about four times the number of neutrons with energy larger than ζ. According to the Fermi distribution, this number is given by

$$\tfrac{1}{4}n=(3/2)C\int_\zeta^\infty\frac{\epsilon^{\frac{1}{2}}d\epsilon}{e^{\beta(\epsilon-\zeta)}+1}=3/2\frac{C\zeta^{\frac{3}{2}}}{\beta}(1-\tfrac{1}{2}+\tfrac{1}{3}-\cdots)$$
$$=3/2\frac{C\zeta^{\frac{3}{2}}}{\beta}\log 2 \quad (45)$$

or, inserting the value of β from (11a), and of C from (9), we have

$$n=\frac{12}{\pi}\log 2\left(\frac{QN}{\zeta}\right)^{\frac{1}{2}}=\frac{6\sqrt{2}\log 2}{\pi}\left(\frac{QA}{\zeta}\right)^{\frac{1}{2}}=1.87\left(\frac{QA}{\zeta}\right)^{\frac{1}{2}}. \quad (46)$$

For $A=100$, $Q=8$ MV and $\zeta=20$ MV, this would be about 12.

With this value for n, and (44a) for $j(j+1)$, we find for the number of nuclear levels with angular momentum I:

$$\rho(Q, I)=(2I+1)\rho(Q)(5/\log 2)^{\frac{3}{2}}\cdot 2^{-9/4}\cdot 3^{-2}(\zeta/Q)^{3/4}A^{-7/4}. \quad (47)$$

Inserting $\rho(Q)$ from (27), we have

$$\rho(Q, I)=2^{-15/4}\cdot 3^{-3}(5/\log 2)^{\frac{3}{2}}(2I+1)\zeta_0^{-1}\times(\zeta_0/QA)^2e^{\pi(AQ/\zeta_0)^{\frac{1}{2}}} \quad (48)$$

or, introducing the abbreviation $x=\pi(AQ/\zeta_0)^{\frac{1}{2}}$ from (31):

$$\rho(Q, I)=\frac{\pi^4\cdot 2^{\frac{1}{2}}}{432}\left(\frac{5}{\log 2}\right)^{\frac{3}{2}}(2I+1)\zeta^{-1}x^{-4}e^x. \quad (49)$$

The spacing between two levels of spin I is the reciprocal of this, *viz.*

$$\Delta=\frac{432}{2^{\frac{1}{2}}\pi^4}\left(\frac{\log 2}{5}\right)^{\frac{3}{2}}\frac{\zeta}{2I+1}x^4e^{-x} \quad (50)$$

or
$$\Delta=\Delta_0/(2I+1) \quad (50a)$$

with
$$\Delta_0=4.1\cdot 10^6x^4e^{-x}\text{ volts} \quad (50b)$$

and
$$x=(AQ/2.20)^{\frac{1}{2}},\quad (Q\text{ in MV}). \quad (50c)$$

In applications to the capture of slow neutrons, it should be remembered that levels with $I=I_0\pm\frac{1}{2}$ of the "final" nucleus are effective if I_0 is the "spin" of the capturing nucleus.

5. Criticism of the Method Used

We have assumed the energy levels of the nucleus as a whole to be given by the sum of the energies of the individual particles. In other words, we have taken into account the interaction between the particles only insofar as it can be expressed in terms of a potential acting on each particle (Hartree method). This method would, of course, lead to a hopelessly wrong result if we wanted to deduce the actual characteristics (wave function) of each nuclear level from it. It might, however, give fairly correct results for the number of levels in a given energy interval.

The interaction between particles will thoroughly mix the wave functions of the various nuclear levels obtained from the Hartree approximation. A group of levels which would have approximately the same energy in the Hartree approximation will, by the interaction between the particles, be drawn out into a spectrum extending over a wide energy range, probably several MV. Conversely, the wave functions of the *actual* nuclear states in a given energy interval will be linear combinations of Hartree wave functions belonging to much lower as well as much higher Hartree levels.

From these considerations, it might seem that the general behavior of the density of levels, as a function of the energy, might *in the average* be not very greatly changed by the interaction between the particles. There is, however, one fact which will somewhat invalidate this conclusion: The lowest level of the nucleus lies certainly lower than the corresponding Hartree level. It might seem that Q should be taken as the energy of a nuclear level as compared to that of the lowest *Hartree* level. Then Q would be smaller than the actual excitation energy counted from the *true* ground state of the nucleus, which we may call Q'. If we inserted Q' into our formulae we should then obtain too small a spacing between the nuclear levels.

