On the Continuous γ-Radiation Accompanying the β-Decay

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Combining Fermi's theory of the β-decay with the ordinary principles of quantum electrodynamics one can treat the continuous radiation, to be expected from the acceleration of charges during the nuclear decay. Formulae for its spectral distribution and its total intensity are developed, applicable to light elements, where the effect of the nuclear Coulomb-field on the electrons can be neglected. The relative amount of energy, liberated in the form of radiation, increases monotonously with increasing total energy; in the case of radioactive boron it amounts to about 0.6 percent.

1. INTRODUCTION

IN his theory of the β-decay Fermi[1] has introduced elementary processes where a neutron in the nucleus is transformed into a proton simultaneously with the creation of a neutrino and an electron. The total energy D liberated by the nuclear transformation in such a process reappears in the sum of the energies of the two created particles. One may think, however, that the transformation could also occur in such a way that besides the two particles a light quantum will be emitted. This idea, that part of the energy D can appear in form of radiation, immediately suggests itself, if, for a moment, one considers the β-decay from a classical point of view: Of the two opposite charges originally neutralizing each other in the nucleus the positive one will remain there (and, since attached to a heavy particle, practically stay at rest) while the negative charge will be set into motion and appear as β-ray. Wherever charges are accelerated, according to classical electrodynamics, radiation will be emitted; although the very process of the creation of an electron is entirely beyond the possibility of a classical description one should expect a corresponding radiation in quantum theory.

In fact it is included in Fermi's theory if only the interaction between the electron and the electromagnetic field is taken into account. Of course the most important part of this interaction is already anticipated in the normal β-decay by the fact that the electron is surrounded by a Coulomb-field and has a mass, part of which must be of electromagnetic origin. The well-known difficulties of relativistic quantum mechanics do not allow at present a consistent account of this aspect and thus a rigorous description of electromagnetic effects must be abandoned off-hand. Nevertheless one can obtain unambiguous and reliable information about the radiation effects to be expected, by restricting the theory to the same approximation in which also atomic radiation processes appear. The total process here to be considered can then be regarded as to happen in two stages:[2]

(1) The nuclear transformation, accompanied by the creation of a neutrino in a state σ and an electron in an "intermediate" state s'.

(2) The transition of the electron from the state s' into a final state s by simultaneous emission of a light quantum.

It shall be the purpose of the next sections to develop a quantitative study of the radiation, thus to be expected, particularly for light nuclei.

It will be found that it forms a continuum, extending from zero to a maximum circular frequency

$$\omega_{\max} = (D - mc^2)/\hbar$$

(mc^2 = rest-energy of the electron, $2\pi\hbar$ = Planck's quantum of action). For $D - mc^2$ of the order of magnitude mc^2 the intensity is such that the ratio of the probabilities of radiative and normal β-decay will be given approximately by the value of Sommerfeld's fine structure constant $e^2/\hbar c = 1/137$; i.e., roughly one percent of the energy of a β-active substance will appear in form of radiation.

Of course it is always possible that after the

[1] E. Fermi, Zeits. f. Physik **88**, 161 (1934).

[2] Similar to the radiation of particles, penetrating a potential barrier, investigated by Heisenberg, Pauli and Oppenheimer. W. Heisenberg and W. Pauli, Zeits. f. Physik **56**, 1 (1929); J. R. Oppenheimer, Phys. Rev. **35**, 939 (1930).

β-decay, the nucleus will be found in an excited state and then emit another quantum. This ordinary γ-radiation is very different in character and intensity from the radiation here discussed and shall not be considered further.

2. Probability of Radiative β-Decay

We consider a nucleus which in its initial state contains a neutron in a state with eigenfunction u_m and energy E_m.[3] After the decay we will have instead of the neutron a proton in a state with eigenfunction v_n and energy E_n. This transformation shall be considered under the action of two perturbing energies H and K, the former being the one introduced by Fermi in his theory of the β-decay, the latter being the interaction energy between electron and radiation field.

We will have to consider three specified states of the unperturbed system:

(I) The nucleus in its initial state and no neutrino, electron, or light quantum present.

(II) The nucleus in its final state, together with a neutrino in a state σ with energy E_σ and an electron in the "intermediate" state s' with energy $E_{s'}$ but still without light quantum.

