A Note on the Possible Effect of Screening in the Theory of Beta-Disintegration

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The present paper considers the possibility that screening by the atomic electrons has an appreciable effect on the energy distribution of β-particles emitted by heavy nuclei. A general formula is derived from which it is possible to conclude that the effect of screening is negligible. This result is also derived by another method. Finally, an explicit calculation based on the model of a charged sphere is shown to lead to the same conclusion.

1.

IN a recent paper Konopinski and Uhlenbeck[1] have remarked that, on the basis of the Fermi theory of β-disintegration with their modified interaction, one finds for heavy elements such as Ra E an energy distribution which predicts too many slow electrons as compared with the experimental results.[2] An attempt by these authors to remove this difficulty by other modifications of the interaction between the heavy particle and the electron-neutrino field was not successful in diminishing the number of slow electrons without at the same time losing the degree of asymmetry in the distribution curve required by experiment.[3] Although there may be some question as to the existence of a real discrepancy between experiment and the theory, as modified by Konopinski and Uhlenbeck, it is of some interest to inquire whether there may be some influence hitherto not considered which could appreciably affect the energy distribution.

In this connection it is plausible to consider the possible effect of screening by the atomic electrons. It is evident, from a qualitative standpoint, that the screening will alter the distribution curve in the direction of the "free particle" distribution[4] which, incidentally, does fit the experimental curve for Ra E as given by Sargent. In the following we shall determine the effect of screening quantitatively.

By enclosing the system in a "sphere" (whose size is afterwards allowed to increase without limit) we may consider the wave function ψ of the β-particle to be of a discrete type. The screening can be represented by a Thomas-Fermi potential[5]

$$V=-(\alpha Z/r)\Phi(\gamma r), \quad \gamma=\alpha(128Z/9\pi^2)^{\frac{1}{3}}. \quad (1)$$

It is to be noted however, that the main argument given below depends on the use of the Thomas-Fermi potential only secondarily.

We now select a point r_0 such that for $r>r_0$ a solution of the wave equation of the W. K. B. type will be valid,[6] and for $r<r_0$ the argument of Φ in (1) is small enough to allow the approximation[7]

$$\Phi=1-\gamma r.$$

Then for $r<r_0$ the wave functions will be Coulombian with the energy W replaced by $W-V_0$ where the constant screening potential $V_0=\alpha Z\gamma$. The phase of the wave function at r_0 will be denoted by χ and the increase in phase from r_0 to the classical limit of motion (or to the boundary of the sphere) by θ. Further we label each state with an index n (radial quantum number). Then

$$\chi_{n+1}-\chi_n+\theta_{n+1}-\theta_n=\pi \quad \text{or} \quad d(\chi+\theta)/dn=\pi. \quad (2)$$

We then apply adiabatically an infinitesimal perturbation at small r producing an energy change

$$\delta W=\lambda\int F(r)|\psi|^2 d\tau_{nuc.}, \quad (3)$$

where $F(r)$ is an arbitrary function of r confined to nuclear dimensions and λ is very small. The integration in (3) is to be taken over the volume of the nucleus.

[1] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **48**, 7 (1935).

[2] B. W. Sargent, Proc. Camb. Phil. Soc. **28**, 538 (1932).

[3] This feature of the experimental results is apparently more reliable than the results for the low energy electrons. See also reference 11.

[4] That is, the distribution obtained by neglecting the Coulomb field of the nucleus.

[5] We use rational relativistic units, length being measured in $\hbar/mc$, energy in mc^2 and momentum in mc. In these units $e^2=\alpha$ (fine structure constant).

[6] Only differences in phase as given by Eq. (2) below rather than the exact values of the phases have to be given correctly by the W. K. B. formulas.

[7] For $Z\sim 80$, $\gamma=0.036$ while the singularity in the W. K. B. solution is at $r\lesssim 1$.

