On the Geomagnetic Analysis of Cosmic Radiation

G. LEMAITRE AND M. S. VALLARTA,* *University of Louvain and Massachusetts Institute of Technology*

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The results of an extensive study of trajectories asymptotic to a certain family of periodic orbits in the earth's magnetic field, carried out by means of Bush's differential analyzer, are presented in this paper. The theory of the region of full light, or main cone, is fully discussed. Attention is then restricted to the section of the main cone in the plane of the geomagnetic meridian and it is shown that the north-south asymmetry furnishes the most direct approach to the analysis of the energy spectrum in a wide region, independently of the particles' sign. Further it is shown that the general shape and the minimum of the north-south asymmetry discovered by Johnson in the course of his Mexican experiments are fully accounted for by the action of the earth's field.

1. INTRODUCTION

RECENT experiments by the method of multiple coincidences carried out by Johnson[1] in Mexico (geomagnetic latitude 29°N) and by Clay[2] in Java (geomagnetic latitude 18°S) have shown that in the northern geomagnetic hemisphere the intensity of cosmic radiation in the geomagnetic meridian is, for equal zenith angles, greater from the south than from the north; conversely in the southern hemisphere it is greater from the north than from the south.[3] That this is a consequence of the action of the earth's magnetic field on the motion of charged particles was pointed out by the present authors as early as 1932.[4] Shortly afterwards Bouckaert[5] was able to calculate this north-south asymmetry for geomagnetic latitudes up to 20° and moderate zenith angles. Considerable difficulties stand in the way of extending these calculations to higher latitudes and larger zenith angles. The use of Bush's differential analyzer,[11] which was made available to us to carry out the investigation reported here, has made it possible to include in the present analysis latitudes as high as 40° and, in most cases, zenith angles as far as the horizon. Thus with the completion of these researches the problem of determining the allowed main cone, or region of full light, for the latitudes mentioned is essentially solved.

The chief results of our present research are, first, that the north-south asymmetry, already discussed in a previous paper,[6] depends for each latitude on a narrow band of energy. Therefore, as further shown in the sequel, the experimental study of this asymmetry provides a direct workable method for the analysis of the spectral distribution of corpuscular cosmic radiation, independent of the latter's sign. Second, we show that the minimum of the north-south asymmetry found by Johnson at a geomagnetic latitude 29°N and at about 45° zenith angle, as well as the general features of his experimental results, are fully accounted for by the action of the earth's field. In fact the theory predicts a slight reversal of the sign of the asymmetry for zenith angles between 45° and 55° which depends on the existence of a very narrow energy band. This reversal begins at latitudes around 20° for for zenith angles about 70° and eventually broadens out so as to include the whole region between the zenith and 40° at latitudes around 35°. Contrary to our earlier[6] suggestion it is not necessary to invoke atmospheric absorption in order to account for this minimum.

The results we wish to present here must be considered as preliminary insofar as a critical examination of their precision is concerned. There is in fact ground for suspecting that our method of finding asymptotic trajectories leads to slightly too large a value for the aperture of

* At present Visiting Professor of the C.R.B. Educational Foundation at the University of Louvain.

[1] T. H. Johnson, Phys. Rev. **47**, 91 (1935).

[2] J. Clay, Physica **8**, 867 (1935).

[3] A northern excess in Eritrea (geomag. lat. 11.5°N) at a zenith angle of 45°, slightly larger than the rather large experimental error, has been reported by B. Rossi, Ricerca Scient. **5**, 583 (1934).

[4] G. Lemaitre and M. S. Vallarta, Phys. Rev. **43**, 87 (1933).

[5] L. Bouckaert, Ann. de la Soc. Sci. de Bruxelles, **A54**, 174 (1934).

[6] G. Lemaitre, M. S. Vallarta and L. Bouckaert, Phys. Rev. **47**, 434 (1935). Full references bearing on our application of Liouville's theorem to the present problem are given in the footnote on p. 435.

the main cone. The whole question as to the precision of our present results cannot be finally settled before more rigorous calculations of asymptotic trajectories in which we are now engaged are carried through to completion; we hope to return to this question shortly.