(III) The nucleus in its final state, together with a neutrino in the state σ, the electron in its "final" state s with energy E_s and a light quantum with circular frequency ω, emitted in the positive z-direction and polarized in the x-direction. (The last two restrictions about the light quantum are obviously made without any loss of generality.)

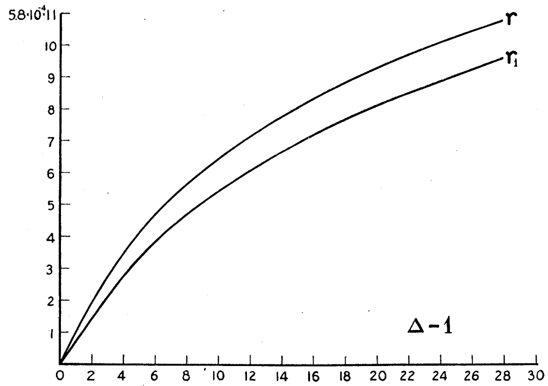

Fig. 1. Values of r and r_1 as functions of $\Delta-1$, the maximum kinetic energy of the electrons divided by mc^2.

The amplitudes of probability of these three states shall be designated by α, $\beta(s', \sigma)$, $\gamma(s, \sigma, \omega)$. They are functions of the time and have to satisfy the initial condition, that for $t=0$:

$$\alpha=1; \qquad \beta(s', \sigma)=\gamma(s, \sigma, \omega)=0.$$

Taking only the terms of lowest order in the perturbation energies H and K, the variation of constants gives for the change of β and γ in course of the time:

$$-(\hbar/i)\dot{\beta}(s', \sigma)=H^m{}_{n;\, s'\sigma}e^{(i/\hbar)(E_{s'}+E_\sigma-D)t}, \tag{1a}$$

$$-(\hbar/i)\dot{\gamma}(s, \sigma, \omega)=\sum_{s'}K_s{}^{s'}(\omega)\beta(s', \sigma)e^{(i/\hbar)(\hbar\omega+E_s-E_{s'})t}, \tag{1b}$$

and therefore after a time t with the initial condition mentioned before

$$\beta(s'\sigma)=H^m{}_{n;\, s'\sigma}\frac{e^{(i/\hbar)(E_{s'}+E_\sigma-D)t}-1}{D-E_{s'}-E_\sigma}, \tag{2a}$$

$$\gamma(s'\sigma\omega)=\sum_{s'}\frac{H^m{}_{n;\, s'\sigma}K_s{}^{s'}(\omega)}{E_{s'}+E_\sigma-D}\left[\frac{e^{(i/\hbar)(E_s+E_\sigma+\hbar\omega-D)}-1}{E_s+E_\sigma+\hbar\omega-D}-\frac{e^{(i/\hbar)(E_s+\hbar\omega-E_{s'})t}-1}{E_s+\hbar\omega-E_{s'}}\right]. \tag{2b}$$

In (1) and (2)

$$D=E_{\mathrm{m}}-E_n \tag{3}$$

stands for the total energy, liberated in the transformation, $2\pi\hbar=h$ is Planck's quantum of action and $H^m{}_{n,\, s'\sigma}$, $K_s{}^{s'}(\omega)$ are the matrix-elements of the perturbation energies H and K, which account for the transitions from the states I to II and II to III respectively. They obtain a simple form if

[3] We consider here only the case of negative electron emission. By interchanging the words "neutron" and "proton" and writing "positron" instead of "electron" our considerations as well as the final results apply immediately also to the case of positron emission.

we neglect from now on the influence of the nuclear charge on the electron; for usual values of the decay-energy we commit thus only an error of the relative order of magnitude $Ze^2/\hbar c$; i.e., for light elements of a few percents. The eigenfunctions of the states s, s' of the electron and σ of the neutrino are then solutions of Dirac's wave equation for free particles, namely plane waves with amplitudes having four components according to the four values of the spin variable. We obtain thus

$$H^m{}_{n;\,s'\sigma} = (gI/V)a_{s'}{}^*\eta b_\sigma{}^*, \tag{4}$$

$$K_s{}^{s'}(\omega) = e\hbar(2\pi c/Vp)^{\frac{1}{2}}a_s{}^*\alpha_x a_{s'}, \tag{5}$$

