MUTUAL INDUCTANCES OF CIRCUITS COMPOSED OF STRAIGHT WIRES.

BY GEORGE A. CAMPBELL.

THE mutual inductance between any two circuits made up of m and n straight wires of negligible diameter may be most simply expressed as the sum of the mn mutual Neumann integrals between the sides taken in pairs, one from each circuit. As such inductances are required in practical computations, it is desirable to have a formula for the mutual Neumann integral between two skew lines of any lengths in any relative location. The Bureau of Standard's collection of "Formulas and Tables for the Calculation of Mutual and Self-Induction (Revised)" does not contain such a formula and the only statement of the result[1] which I have seen is involved and unsatisfactory for actual use. A general formula in convenient form, formulas for a number of special cases, a diagram for use in calculations, and, finally, the deduction of the formulas follow.

To speak of the self and mutual inductances of circuits one or both of which are unclosed, is logically inexact and practically unsafe, for it tends to vague thinking and the entire neglect of the return circuit in cases where the effect of the return is easily lost sight of because it is at a remote distance. Heaviside advocated the exclusive consideration of closed circuits, securing external continuity in every case by means of two superposed uniform radial systems, one diverging from the positive terminal in all directions, the other converging on the negative terminal from all directions. This means subtracting one half of the second order difference $(-Aa + Ab + Ba - Bb)$ of the distances between the terminals A, B; a, b, from the Neumann integral for any unclosed circuits between these terminals. A better way, it seems to me, is to continue using the Neumann integrals for unclosed circuits but to refer to them as the self or mutual Neumann integrals according as the two unclosed circuits are or are not identical. This reserves the terms self and mutual inductances for use with closed circuits exclusively. The following results are expressed in this way.

[1] Martens, F. F., Ann. der Phys., *29*, p. 959, 1909.

Neumann Integral for Skew Lines.

The mutual Neumann integral between any two straight filaments AB, ab, *the positive directions* A *to* B *and* a *to* b *making with each other the angle* e $(0 \ngtr e \ngtr \pi)$, *is*

$$\left.\begin{aligned} N &= \underline{pB'} \log \frac{Bb + \underline{\underline{B'b}}}{Ba + \underline{B'a}} - \underline{pA'} \log \frac{Ab + \underline{\underline{A'b}}}{Aa + \underline{A'a}} + \underline{Pb'} \log \frac{bB + \underline{\underline{b'B}}}{bA + \underline{b'A}} \\ &\qquad - \underline{Pa'} \log \frac{aB + \underline{\underline{a'B}}}{aA + \underline{a'A}} - \frac{Pp\Omega}{\tan e} \\ &= 2\underline{pB'} \tanh^{-1} \frac{ab}{aB + Bb} - 2\underline{pA'} \tanh^{-1} \frac{ab}{aA + Ab} \\ &\qquad + 2\underline{Pb'} \tanh^{-1} \frac{AB}{Ab + bB} - 2\underline{Pa'} \tanh^{-1} \frac{AB}{Aa + aB} - \frac{Pp\Omega}{\tan e} \end{aligned}\right\} \quad (1)$$

where $A'B'$, $a'b'$ *are the projections of* AB *and* ab *on each other;* Pp *is the common perpendicular to* AB, ab *taken positive, as are all other distances except those (underscored) measured along* AB *and* ab *which are taken algebraically positive in the directions* AB *and* ab *respectively;* Ω *is the (positive) solid angle subtended at* B *by a parallelogram* $abcd$ *constructed on* ab *with* bc *parallel and equal to* AB.[1]

Special Cases.

1. Filaments mutually perpendicular, $N = 0$.
2. Filaments starting from a common point $(P = A = p = a)$,

$$N = \underline{A'B'} \log \frac{Bb + \underline{\underline{B'b}}}{AB + \underline{B'a}} + \underline{a'b'} \log \frac{bB + \underline{\underline{b'B}}}{ab + \underline{b'A}}. \quad (2)$$

3. Filaments mutually parallel ($e = 0$ or π),

$$N = AB \log \frac{aB + \underline{\underline{a'B}}}{bB + \underline{b'B}} + \underline{Ab'} \log \frac{bB + \underline{\underline{b'B}}}{bA + \underline{b'A}} - \underline{Aa'} \log \frac{aB + \underline{\underline{a'B}}}{aA + \underline{a'A}} - (-Aa + Ab + Ba - Bb) \quad (3)$$

3*a*. If $e = 0$, and the midpoints of AB and ab are opposite each other.