2. Outline of the Theory of the Allowed Cone

Let us now review briefly the statement of our problem and our method of attack. We start from the assumption of an isotropic distribution of charged particles at very large distances from the earth. Because of the action of the earth's magnetic field some of these particles are shot back to infinity, others are allowed to reach the earth. We have then shown[4, 7] that at any point a given distance from the earth's magnetic center all particles of a given energy reaching that point must come within a cone, generally of very involved shape, which forms the boundary between the region where all directions are allowed—the full light region, to borrow from the terminology of optics—and the region where only some or no direction are allowed. In any allowed direction, however, we have shown that[6] the intensity is precisely the same as would exist in the absence of the field and hence the same as it is at infinity.

The problem is thus to determine the generators of the cone. These generators, as shown by one of us[7] following ideas already outlined in our first paper,[4] may be of either of two kinds. The trajectories of the first kind are asymptotic to a periodic orbit, in fact asymptotic to periodic orbits forming a family first studied by Störmer.[8] This family has been calculated in detail by one of us,[9] as well as trajectories asymptotic to the members of this family in the immediate vicinity of the equatorial plane.[7] It was there shown, moreover, that some of these trajectories are doubly asymptotic, i.e., asymptotic to the same periodic orbit in the past as well as in the future. It follows further that asymptotic trajectories in the vicinity of these doubly asymptotic orbits separate again into trajectories going to infinity and trajectories returning to the earth. The latter may return infinitely near the first asymptotic trajectory; further among them is an infinite number of other doubly asymptotic trajectories. One of us has carried out the analysis of this maze of trajectories in the immediate vicinity of the equator.[7] The consequence is that the full light region is surrounded by an infinity of infinitely narrow bands of extremely complicated structure. This region we have called, borrowing again from the language of optics, the region of penumbra.

We have thus been led to distinguish among the trajectories of the first kind those which directly reach the earth, i.e., without returning to the vicinity of the periodic orbit. These asymptotic trajectories form the generators of what we have called the main cone or region of full light. As mentioned above the latter is surrounded by the penumbra. The distinction between full light and penumbra is of course only a matter of convenience and in no wise is to be looked upon as fundamental.

In addition to the trajectories of the first kind which are asymptotic to a periodic orbit there are also trajectories of the second kind which can form part of the boundary of the main cone.[7] These trajectories are tangent to the earth before reaching the point of observation. They limit a part of the main cone because of what we might call, again to borrow from the vocabulary of optics, the shadow of the earth.

We have finally to recall briefly some of the properties of the differential equations of motion of a charged particle in the earth's dipole field to which the above discussion applies. These equations have been first set up, studied and made generally well known by Carl Störmer to whom the results we are about to recall are also due.

First by suitably defining the units of time and length one may study at once trajectories of different particles and correspondingly different energies. For a given particle and for each energy the standard earth's radius is represented by a certain length which measures the energy of the particle. We suggest that this unit of length be called in the future a Störmer unit, or briefly a Störmer, in honor of its discoverer. For high energy particles this length is very nearly proportional to the square root of the energy.

[7] G. Lemaitre, Ann. de la Soc. Sci. de Bruxelles **A54**, 165 (1934).
[8] C. Störmer, Zeits. f. Astrophys. **1**, 237 (1930).
[9] G. Lemaitre, Ann. de la Soc. Sci. de Bruxelles **A54**, 194 (1934).

Second, the motion of a particle may be resolved into two components: one in the meridian plane and the other a rotation of the meridian plane around the magnetic axis. The calculation of this last motion has no bearing on our problem and consequently is left out in the sequel.[10] If r and λ are polar coordinates fixing the position of the particle in the meridian plane then the space trajectory intersects this plane at an angle θ given by Störmer's equation

$$\sin \theta = -2\gamma_1/(r \cos \lambda) + (\cos \lambda)/r^2. \tag{1}$$

The constant γ_1 is a constant of the motion; it is the component of momentum of the particle conjugate to the ignorable coordinate of the meridian.