$g = 4\cdot 10^{-50}$ cm^3 erg is the constant, appearing in Fermi's theory,
$I = \int v_n{}^* u_m d\tau$ the overlapping-integral of the eigenfunctions of neutron and proton.[4]

$a_{s'}$, α_s and b_σ are the amplitudes of the plane waves, representing the states s, s', and σ of the electron and the neutrino with momenta $\mathbf{p}_{s'}$, $\mathbf{p}_s$, $\mathbf{p}_\sigma$, respectively. $p = \hbar\omega/c$ is the absolute value of the momentum $\mathbf{p}$ of the light quantum with components $p_x = p_y = 0$; $p_z = \hbar\omega/c$. It is connected with $\mathbf{p}_s$ and $\mathbf{p}_{\sigma'}$ by the equation

$$\mathbf{p}_{s'} = \mathbf{p} + \mathbf{p}_s. \tag{6}$$

η is the matrix $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ introduced by Fermi.[5] $\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is one of Dirac's matrices, which appears in the interaction energy between the electron and a light quantum, polarized in the x-direction. e is the elementary charge, c the velocity of light and V is a large volume within which the states of electron, neutrino, and light quantum are quantized and normalized. Furthermore, as in Fermi's theory, it is assumed that the wave-lengths p_s/h, $p_{s'}/h$, p_σ/h of electron and neutrino are large compared to nuclear dimensions.

From (2b) we find now in the usual way that the number of transitions per unit time, which lead to the final state III, is given by

$$\frac{dP}{dt} = \frac{2\pi}{\hbar}\delta(\hbar\omega + E_s + E_\sigma - D)\left|\sum_{s'}\frac{H^m{}_{n;\,s'\sigma}K_s{}^{s'}(\omega)}{E_{s'} - E_s - \hbar\omega}\right|^2, \tag{7}$$

where δ is Dirac's singular δ-function.[6] The summation over s' has to be performed over all intermediate states s' of the electron; since for given values $\mathbf{p}$ and $\mathbf{p}_s$ the matrix element $K_s{}^{s'}(\omega)$ is only different from zero if the momentum $\mathbf{p}_{s'}$ has the value given by (6), this summation extends only over the four states, which, according to Dirac's equation, belong to the momentum $\mathbf{p}_{s'}$. We designate these four states by an index λ and obtain thus from (4) and (7)

$$\frac{dP}{dt} = \delta(\hbar\omega + E_s + E_\sigma - D)\frac{4\pi^2 g^2|I|^2 e^2\hbar c}{V^3 p}\left|\sum_{\lambda=1}^{4}\frac{a_s{}^*\alpha_x a_{\lambda s'} a_{\lambda s'}{}^*\eta b_\sigma{}^*}{E_{\lambda s'} - \hbar\omega - E_s}\right|^2 \tag{8}$$

with

$$\begin{aligned} E(p_s) = E_s = +c(p_s{}^2 + m^2c^2)^{\frac{1}{2}}, \quad & E(p_\sigma) = E_\sigma = +cp_\sigma,^7 \\ E_{1s'} = E_{2s'} = +c(p_s{}^2 + m^2c^2)^{\frac{1}{2}}, \quad & E_{3s'} = E_{4s'} = -c(p_{s'}{}^2 + m^2c^2)^{\frac{1}{2}}.^8 \end{aligned} \tag{9}$$

[4] We assume this integral to be different from zero; the changes that have to be made otherwise are quite analogous to the corresponding ones in Fermi's theory and do not essentially alter our considerations.

[5] Fermi uses δ instead of η.

[6] Defined by

$$\int_a^b \delta(x)dx = \begin{cases} 1 & \text{for } a < 0 < b, \\ 0 & \text{otherwise.} \end{cases}$$

[7] We assume the mass of the neutrino to be zero.