However, we believe that this error is compensated by the fact that we are considering only nuclear levels of low angular momentum. From experience, and from some calculations made recently on light nuclei,[7] we know that the nuclear levels of low angular momentum usually lie lowest, those with high momentum highest among the levels arising from a given configuration. It may be expected that the levels of low momentum arising from "excited configurations" are depressed by the same amount, as compared to their position in the Hartree approximation, as the ground level. Thus we may expect our formulae to give us about the correct density of levels of low angular momentum if we insert for Q the actual energy of excitation above the ground state. Of course, the formulae would give us too high a density of levels of high momenta.

Another reason why we believe our formulae to be not too far wrong, is the fact that the actual levels in a given energy interval will mostly arise from Hartree levels of higher energy, simply because there is a rapid increase of the density of Hartree levels with increasing energy. Most of the levels will therefore be related to the corresponding Hartree levels in a similar way as the ground state.

6. Discussion of the Spacing of Nuclear Levels

The spacing of nuclear energy levels depends, according to formula (50), only on the product of the mass number A of the nucleus and the excitation energy Q. Light nuclei, and heavy nuclei at energies just above the ground level, should possess very few quantum states while highly excited heavy nuclei ought to have an enormous number of closely spaced levels.[1]

Table I gives the spacing between the levels with $I=0$ for various values of the product AQ. E.g., if a nucleus of atomic weight 112(Cd) captures a neutron with the evolution of about

Table I. *Spacing Δ_0 of nuclear energy levels of zero angular momentum in volts.**

AQ (in MV)	100	200	400	600	800	1000	1200	1500	1800
Δ_0 (in Volts)	10^7	$2.4\cdot10^6$	$1.9\cdot10^5$	$2.1\cdot10^4$	2800	450	85	8.5	1.0

*For angular momentum I, the spacing would be $\Delta=\Delta_0/(2I+1)$.

[7] Bethe and Bacher, Rev. Mod. Phys. **8**, 82 (1936) (quoted as B), §36, Wigner and Feenberg, to appear shortly in the Phys. Rev., and unfinished calculations of Bethe and Rose.

9 MV energy, QA would be 1000 and the spacing of S levels of that nucleus only 450 volts. For higher angular momenta, the spacing between levels becomes even less, e.g., for $I=3$ it would be only 1/7 of the previous value, i.e., 60 volts. Since two values of the angular momentum ($I=I_0\pm\frac{1}{2}$, see section 4) lead to neutron capture, the distance between neutron resonance levels would be only one half of the values given. The distance between the highly excited levels of fairly heavy nuclei is thus very small indeed.

For smaller charge, the spacing between levels becomes very much larger. If we let Q be again of the order 9 MV, the spacing of levels for a nucleus such as Fe($A=55$) will be of the order of ten thousand volts, and for really light nuclei such as O($A=16$) of the order of a million volts. This explains why simple capture of neutrons is practically never found with any great intensity for really light nuclei. It would also mean that the lowest neutron resonance level for a nucleus of atomic weight around 50 will, in the average, lie at very much higher energy than for atomic weights of the order 100.

Another cause for an increased spacing of levels would be a smaller value of the excitation energy Q. In connection with the capture of slow neutrons, this would mean that the energy set free in the capture process would have to be smaller than 9 MV. This energy is given by

$$Q=M_{A-1}+M_n-M_A, \tag{51}$$

where M_{A-1}, M_n and M_A denote the (exact) masses of the capturing nucleus, the neutron and the product nucleus respectively, in energy units. In the average, nuclei of medium atomic weight have packing fractions of $-1/1000$, so that M_A-M_{A-1} will be about 0.999 mass unit. The neutron mass being almost 1.009, we find *in the average* the above-mentioned figure $Q=9$ MV (For details, see below).

However, deviations from this figure are to be expected if either of the nuclei $A-1$ or A is exceptionally stable or unstable. The greatest variations will in general come in through the product nucleus A. If this nucleus is radioactive, it is obviously less stable than if it is not. Thus it is to be expected that the energy evolution Q is smaller (in the average) if a radioactive nucleus A is produced, than otherwise. Therefore the spacing of neutron energy levels will be larger in the case of the production of a radioactive nucleus, and the first resonance level will lie at a higher neutron energy. Now it is known from the Breit-Wigner theory[2] of neutron capture that the capture probability for thermal neutrons is, cet. par., the larger the lower the energy of the first resonance level. *The probability of capture of thermal neutrons will, therefore, in the average be smaller if the capture leads to a radioactive nucleus than if it leads to a stable nucleus.*

This fact has been known for some time experimentally and has puzzled investigators to some extent. All the very large cross sections (of the order 10^{-21} cm^2 and more, e.g., Cd, Sm) for the absorption of slow neutrons are connected with the formation of stable nuclei whereas the cross section for the formation of radioactive nuclei are in general only moderately large (about 10^{-22} cm^2 or smaller). Even more marked differences should be found in the positions of the lowest resonance level for neutrons: This level should lie, in the average, at higher neutron energies for the formation of radioactive nuclei than for capture processes leading to stable nuclei.