For purposes of calculation it is convenient as noted by Störmer to introduce a new variable x by the transformation

$$e^x = 2\gamma_1 r. \tag{2}$$

If now x and λ are looked upon as rectangular coordinates a conformal map of the meridian plane is obtained where the radius vector r becomes a line parallel to the λ axis and the circles $r=$const. become lines parallel to the λ axis. As already explained above the earth is represented on this map by a line parallel to the λ axis for each value of the energy r in Störmer units. The angle η between a trajectory and the zenith direction is therefore read off directly on the conformal map since it is the slope defined by

$$\tan \eta = d\lambda/dx. \tag{3}$$

Finally in terms of an independent variable σ, function of the time, the equations of motion become

$$d^2x/d\sigma^2 = \tfrac{1}{2}\partial P/\partial x, \qquad d^2\lambda/d\sigma^2 = \tfrac{1}{2}\partial P/\partial \lambda, \tag{4}$$

where

$$\begin{aligned} P &= (dx/d\sigma)^2 + (d\lambda/d\sigma)^2 \\ &= e^{2x}/16\gamma_1^4 - (e^{-x}\cos\lambda - 1/\cos\lambda)^2 \end{aligned} \tag{5}$$

and from (1) and (2) it is readily seen that the locus of the points for which $\theta = 0$ is given by

$$x = 2 \log \cos \lambda \tag{6}$$

and represents on the conformal map those points of the cone lying on the meridian plane.

3. The Determination of Asymptotic Trajectories

The capacity of the differential analyzer[11] now available at the Massachusetts Institute of Technology is just sufficient to integrate the system of differential equations (4). An ingenious set of machine connections was devised for this problem by S. H. Caldwell, but for lack of space the details of operation must be omitted here. Besides a preliminary attempt in the winter of 1932 the work was carried out at two different periods. During the first period (December, 1933) we found about a score asymptotic trajectories for $1/16\gamma_1^4 = 0.07$, $\gamma_1 = 0.972$. The initial slopes were calculated from the first terms of an analytical expansion of the asymptotic trajectory valid in the vicinity of the periodic orbit. This method of attack provided valuable information on the general appearance and properties of asymptotic trajectories and gave one of us the incentive for the introduction of harmonic expansions such as those used later by himself[7] and by Bouckaert.[5] It suffers from the serious disadvantage that the trajectory is started from a region of instability particularly sensitive to small errors either of the machine or in the initial data. During the second period (spring of 1935) we preferred, for this reason, to start the trajectory outwards from a suitably chosen point on the equator and then to adjust the initial slope so that the trajectory would neither intersect nor fall short of the periodic orbit, which was already accurately known.[9] It was found possible to adjust the initial slope after a few runs with a precision usually around one-thousandth of a radian. When the initial slopes had been found for several points on the equator the initial data at intermediate points were determined by graphical interpolation and a score trajectories were started inwards. Later it became obvious that certain interesting trajectories would be missed in this way and so, instead of starting from the equator, appropriate

[10] The motion of the meridian plane is important in connection with the variation of intensity due to the rotation of the galaxy which has been discussed by A. H. Compton and I. A. Getting, Phys. Rev. **47**, 817 (1935). This point has been partially studied by L. A. van Wijk and H. Zanstra, Physica **3**, 75 (1936). See also C. Störmer, Astrophys. Norv. **1**, 157 (1935).

[11] For a full description of the differential analyzer and its operation see V. Bush, J. Frank. Inst. **212**, 447 (1931).

points were chosen on the line $\theta=0$ and trajectories were run from these points outwards. The initial slope was then adjusted so that the trajectory would be tangent to the envelope of the asymptotic family, which had been previously determined from those asymptotic trajectories already known. As an additional verification a few trajectories were run as far as the periodic orbit. After the initial data at intermediate points had been found by graphical interpolation as in the case of the equatorial runs, a score or more asymptotic trajectories were continued inwards. We have thus studied the following values of γ_1: 0.81, 0.83, 0.85, 0.87, 0.89, 0.91, 0.93, 0.95 and 0.99. Runs from equatorial points were made for the following intermediate even values of γ_1: 0.88, 0.90, 0.92, 0.94, 0.96, 0.98. In all some three hundred asymptotic trajectories were traced.