[8] Obviously, in this scheme both "positive" and "negative" intermediate states have to be taken into account. If, instead, one assumes (with Dirac) all negative states to be filled, due to the exclusion principle only positive intermediate

The summation over λ in (8) can easily be carried out if one takes into account the relation of completeness of the amplitudes $a_{\lambda s'}$ with respect to the spin-variable, and leads to

$$\frac{dP}{dt}=\delta(\hbar\omega+E_s+E_\sigma-D)\frac{4\pi^2g^2|I|^2e^2\hbar c}{V^3p[E_{s'}{}^2-(\hbar\omega+E_s)^2]^2}|a_s{}^*\alpha_x\Gamma\eta b_\sigma{}^*|^2 \tag{10}$$

with

$$\Gamma=\hbar\omega+E_s+c(\boldsymbol{\alpha}\mathbf{p}_{s'})+\beta mc^2 \tag{11}$$

and

$$E_{s'}{}^2=c^2p_{s'}{}^2+m^2c^4,$$

the vector matrix $\boldsymbol{\alpha}$ and the matrix β being the same as in Dirac's relativistic theory of the electron. In formula (10) not only the momenta $\mathbf{p}_s$ and $\mathbf{p}_\sigma$ of electron and neutrino but also their spin-directions are supposed to be given. For the total transition probability we have to sum over these spin directions. Indicating this summation by $\sum_{\text{s.d.}}$ we obtain

$$\sum_{\text{s.d.}}\frac{dP}{dt}=\delta(\hbar\omega+E_s+E_\sigma-D)\frac{4\pi^2g^2|I|^2e^2\hbar c}{V^3p[E_{s'}{}^2-(\hbar\omega+E_s)^2]^2}\overline{(M\alpha_x\Gamma N\Gamma\alpha_x)}. \tag{12}$$

M and N are the two matrices

$$M=1+[c(\boldsymbol{\alpha}\mathbf{p}_s)+\beta mc^2]/E_s,\qquad N=1+c(\boldsymbol{\alpha}\mathbf{p}_\sigma)/E_\sigma. \tag{13}$$

The bar over the last matrix product in (12) indicates the average value, i.e., one-fourth of the sum of its diagonal elements. It is

$$\overline{(M\alpha_x\Gamma N\Gamma\alpha_x)}=f(\mathbf{p}_s,\boldsymbol{\pi}_\sigma,\boldsymbol{\pi}_{s'})=c^2(\pi_{s'}{}^2+m^2c^2)+2(\hbar\omega+E_s)\left(\frac{c^2(\boldsymbol{\pi}_\sigma\boldsymbol{\pi}_{s'})}{E_\sigma}+\frac{c^2(\mathbf{p}_s\boldsymbol{\pi}_{s'})-m^2c^4}{E_s}\right)$$
$$+(\hbar\omega+E_s)^2\left(1+\frac{c^2(\mathbf{p}_s\boldsymbol{\pi}_\sigma)}{E_sE_\sigma}\right)+\frac{c^4}{E_sE_\sigma}\left(2(\boldsymbol{\pi}_\sigma\boldsymbol{\pi}_{s'})[(\mathbf{p}_s\boldsymbol{\pi}_{s'})-m^2c^2]-(\mathbf{p}_s\boldsymbol{\pi}_\sigma)[\pi_{s'}{}^2-m^2c^2]\right), \tag{14}$$

where $\boldsymbol{\pi}_\sigma$ and $\boldsymbol{\pi}_{s'}$ are vectors with components $p_{\sigma x}$, $-p_{\sigma y}$, $-p_{\sigma z}$ and $p_{s'x}$, $-p_{s'y}$, $-p_{s'z}$, respectively. Integrating (12) over the momentum-space of the neutrino and over the directions of the emitted electron, we finally obtain the rate of transition (probability per unit time) of a process, in which an electron is emitted with an energy between E and $E+dE$ [9] simultaneously with a light quantum of circular frequency between ω and $\omega+d\omega$. The light quantum shall be emitted within an angle between θ and $\theta+d\theta$ against the direction of the electron and be polarized in the plane, connecting the momenta of electron and light quantum.[10]

With (6), (9), (12), and (14) this rate of transitions becomes:

$$dR_x=A(E+\hbar\omega-D)^2\frac{u}{(1-u\cos\theta)^2}\left[\frac{\hbar^2\omega^2}{2}(1-u\cos\theta)+E(E+\hbar\omega)u^2\sin^2\theta\right]\frac{d\omega}{\omega}dE\sin\theta d\theta, \tag{15}$$

where

$$A=g^2|I|^2e^2/4\pi^4\hbar^8c^7 \tag{16}$$

and $u=v/c=(1/E)(E^2-m^2c^4)^{\frac{1}{2}}$ is the velocity of the electron in units of the velocity of light. If on

states of the electron have to be taken into account. In this case, however, one has to consider that the final state III can also be reached by a double process, in which first an electron makes a transition from a negative state s' into the positive state s by emission of a light quantum and where afterwards by the nuclear transition another electron is created in such a way that it fills the hole in the negative states, left by the transition of the first electron. It is easy to show that the net result will be the same as given by (8).