$$N = AB \log \frac{2Ab + AB + ab}{2Aa + AB - ab} + ab \log \frac{2Ab + AB + ab}{2Aa + ab - AB} + 2Aa - 2Ab, \quad (3a)$$

4. Filaments mutually parallel beginning at a common perpendicular, with positive direction either the same or opposite ($e = 0$ or π, $Aa = Pp$),

$$N = AB \log \frac{aB + AB}{bB + \underline{b'B}} + ab \log \frac{Ab + ab}{Bb + \underline{B'b}} - (-Aa + Ab + Ba - Bb) \quad (4)$$

[1] Formula 10 may be used for Ω.

$$\left.\begin{aligned} &= 2AB \log \frac{aB + AB}{Aa} - 2(Ab - Aa) \\ &= 4AB \tanh^{-1} \frac{AB}{aB + Bb} - 2Ab + 2Aa \\ &= 2AB \left(\log \frac{2}{t} - 1 + t - \frac{t^2}{4} + \frac{t^4}{32} - \frac{t^6}{96} \right. \\ &\qquad \left. + \frac{5t^8}{32.32} - \cdots \right), \quad t = \frac{Aa}{AB} \end{aligned}\right\} \text{ if } AB = ab,\ e = 0 \quad (5)$$

$$\left.\begin{aligned} &= 2AB \log \frac{aB + AB}{bB + 2AB} - (-Aa + 2Ab - Bb) \\ &= 2AB \left(-\log 2 + \frac{t}{2} - \frac{3t^2}{16} + \frac{15t^4}{16.32} - \frac{63t^6}{64.96} \right. \\ &\qquad \left. + \frac{5.255t^8}{16.16.32.32} - \cdots \right), \quad t = \frac{Aa}{AB} \end{aligned}\right\} \text{ if } AB = ab,\ e = \pi. \quad (6)$$

5. Non-overlapping portions of a straight filament $ABab$ ($e = 0$, $Pp = 0$)

$$\begin{aligned} N &= AB \log \frac{Ab}{Aa} + ab \log \frac{Ab}{Bb} + Ba \log \frac{Ab \cdot Ba}{Aa \cdot Bb} \\ &= -Aa \log Aa + Ab \log Ab + Ba \log Ba - Bb \log Bb. \end{aligned} \quad (7)$$

6. Filament ab with the element at the point s having the algebraic projection dx on filament AB.

$$\begin{aligned} dN &= \log \frac{sB + s'B}{sA + s'A} dx \\ &= 2dx \tanh^{-1} \frac{AB}{As + sB}, \end{aligned} \quad (8)$$

RULE FOR USING THE DIAGRAM OF CONFOCAL ELLIPSES (FIG. 1) FOR FINDING THE MUTUAL NEUMANN INTEGRAL BETWEEN A FINITE STRAIGHT LINE AB AND ANY OTHER LINE ab.

Draw the two lines on such a scale that A, B coincide with the foci; if AB and ab do not both lie in one plane bring ab into the plane of the paper by rotating it point by point about the right line through AB. Consider the ellipses to be contour lines with the elevations noted upon them. Determine the projection upon the vertical plane through the foci A, B of the vertical cylindrical surface erected upon ab and bounded by the contour surface, considering areas to be positive or negative according as the projection of ab has or has not *the same positive direction as AB. If horizontal distances have been measured in* 10^9 *cm. the area found is the mutual integral in henries.*