As mentioned in the introduction, a critical study of the precision of the results thus obtained must in the final analysis await the outcome of calculations now in progress. We should mention at the outset that fairly systematic checks based on the use of the energy Eq. (5) do not reveal serious systematic errors. Only in one instance did we find a small systematic error which was later traced to incorrect plotting of one of the known functions introduced in the machine. When this mistake was corrected and a part of the previous work was repeated, the observed changes in the trajectories were so small that it was not deemed necessary to repeat all of the previous runs, some twenty-five trajectories in all. One can, however, mostly judge of the accuracy of the present material from the point of view of its self-consistency. Asymptotic trajectories possess, as a matter of fact, a complicated system of envelopes to be more fully discussed below and even small accidental errors are immediately visible. It should be borne in mind that the final precision of a curve traced by the differential analyzer depends not only on the mechanical trustworthiness of the machine itself, but also on the accuracy of the laborious and often complicated numerical calculations required to introduce the initial conditions into each of the members of a coupled system of six integrating units and five input tables. It even depends on the care and faithfulness with which a crew of five assistants keep in step their respective cross hairs on the known functions plotted on metallic plates on the input tables. Our guiding principle for the reduction of the data obtained with the help of the differential analyzer has thus been to accept only such conclusions as are based on the study not of a single, but of a good many consistent trajectories.

4. General Description of Asymptotic Trajectories and Their Envelopes

After these considerations we pass on to a description of some general features of asymptotic trajectories. To begin with we should mention the trajectories calculated by Carl Störmer[12] by the method of numerical integration which reach the pole $r=0$ or $x=-\infty$. These trajectories are osculating to the line $\theta=0$ already mentioned above. Although most of them belong to a family different from the one we are concerned with here, there are a few which are members of both, in particular one for γ_1 between 0.9313 and 0.9314 and a few which are of importance for the study of the penumbra. Störmer has shown that for sufficiently small values of x all other trajectories oscillate about the nonoscillating orbit going to the pole. This is precisely the general character of the trajectories we have obtained. They oscillate about Störmer's nonoscillatory orbit; for decreasing values of x the loops come closer together and the trajectory oscillates outwards in the same manner as inwards. An example of a family of asymptotic trajectories is shown in Fig. 1, where for the sake of completeness the locus of the vertices of the family of periodic orbits and the line $\theta=0$ are also drawn.

A family of asymptotic trajectories possesses an extensive family of envelopes which we are now about to describe (Figs. 1 and 2). Beginning at the periodic orbit there is an envelope denoted by E_0 which first proceeds in a direction fairly parallel to the x axis and then rises rapidly towards a cusp C_0 where it meets a corresponding envelope proceeding upwards, which we denote by F_0. The system of envelopes continues in a similar manner; corresponding branches are

[12] C. Störmer, Terr. Mag. and Atmos. Elec. **37**, 375 (1932). Astrophys. Norv. **1**, 145 (1935).

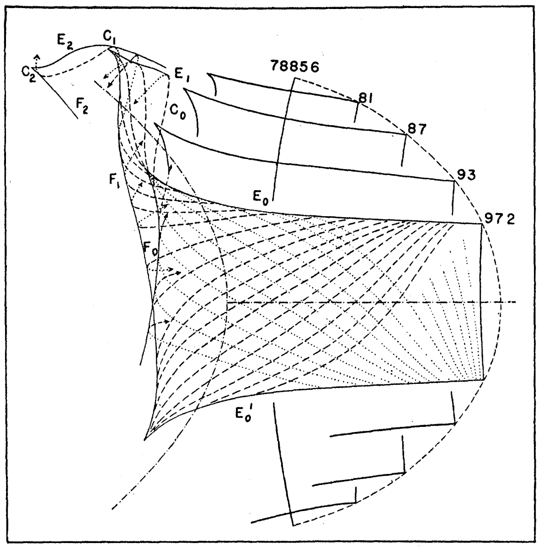

FIG. 1. The family of envelopes and periodic orbits, and a family of asymptotic trajectories ($\gamma_1 = 0.972$).

denoted by C_1, $C_2 \cdots$, F_1, F_2, $\cdots$. These envelopes are tangent to the line $P=0$ (not drawn in the figure) at points which are denoted by S for the E envelopes and T for the F envelopes. At any S or T there is an asymptotic trajectory turning back on itself, i.e., a self-reversing doubly asymptotic trajectory. On either side of S or T an asymptotic trajectory touches the envelope and then proceeds in directions away from S or T. In particular the point S_0 is the vertex of the periodic orbit. The corresponding vertex for negative value of λ, and in general the corresponding points of tangency below the x axis will be denoted in the following by primed symbols.