[9] We now omit the index s.

[10] We have thus chosen here this plane to be the $x-z$-plane of our coordinate system.

the other hand the light quantum is polarized perpendicularly to the direction of electron, we obtain similarly

$$dR_y=A(E+\hbar\omega-D)^2\frac{u}{(1-u\cos\theta)^2}\left[\frac{\hbar^2\omega^2}{2}(1-u\cos\theta)\right]\frac{d\omega}{\omega}dE\sin\theta d\theta, \tag{17}$$

and thus for the rate of transition without specifying the polarization

$$\begin{aligned} dR&=dR_x+dR_y \\ &=A(E+\hbar\omega-D)^2\frac{u}{(1-u\cos\theta)^2}[\hbar^2\omega^2(1-u\cos\theta)+E(E+\hbar\omega)u^2\sin^2\theta]\frac{d\omega}{\omega}dE\sin\theta d\theta. \end{aligned} \tag{18}$$

This result has been obtained by assuming Fermi's form for the interaction between nucleus and electron-neutrino field. Konopinski and Uhlenbeck[11] have proposed another form, which contains as a factor the energy of the neutrino. With the exception of a few minor changes the calculation remains practically the same, if we accept this second form of the interaction; we shall briefly indicate these changes:

Instead of (4), one has to take

$$H^m{}_{n;\, s\sigma}=\frac{gI}{V}\frac{E_\sigma}{mc^2}a_{s'}{}^*\eta\beta b_\sigma{}^* \tag{4a}$$

and thus finds instead of (12)

$$\sum_{\text{s.d.}}\frac{dP}{dt}=\delta(\hbar\omega+E_s+E_\sigma-D)\frac{4\pi^2g^2|I|^2e^2\hbar c}{V^3p[E_{s'}{}^2-(E_s+\hbar\omega)^2]^2}\left(\frac{E_\sigma}{mc^2}\right)^2\overline{(M\alpha_x\Gamma N_1\Gamma\alpha_x)} \tag{12a}$$

with $N_1=1-c(\boldsymbol{\alpha}\mathbf{p}_\sigma)/E_\sigma$. Instead of (14) it is

$$\overline{(M\alpha_x\Gamma N_1\Gamma\alpha_x)}=f(p_s,\ -\pi_\sigma,\ \pi_{s'}), \tag{14a}$$

where f is the same function as in (14). Finally instead of (18) one obtains for the rate of transition

$$dR_1=A\frac{(E+\hbar\omega-D)^4}{m^2c^4}\frac{u}{(1-u\cos\theta)^2}[\hbar^2\omega^2(1-u\cos\theta)+E(E+\hbar\omega)u^2\sin^2\theta]\frac{d\omega}{\omega}dE\sin\theta d\theta. \tag{18a}$$

3. Discussion: Spectral Distribution and Total Amount of Radiated Energy

Formulae (18) and (18a) show that there is to be expected an anisotropy of the intensity of radiation with respect to the direction of the emitted electron very similar to that of the continuous radiation, caused by the impact of electrons. In the non-relativistic case, i.e., if

$$\hbar\omega/mc^2\ll u\ll 1,$$

the intensity per unit solid angle will be proportional to $\sin^2\theta$ as one should expect it classically from a dipole radiation. In the highly relativistic case, however, i.e., for $u\cong 1$ the anisotropy will be essentially determined by the denominator $(1-u\cos\theta)^2$ which then becomes very small for small angles θ. In this case, therefore, most of the radiation will be emitted nearly parallel to the direction of the electron.