In general χ and θ could depend on λ and on W. Since the W. K. B. method is supposed to apply for $r>r_0$ and since $F=0$ in this region one may consider θ as depending only on W and not on λ while χ naturally depends on both λ and W. The perturbation formula (3) is supposed to apply to a constant n, i.e., to such changes of χ, θ that

$$\delta\chi+\delta\theta=0.$$

Since θ is a function only of W and not of λ this means that

$$(\partial\chi/\partial W+\partial\theta/\partial W)\delta W=-\lambda\partial\chi/\partial\lambda. \quad (4)$$

Substituting for δW and $\partial(\chi+\theta)/\partial W$ by means of (2) and (3) we have

$$(dn/dW)\int F(r)|\psi|^2 d\tau_{nuc.}=-(1/\pi)\partial\chi/\partial\lambda. \quad (5)$$

The right side of this equation is the same for a given F as long as the wave equation is the same. Thus it is the same for an electron of energy W in the field of a screened nucleus and an electron of energy $W-V_0$ in the field of an unscreened nucleus. Such states of screened and unscreened nuclei may be called corresponding states and the essence of the discussion lies in the possibility of establishing such a correspondence. The left side of the equation is the product of the number of states per unit energy range dn/dW and a weighted mean of the density $|\psi|^2$. Since F is arbitrary $(dn/dW)|\psi|^2$ at any point in the nucleus is the same for corresponding screened and unscreened states. This latter quantity we shall refer to as the electron density per unit energy range.

It should be noted that dn/dW and $|\psi|^2$ change by large factors if the standard normalization to the same value of $r^2|\psi|^2$ at infinity is used. Also the function F can be generalized to an operator which may involve the spin indices of the function ψ without altering the above proof. It thus follows that

$$(dn/dW)\Sigma\psi_\mu{}^*G_{\mu\nu}\psi_\nu \quad (5')$$

is the same for corresponding screened and unscreened states for any point inside the nucleus and for an arbitrary Hermitian $G_{\mu\nu}$.

The essential quantities which enter in the theories of β-decay are all of the form $(5')$[8] and the result of the above consideration can be expressed without formulas by saying that the electron distribution is always such as though the nucleus were not conscious of the screening and as though it emitted electrons into its immediate vicinity always in the same way; the only effect of the screening is then to accelerate the electrons by the Coulombian repulsion of the screening electrons and to change the kinetic energy $W-V_0$ into W by this acceleration. From this point of view the result has superficially an appearance of being self-evident. Actually it requires proof because it is possible to find examples in which it is not correct. One such example is given in section 3 of this note. In fact conclusions reached in (5) and (5′) may be inexact because interference effects in the space between the nucleus and the screening charge can change the density $|\psi|^2$. These interference effects are neglected in the W. K. B. method used in obtaining (5). In the applications to β-decay it is reasonable to use such an approximation because the screening potential changes little within a wave-length of the electron.

Proceeding to the disintegration formula we write the Konopinski-Uhlenbeck result for the probability of emission of an electron with energy between W and $W+dW$ as

$$P_{KU}(W)dW=W(W_0-W)^4u(W)dW, \quad (6)$$

W_0 being the maximum emission energy. Then it follows from the above considerations that the effect of screening is to change this to

$$P(W)dW =(W-V_0)(W_0-W)^4u(W-V_0)dW. \quad (6')$$

Since for $Z\sim 80$, $V_0=0.02$, the change in P is of the order 3 percent or less. Hence one may conclude that the effect of screening is negligible.[9]

2.

It is possible to arrive at the conclusion expressed in (5) by another method in which explicit use of the W. K. B. solution is made. We denote by φ_1/r and φ_2/r the radial functions of the Dirac ψ which can be made to fulfill the normalization condition

$$\int_0^R(|\varphi_1|^2+|\varphi_2|^2)dr=1, \quad (7)$$

where R is the radius of a large sphere in which the system under consideration is enclosed. By

[8] Cf. E. Fermi, Zeits. f. Physik **88**, 161 (1934).