The trajectories are divided by their points of contact with their envelopes into sections (Figs. 1 and 2). According to a known theorem in the theory of kinetic foci[13] any trajectory infinitely near an asymptotic orbit intersects each one of the sections once only in any finite domain of variation of σ. All asymptotic trajectories tend towards periodic orbits while performing an infinite number of oscillations between the envelope E_0 and its symmetrical counterpart E_0'. The trajectory in its infinitely close vicinity will therefore intersect each of the sections into which it is divided by its points of contact with its envelope. One may therefore choose the neighboring trajectory sufficiently close to the asymptotic orbit that the former may intersect the latter's sections as many times as desired but finally the former will pull away from the latter either by cutting the periodic orbit or by returning inwards. We thus have to distinguish between two kinds of sections (Figs. 1 and 2): northern sections (dashed) are those for which an infinitely close trajectory intersecting the asymptotic trajectory at an angle *smaller* than its own slope at that point finally cuts the periodic orbit. They form the northern boundary of the full light region or main cone. Southern sections (dotted) are such that an infinitely close trajectory intersecting the asymptotic trajectory at an angle *greater* than its own slope finally cuts through the periodic orbit. They form the southern boundary of the main cone. We have concluded from a study of the trajectories drawn by the differential analyzer that the sections coming from E_0 are northern sections, those coming from E_0' are southern sections. If we denote their slopes by η and ζ, respectively, we have the result that all trajectories included

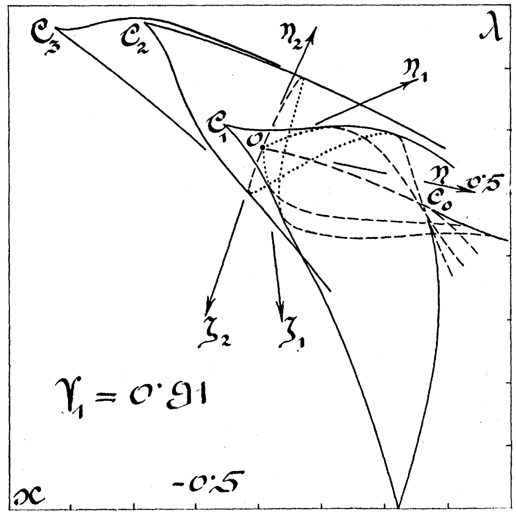

FIG. 2. Northern and southern sections of asymptotic trajectories.

[13] It is well known that the equation of the distance between trajectories infinitely close to some trajectory and this trajectory, as a function of the arc measured along the trajectory in question, is linear, homogeneous and of the second order, cf. E. T. Whittaker, *Analytical Dynamics* (Cambridge, 1904), p. 382. By a known result due to Sturm the zeros of such an equation separate one another, cf. M. Bocher, *Leçons sur les méthodes de Sturm* (Paris, 1917), p. 46. See also H. Poincaré, *Méthodes nouvelles de la mécanique céleste*, Vol. 3, p. 261.

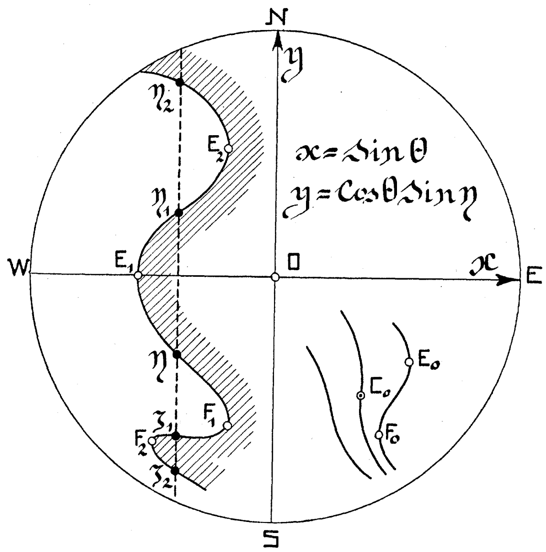

FIG. 3. Schematic representation of main cone.