While (18) and (18a) are valid for all velocities of the electron[12] they obviously break down for very small values of the frequency ω. This can be seen from the fact that the frequency appears once in the denominator so that by integrating over the frequency from zero to its maximum value $(D-E)/\hbar$ one would get an infinite result for the probability of radiative β-decay. The same situation is found wherever not only the radiationless process is possible but also the same process, simul-

[11] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **47**, 202 (1935).
[12] Except for the fact that for very small velocities the Coulomb-field of the nucleus becomes important.

taneous with the emission of an arbitrarily soft light quantum,[13] and has its origin in the treatment of the electromagnetic field as a small perturbation. It is due to this treatment that there appears (as in formula (7)) the denominator $E_{s'}-E_s-\hbar\omega$, which for sufficiently small values of ω becomes arbitrarily small and ultimately leads to a divergent result for the probability of the radiative process. Although strictly speaking incorrect, when applied to transition probabilities, our formulae (18) and (18a) seem to give trustworthy results for the average energy of the radiation. In this case indeed we have to multiply them by another factor $\hbar\omega$, thus avoiding the divergence, mentioned before.

Integrating (18) over the angle θ from 0 to π and over the energy of the electron E from mc^2 to $D-\hbar\omega$ we obtain

$$\hbar\omega\int_\theta\int_E dR=i(\omega)d\omega$$

as the radiation energy emitted into the frequency range $d\omega$ per unit time. It is

$$i(\omega)=A\hbar(mc^2)^5F((D-\hbar\omega)/mc^2) \tag{19}$$

with

$$F(x)=\left[\Delta^2\left(\frac{2}{3}x^3+x\right)-\Delta\left(x^4+x^2-\frac{1}{8}\right)+\frac{7}{15}x^5-\frac{3}{8}x\right]\lg\,(x+(x^2-1)^{\frac{1}{2}})$$
$$-\left[\Delta^2\left(\frac{11}{9}x^2+\frac{4}{9}\right)-\Delta\left(\frac{7}{4}x^3+\frac{1}{8}x\right)+\frac{689}{900}x^4-\frac{1021}{1800}x^2-\frac{8}{75}\right](x^2-1)^{\frac{1}{2}}$$

and $\Delta=D/mc^2$. Similarly from (18a)

$$i_1(\omega)=A\hbar(mc^2)^5F_1((D-\hbar\omega)/mc^2) \tag{19a}$$

with

$$F_1(x)=\left[\Delta^2\left(\frac{2}{5}x^5+2x^3+\frac{3}{4}x\right)-\Delta\left(\frac{2}{3}x^6+3x^4+\frac{3}{4}x^2-\frac{1}{24}\right)+\frac{32}{105}x^7+x^5-\frac{1}{2}x^3-\frac{5}{24}x\right]\log\,(x+(x^2-1)^{\frac{1}{2}})$$
$$-\left[\Delta^2\left(\frac{137}{150}x^4+\frac{607}{300}x^2+\frac{16}{75}\right)-\Delta\left(\frac{3}{2}x^5+\frac{17}{6}x^3+\frac{1}{24}x\right)+\frac{2449}{3675}x^6+\frac{9413}{14700}x^4-\frac{19603}{29400}x^2-\frac{32}{735}\right](x^2-1)^{\frac{1}{2}}.$$

The intensity distribution of the emitted radiation, as given by (19) and (19a), shows a rather rapid decrease from its maximum at $\omega=0$ to the zero value at the high frequency limit $\omega_{\max}=(D-mc^2)/\hbar$. In the limiting case $\Delta\gg mc^2$ about 50 percent of the total radiation according to (19), 60 percent according to (19a) will be found at a frequency less than one-fourth of the high frequency limit.

Finally, we find for the total intensity of the radiation from (19)

$$I_{\text{tot}}=\int_0^{\omega_{\max}}i(\omega)d\omega=\frac{A}{2}(mc^2)^6\phi(\Delta) \tag{20}$$

with $$\phi(\Delta)=\left[\frac{4}{45}\Delta^6+\frac{1}{3}\Delta^4-\frac{1}{72}\right]\log\,(\Delta+(\Delta^2-1)^{\frac{1}{2}})-\left[\frac{44}{225}\Delta^5+\frac{27}{100}\Delta^3-\frac{103}{1800}\Delta\right](\Delta^2-1)^{\frac{1}{2}}$$

and from (19a)

$$(I_{\text{tot}})_1=\int_0^{\omega_{\max}}i_1(\omega)d\omega=(A/2)(mc^2)^6\phi_1(\Delta) \tag{20a}$$

[13] It also appears for example in the theory of radiative impacts. (Bethe and Heitler.)