[9] I am indebted to Professor G. Breit who pointed out to me the derivation of this result.

the substitution

$$v_2=(1+W-V)^{\frac{1}{2}}\varphi_2$$

one finds that v_2 satisfies the differential equation[10]

$$d^2v_2/dr^2+f^2(r)v_2=0. \quad (8)$$

To (8) the W. K. B. method of solution may be applied. With

$$R[|\varphi_1(\infty)|^2+|\varphi_2(\infty)|^2]=1$$

we find

$$\varphi_2(\infty)=[(T+1)/2RT]^{\frac{1}{2}}\sin(kr+\text{phase}), \quad (9)$$

where $T=W-V$ and the wave number k is given by

$$T^2=k^2+1. \quad (10)$$

In a region where the W. K. B. solution just begins to be applicable we have

$$\varphi_2=\left[\frac{T'+1}{2RT}\frac{k}{k'}\right]^{\frac{1}{2}}\sin(k'r+\text{phase}), \quad (11)$$

where k' is the wave number and T' the kinetic energy in this region.

Since R can be made much larger than any other length entering the problem, the phase shift in the wave function is essentially given by

$$R\Delta k=\pi. \quad (12)$$

From (10), (11) and (12) we find that

$$|\varphi_2|^2/\Delta T=(1+T')/2\pi k', \quad (13)$$

which is independent of k and T. Hence keeping k' the same it follows that $|\psi|^2/\Delta T$ is independent of the screening. Since the screening has the effect of changing k' in accordance with the energy shift $W\rightarrow W-V_0$, it is evident that the electron density per unit energy range is given by the corresponding expression in the case of no screening with the energy W replaced by $W-V_0$.

3.

An alternative way in which we could represent the screening is to use a charged sphere potential.

$$V=-\alpha Z(1/r-1/r_s) \quad \text{for} \quad r<r_s,$$
$$V=0 \quad \text{for} \quad r>r_s; \quad r_s=1/\gamma.$$

With this model the calculation of the disintegration probability may be carried out with exact wave functions and thus it is possible to take into account the interference effects arising from the reflection of the electron waves by the charged sphere. The result one obtains is not the disintegration probability (6′) but a more complicated expression which contains (6′) as a first approximation.

For $r<r_s$ the wave function is again the Coulombian one with the energy displaced by an amount V_0 while for $r>r_s$ linear combinations of Bessel functions are used. The normalization constant entering in $|\psi|^2$ will depend on the wave functions at the point r_s. For large Z the "atomic radius" r_s is large compared to unity ($Z\sim 80$, $r_s=28$) so that we may use the asymptotic expansions of the wave functions. The disintegration probability thus obtained is

$$P_s(W)dW=P(W)[1+O(r_s^{-2})]dW, \quad (14)$$

where $P(W)$ is given in (6′). The correction term in (14) denoted by $O(r_s^{-2})$ arises from terms beyond the first in the asymptotic expansions of the wave functions. These higher terms are of an oscillatory nature and an average over a range of large values of the radius r_s has been carried out. Actually the term $O(r_s^{-2})$ contributes a correction much smaller than the relative difference between $P(W)$ and $P_{KU}(W)$ so that to a first approximation the screening is represented in the case of the present model in the same manner as before.[11]

I wish to express my sincere thanks to Professor G. Breit for his many courtesies to me.

[10] Cf. G. Breit, Phys. Rev. **38**, 463 (1931). The function $f^2(r)$ is defined on p. 470 of this paper.

[11] Since the completion of these calculations there has appeared a notice of a new investigation of the continuous β-ray spectrum of Ra E by A. I. Alichanow, A. I. Alichanian and B. S. Dzelepow, Nature **137**, 314 (1936). In these experiments particular attention has been given to the low energy portion of the distribution and it was found that a large number of slow electrons are emitted. These authors have made measurements starting at an energy (not including the rest energy) of 30 kv and have obtained a distribution curve which appears to be in agreement with the Konopinski-Uhlenbeck theory. The appearance of their curve suggests that the probability does not tend to zero at the origin, as is the case with the Sargent curves, but rather to a finite value comparable with the maximum probability. With these new measurements as a criterion in the case of heavy nuclei and with the results for light nuclei, see especially F. N. D. Kurie, J. R. Richardson and H. C. Paxton, Phys. Rev. **49**, 368 (1936), it seems that the Konopinski-Uhlenbeck theory is capable of accounting for all the features of the continuous β-ray spectrum.