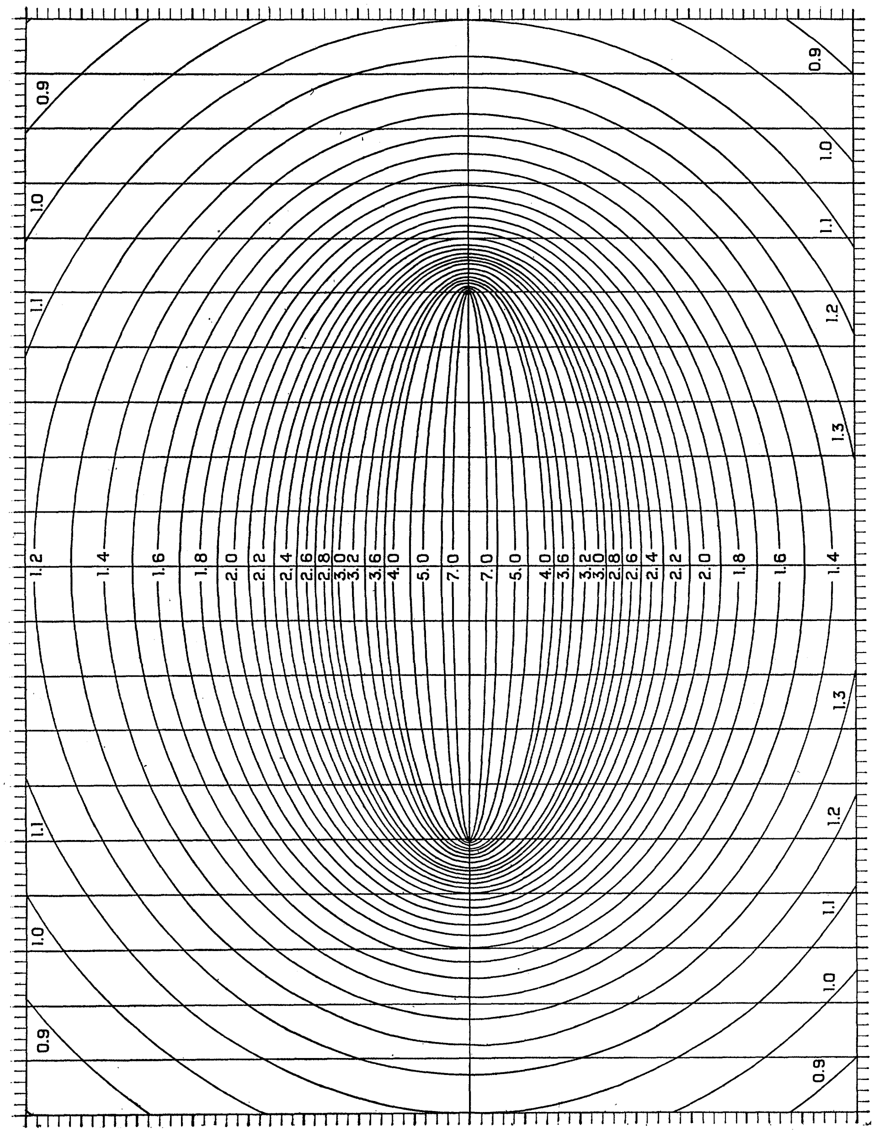

Fig. 1.

Confocal Ellipse Diagram for Calculating Mutual Neumann Integrals.

Transformations of Formula 1.

Formula (1) may be thrown into a variety of equivalent forms by means of the following geometrical relationships for the first ratio occurring in the logarithmic terms, which is typical of all, and for the solid angle, the expression for which is in terms of the dihedral angles of the tetrahedron on AB, ab.

$$\frac{Bb+\underline{B'b}}{Ba+\underline{B'a}} = \frac{Ba-\underline{B'a}}{Bb-\underline{B'b}} = \frac{(Bb+\underline{B'b})(Ba-\underline{B'a})}{B'B^2}$$

$$= \cot\frac{\angle Bab}{2}\cot\frac{\angle Bba}{2} = \frac{Ba+Bb+ab}{Ba+Bb-ab} = e^{2\tanh^{-1}\frac{ab}{aB+Bb}}, \qquad (9)$$

$$\Omega = \tan^{-1}\left(\frac{Pp}{Bb}\cot e + \frac{PB}{Pp}\frac{pb}{Bb}\sin e\right) - \tan^{-1}\left(\frac{Pp}{Ba}\cot e + \frac{PB}{Pp}\frac{pa}{Ba}\sin e\right)$$

$$- \tan^{-1}\left(\frac{Pp}{Ab}\cot e + \frac{PA}{Pp}\frac{pb}{Ab}\sin e\right) + \tan^{-1}\left(\frac{Pp}{Aa}\cot e + \frac{PA}{Pp}\frac{pa}{Aa}\sin e\right). \qquad (10)$$

PROOFS.

The Neumann integral is

$$N = \cos e \int\int \frac{dSds}{r}, \qquad \text{where } r^2 = Pp^2 + S^2 - 2Ss\cos e + s^2$$

if S, s are measured in the positive directions along AB, ab respectively from the common perpendicular Pp. As is easily shown by performing the indicated differentiations, the integral for N may be written in the following directly integrable form.