with

$$\phi_1(\Delta)=\left[\frac{2}{105}\Delta^8+\frac{2}{15}\Delta^6-\frac{1}{12}\Delta^2-\frac{1}{192}\right]\log\left(\Delta+(\Delta^2-1)^{\frac{1}{2}}\right)$$
$$-\left[\frac{691}{14700}\Delta^7+\frac{171}{1176}\Delta^5-\frac{9589}{117600}\Delta^3-\frac{11063}{235200}\Delta\right](\Delta^2-1)^{\frac{1}{2}}.$$

In order to get a convenient measure for the total intensity, we will divide (20) and (20a) by the average kinetic energy $\bar{E}_{\text{kin}}=\overline{(E-mc^2)}$ given to the electrons per unit time. According to Fermi's theory this is[14]

$$\bar{E}_{\text{kin}}=A(mc^2)^6(hc/e^2)\psi(\Delta) \tag{21}$$

with $\psi(\Delta)=\left[\frac{1}{60}\Delta^5-\frac{1}{30}\Delta^4-\frac{1}{30}\Delta^3+\frac{3}{20}\Delta^2+\frac{49}{240}\Delta+\frac{2}{15}\right](\Delta^2-1)^{\frac{1}{2}}-\left[\frac{1}{8}\Delta^2+\frac{1}{4}\Delta+\frac{1}{16}\right]\log\left(\Delta+(\Delta^2-1)^{\frac{1}{2}}\right)$,

according to Uhlenbeck and Konopinski

$$\bar{E}_{\text{kin}}=A(mc^2)^6(hc/e^2)\psi_1(\Delta) \tag{21a}$$

with

$$\psi_1(\Delta)=\left[\frac{1}{280}\Delta^7-\frac{1}{105}\Delta^6-\frac{5}{336}\Delta^5+\frac{2}{21}\Delta^4+\frac{1913}{6720}\Delta^3+\frac{247}{420}\Delta^2+\frac{3571}{13440}\Delta+\frac{8}{105}\right](\Delta^2-1)^{\frac{1}{2}}$$
$$-\left[\frac{1}{8}\Delta^4+\frac{1}{2}\Delta^3+\frac{3}{8}\Delta^2+\frac{1}{4}\Delta+\frac{5}{128}\right]\lg\left(\Delta+(\Delta^2-1)^{\frac{1}{2}}\right).$$

Dividing (20) by (21) and (20a) by (21a) we find now the percentage of energy liberated in form of γ-radiation by the β-decay in the form

$$r(\Delta)=I_{\text{tot}}/\bar{E}_{\text{kin}}=(e^2/2hc)(\phi(\Delta)/\psi(\Delta))=5.81\cdot10^{-4}\chi(\Delta) \tag{22}$$

with $\chi(\Delta)=\phi(\Delta)/\psi(\Delta)$

and $r_1(\Delta)=5.82\cdot10^{-4}\chi_1(\Delta)$ with $\chi_1(\Delta)=\phi_1(\Delta)/\psi_1(\Delta)$, respectively. (22a)

In the nonrelativistic case, i.e., for $\Delta=1+\epsilon$ with $\epsilon\ll1$, we have $\chi(\Delta)=(32/33)\epsilon$ and $\chi_1(\Delta)=(32/45)\epsilon$. On the other hand for $\Delta\gg1$, we obtain the asymptotic formulae

$$\chi(\Delta)=\frac{16}{3}\left(\log 2\Delta-\frac{11}{5}\right)\quad\text{and}\quad\chi_1(\Delta)=\frac{16}{3}\left(\log 2\Delta-\frac{691}{280}\right).$$

The two values r and r_1 are plotted in Fig. 1 as functions of $\Delta-1$, i.e., of the maximum kinetic energy of the electrons, divided by mc^2.[15]

While for example for radiophosphorus ($\Delta=5$) the radiation contains only about 0.17 percent of the total average energy, liberated in the decay, it amounts to about 0.6 percent in the case of the radioactive boron isotope B^{12} ($\Delta=26$).

Note added in proof: While this paper was in press an article on the same problem has appeared in Physica **3**, 425 (1936). J. K. Knipp and G. E. Uhlenbeck reach essentially the same conclusions as obtained in this paper.

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[14] Neglecting again the Coulomb-field around the nucleus.

[15] I am indebted to Mr. R. D. Gordon for valuable help in computation.