Apart from irregularities for the individual nuclei, the value of the "dissociation energy" Q will depend on charge and mass number of the nuclei A and $A-1$. We know that generally nuclei with even charge and even mass are most stable, such with odd mass number less stable, and nuclei of odd charge and even mass number unstable to the extent of being radioactive.[8] Therefore we have to distinguish three cases:

(1) The capturing nucleus $A-1$ has *odd charge*. Its mass must then be also odd. Then the capture of a neutron will certainly lead to a radioactive nucleus of odd charge and even mass. The energy Q evolved will be comparatively small, the spacing between the neutron levels fairly large and the capture cross section for temperature neutrons only moderately large.

(2) The capturing nucleus $A-1$ has *even charge and even mass*. The nucleus produced will then have even charge and odd mass. It may be radioactive or stable. But in any case, it will have relatively higher energy in its ground state than the capturing nucleus. Therefore the energy evolved will again be comparatively small, and probably of the same order as in case (1), irrespective of whether the nucleus A is radioactive or not.

(3) The capturing nucleus $A-1$ has *even charge and odd mass*. The nucleus produced will then be of the most stable

[8] For a discussion and explanation of this fact, see B, §10.

type, i.e. even charge and even mass. The energy Q evolved will be exceptionally large, the lowest resonance level will lie at very low neutron energy and the cross section for temperature neutrons will be exceedingly large.

From these considerations it would seem that very strong capture of temperature neutrons can only be due to nuclei of even charge and odd mass number. We suggest that the neutron absorption levels at very low neutron energies which have been observed[9] for Cd, Sm and Hg are due to the abundant "odd isotopes" Cd^{111} or Cd^{113}, Sm^{147} or Sm^{179}, and Hg^{199} or Hg^{201}, respectively.

Another factor which will also increase the capture in case 3 is the factor $2I+1$ in the number of energy levels per unit energy (cf. 49). The spin I of the "resonance level" of the product nucleus A may be either $I_0-\frac{1}{2}$ or $I_0+\frac{1}{2}$ if I_0 is the angular momentum of the original nucleus $A-1$ (cf. beginning of section 4). The number of levels of the nucleus A suitable for the capture of neutrons will therefore be proportional to $2(2I_0+1)$, i.e., the larger the greater the spin of the original nucleus. Now all nuclei with even charge and even mass (class 2 above) seem to have spin $I_0=0$, while nuclei with odd mass (classes 1 and 3) have spins different from zero and therefore are more likely to capture slow neutrons.

We shall now try to compute roughly the actual values of Q to be expected and the differences in the Q values between the above-mentioned cases 1, 2 and 3. An estimate of the latter may perhaps be based on the average energy of the β-particles obtained from neutron captures of class 1. The lifetimes of the radioactive nuclei obtained from neutron capture vary from about 20 sec. to some hours or days, at least for the radioactivities known at present. According to the Sargent rule, this corresponds to energies from about 1 to 3 MV. In the average, we find therefore that nuclei of even mass number and odd nuclear charge have energies by 2 MV greater than their neighboring isobars of even charge into which they transform by emitting a β-ray. Now we may safely assume that the average packing fraction of nuclei of *odd* mass number is independent of whether their nuclear charge is even or odd, because firstly there seems to be experimentally no difference between the number of species and the abundance of these two types of nuclei, and secondly there is no theoretical reason for assuming any difference (cf. B, §10). Consequently, the energy Q evolved in the capture of neutrons by nuclei of odd weight and even charge (class 3 above) ought to be about 2 MV more than for nuclei of odd weight and odd charge (class 1). For nuclei of class 2 (even weight, even charge) we may expect about the same Q's as for class 1, because in both cases the transition goes from a more stable to a less stable type of nucleus.

A rough estimate of the *average* value of Q, i.e. the mean between the cases (2) and (3) above, may be obtained from an empirical formula for the average nuclear mass defects as a function of mass number A and charge Z, such as that derived by Weizsäcker[10] or by the author (B, §30). The energy (excess of the exact mass value over the mass number) of the most stable nucleus of atomic weight A is approximately given by (B, Eq. (186))

$$E(A)=-6.6_5A+14.2A^{2/3}+0.156A^{5/3}\cdot 135/(134+A^{2/3}), \quad (52)$$

the unit of energy being a thousandth of a mass unit. The difference in energy between the nuclei A and $A-1$ is therefore (cf. B (186b))

$$E(A)-E(A-1)=-6.6_5+9.5A^{-1/3}+0.156A^{2/3}\cdot 135(223+A^{2/3})/(134+A^{2/3})^2. \quad (52a)$$