$$N = \cos e \int\int \left(D_S\frac{S}{r} + D_s\frac{s}{r} - \frac{Pp^2}{r^3}\right) dSds$$

$$= \cos e\left(\left|S\int\frac{ds}{r}\right| + \left|s\int\frac{dS}{r}\right| - \frac{Pp}{\sin e}\int\int\frac{Pp}{r^3}\sin e\, dSds\right)$$

$$= \cos e \,\Big|\Big|\, S\log(r+s-S\cos e) + s\log(r+S-s\cos e)$$

$$- \frac{Pp\text{ (solid angle)}}{\sin e}\;\Bigg|_{S=\underline{PA}}^{S=\underline{PB}}\;\Bigg|_{s=\underline{pa}}^{s=\underline{pb}}.$$

the solid angle being given by the last integral, as ($\sin e\, dSds$) may be taken as an element of area of the parallelogram $abcd$ in oblique coördinates (if $\sin e$ is positive which requires that e shall lie between 0 and π), the factor Pp/r reduces this to its projection normal to r, and division by r^2 gives the solid angle subtended by the element. Introducing the limits and substituting $\underline{PB}\cos e = \underline{pB'}$, etc., give formula (1).

This Neumann integral is also (1) the Newtonian potential at B of the parallelogram $abcd$ with a uniform mass equal to $\cot e$ per unit area, which suggests other ways of performing the integration; (2) the mutual potential of two rods AB, ab with the mass $\sqrt{\cos e}$ per unit length; (3) the mutual Neumann integral for any arc of a hyperbola and any part of the conjugate axis as is shown below.

Formula (3) is obtained by taking Pp coincident with AA' and changing the first term so as to substitute algebraical quantities measured along AB in place of the algebraical quantities measured along ab. This change is made by noting that for any pair of points X, y on parallel filaments and their projections X', y' the following relations hold, $\underline{X'y} = \pm \underline{Xy'} = \mp \underline{y'X}$, the upper and lower signs applying to positive directions in the same or opposite sense respectively; whence

$$\underline{A'B'} \log \frac{Bb + \underline{B'b}}{Ba + \underline{B'a}} = \pm AB \log \frac{Bb \mp \underline{b'B}}{Ba \mp \underline{a'B}} = AB \log \frac{aB + \underline{a'B}}{bB + \underline{b'B}}.$$

The solid angle term becomes 0/0 but may readily be evaluated, or it may be derived directly from the indefinite integral for this term, which is found to reduce to $\| r \|$ for parallel filaments as $D_S D_s r = -Pp^2 r^{-3} \cos e$ for this case, and substituting this makes the term directly integrable. As the location of Pp is arbitrary, any one of the four terms in (1) may be made to vanish by locating Pp at A, B, a or b. The logarithmic terms may, if desired, be so combined as to be symmetrical in the quantities involved.

Formula (4) is a special case of (3). Formula (5) is the well-known result for the opposite sides of a rectangle, and formulas (3), (4) and (6) may all be derived from it, in spite of the fact that it is merely a special case of (3) and (4).

Formula (7) holds only for the particular sequence of points indicated, but any non-overlapping sequence may be readily reduced to this. The second expression is the one which is most readily obtained by direct integration of Neumann's integral, for that gives $\| -r \log r \|$ for the indefinite integral in this special case.

Formula (10) (which corresponds to the indefinite integral $\| \tan^{-1} ((Pp^2 \cot e + Ss \sin e)/Ppr)) \|$ for the solid angle expressed in terms of the dihedral angles may be checked by differentiating with respect to both S and s, or it may be derived by geometrical considerations.

Formula (8) expresses the mutual Neumann integral between any filament ab and any straight filament AB in terms of the projections of the elements of ab on AB multiplied by factors which depend only upon the location of the elements with respect to AB. A diagram showing the value of this factor at every point may be used for determining the value of the integral for any particular filament ab. To find the locus of the point s for constant values of this factor, it is sufficient to notice that formula (8) (second form) remains unchanged if the denominator $(As + sB)$ is constant, which gives an ellipse with foci at A and B; the ellipticity $= AB/(As + sB)$ is, by formula (8), equal to tanh $(dN/2dx)$. A family of confocal ellipses has therefore been drawn for different values of dN/dx, and is reproduced in the accompanying figure. Since rotating the elements of ab about AB does not change formula (8), the diagram may be used for all lines including those which do not lie in the plane of AB by following the directions given above. The accuracy obtainable in this method of calculating Neumann integrals depends only upon the number of ellipses shown on the diagram and the precision of the graphical work.

The rotation of a finite straight skew filament ab point by point about AB into the plane of AB, Pp changes it into an arc of a hyperbola having the common perpendicular Pp for a semi-transverse axis and asymptotes through P making the angle e with AB. For, take any point s on ab and set $ss' = x$, $Ps' = y$, then

$$x^2 = y^2 \tan^2 e + Pp^2$$

or

$$\frac{x^2}{Pp^2} - \frac{y^2}{Pp^2 \cot^2 e} = 1.$$

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