The mass of the neutron may be calculated from the following data:

(1) The mass spectroscopic comparison of the deuteron and the proton by Bainbridge and Jordan,[11] giving

$$2H-D=0.00153\pm 0.00004 \text{ mass unit.}$$

(2) The binding energy of the deuteron as measured by Feather[12]

$$H+n-D=2.22\pm 0.06 \text{ MV}=0.00238.$$

(3) The mass of the deuteron as derived from

[9] Rasetti, Fink, Goldsmith and Mitchell, Phys. Rev. **49**, 869, 1936. Placzek and Frisch (private communication). Amaldi and Fermi, Ricerca Scientifica **1**, 11–12 (1936).

[10] Weizsäcker, Zeits. f. Physik **96**, 431 (1935).

[11] Bainbridge and Jordan, Bull. Am. Phys. Soc., 1936, Washington meeting, report 123.

[12] Feather, Nature **136**, 467 (1935). A correction of 40,000 volts has been applied to Feather's value because of the range energy relation (B, p. 123).

TABLE II. *Mass excesses of nuclei and energy evolved in neutron capture.*

A	20	50	100	150	200	240
$E(A)-E(A-1)$ − (mass units)	−1.45	−1.0	0	0.75	1.4	1.9
Q(MV)	9.5	9.1	8.2	7.5	6.9	6.4

disintegration data[13]

$$D=2.01445.$$

A combination of these three data gives

$$n=H+0.00085=\tfrac{1}{2}D+0.00161=1.00884. \quad (53)$$

Thus the energy $\overline{Q}$ evolved in the neutron capture will be 8.84 thousandths of a mass unit, minus the energy difference (52a). Converted into MV, this gives

$$\overline{Q}=14.4-8.8A^{-1/3}-0.145A^{2/3}\cdot 135(223+A^{2/3})/(134+A^{2/3})^2. \quad (54)$$

Table II gives the average difference between the mass excess of neighboring isotopes, $E(A)-E(A-1)$, according to the semi-empirical formula (52a), in thousandths of a mass unit, and the average energy evolved in the capture of a neutron, $\overline{Q}$, in MV. The values in the table are of course only averages, and in individual cases large deviations ought to be expected. Moreover, it seems from the observed energies of radioactive α-particles that the mass excess of heavy nuclei increases actually somewhat faster with increasing A, so that $\overline{Q}$ for $A=200$ or more may actually be about 0.5 MV smaller than indicated in the table.

Accepting the energies given in the table for the *average* energy evolved in neutron capture, we should expect values by about 1 MV *higher* for the capture by nuclei of even charge and odd mass number (class 3 above) and about 1 MV *lower* than the values of the table for the other cases (classes 1 and 2 above). Thus the odd isotopes of Cd would probably correspond to Q values slightly over 9 MV, giving for AQ a value somewhat over 1000. With a spin of $I_0=\frac{1}{2}$ (B, Table 19) for both the "odd" isotopes of Cd, we should thus expect a spacing of the nuclear levels of about 50 volts (cf. Table I, divide by $2(2I_0+1)$). For Ag, we would expect $Q=7$ MV, approximately, $AQ=800$, and, with $I_0=3/2$, a spacing of about 300 volts between neighboring levels. These figures seem reasonable, although perhaps a little high.

Turning now to the very heavy nuclei (A around 200), we should expect very many very closely spaced levels. The increase in the density of levels due to the increased number of particles A in the nucleus is, however, partly offset by the decrease in the energy evolved in the neutron capture,[14] as shown in Table II. Even so, we should expect values of QA of the order 1400 for $A=200$ (Hg, cf. Table II). This would correspond to a spacing of about 20 volts between levels with $I=0$. Now Hg^{201} has a spin of 3/2; therefore the average spacing between the resonance levels of neutrons captured by Hg^{201} ought to be about two volts. An element of such high atomic number should have an almost continuous absorption spectrum for slow neutrons if the element belongs to class 3, i.e., has even charge and odd mass.

For elements of odd charge (or of even charge and even mass), the increase in the density of levels for high mass number should be less marked. Suppose the Q value is 1 MV less than the "average" given in Table II; which may easily happen. Then AQ is reduced to 1200 for $A=200$, i.e., not much more than for elements of medium atomic weight such as Cd. The distance between adjacent levels would, accordingly, be of the order of 10 volts. This is compatible with the observed resonance level of Au (2.5 volts.)

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[13] Cockcroft and Lewis, Proc. Roy. Soc. **A154**, 261 (1936).

[14] The importance of this factor was first pointed out to me by Dr. Nordheim.