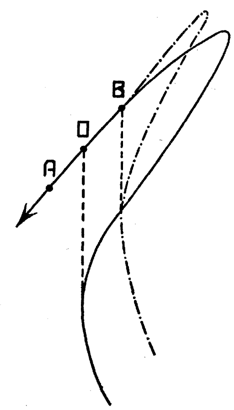

FIG. 4. Typical trajectories of second and third kinds.

between η and ζ come from infinity. The region between η and ζ lies within the main cone of which, in agreement with the results outlined in a preceding section, asymptotic trajectories are the generators. Each time that an asymptotic trajectory has a point of contact with its envelope the sections bounded by this point of contact are of different kinds. Sections of asymptotic trajectories diverging from the first cusp C_0 are therefore northern sections. After having touched the envelopes E_1 or F_1 they become southern sections and therefore the sections diverging from the cusp C_1 are southern sections. In the region between the two cusps C_0 and C_1 we may therefore have *three* asymptotic trajectories through the same point. We denote by η the slope of the trajectory coming from the cusp C_0, by η_1 the slope of the trajectory tangent to E_1 and by ζ_1 that of the curve tangent to F_1. At the next cusp C_1 all three slopes η, η_1 and ζ_1 become the same. Moreover, even before reaching C_1 our trajectories may have touched other envelopes E_2 or F_2 returning with slopes η_2 and ζ_2. What conclusions are to be drawn from these facts regarding the structure of the main cone?

Let us assume for instance that we have obtained, for a certain point x, λ and a given value of γ_1 the directions η, η_1, ζ_1, η_2, ζ_2. The latter belong to the given value of the latitude λ and are related to the energy r and the angle θ by (2) and (1), respectively. For positive values of λ, i.e., in the northern hemisphere and for positive particles θ will be taken positive eastwards. We have adopted for our representation of the cone (Fig. 3) the orthogonal projection on the horizontal plane of its trace on the unit sphere centered at the point in question. In accordance with the conventions stated above we take the coordinates of our representative point, i.e., $\sin\theta$ and $\cos\theta \sin\eta$ as positive eastwards and northwards, respectively. Then the cone will have the general appearance shown *schematically* in Fig. 3, the full light region being to the west. Let us now see the significance of the envelopes and cusps. The envelope E_2 is such that on it η_1 and η_2 coincide. Generally envelopes are double points on the trace of the cone and there is a tangent parallel to the meridian plane. Cusps are triple points where the trace of the cone has a point of inflection with a tangent parallel to the meridian plane. These general principles suffice to interpret asymptotic trajectories in order to determine the main cone or region of full light.

It remains to show how the shadow of the earth limits a part of the main cone. In order that the preceding considerations remain valid it is necessary of course that all the asymptotic trajectories we have considered remain outside the earth. Hence if any of them has anywhere a tangent parallel to the λ axis it retains its previously discussed significance only for sufficiently small values of the energy, less than that which corresponds to the position of the tangent.

At a point such as A (Fig. 4) the η_1 direction gives a generator of the main cone, but at B the latter is already bounded by a trajectory of the second kind. At a point such as O the main cone is bounded by a trajectory which is at once of the first and of the second kinds. Such points indicate where the region of shadow, or region of the second kind, begins. A trajectory which is at once of the first and of the second kinds we shall briefly refer to as a trajectory of the third kind.

The complete main cones, as well as a fuller discussion of the azimuthal effect than has been hitherto possible, are reserved for a forthcoming companion paper.

5. Calculation of the North-South Asymmetry

Directions in the meridian plane correspond on the conformal map to the line $x=2 \log \cos \lambda$ Eq. (6) for which $\theta=0$ by Störmer's formula (1). It is quite feasible to measure for each asymptotic trajectory cutting through this line the slope η and the corresponding value of the latitude λ. These values of η and λ can then be plotted on a diagram, for each value of γ_1, and a smooth curve drawn through the plotted points (Fig. 5). As is to be expected in accordance with our previous discussion a few of these points fall clearly outside the graph indicated by the other points, but in general the plotted curves are determined without ambiguity. For regions where sufficient data are missing it is easy to interpolate between neighboring well-determined curves. Particular attention should be paid in connection with this diagram to the significance of the envelopes and cusps. The intersection of an envelope with the line $\theta=0$ is shown as a point where the tangent to the plotted curve is parallel to the η axis. These are the "loci of maximum latitude" shown on the diagram. A point where a cusp lies on the line $\theta=0$ is a double point of the curves plotted. Its characteristic feature is that two branches of the locus $\gamma_1=\text{const.}$ intersect, one of them having a tangent parallel to the η axis. This is clearly so because on one side of the cusp, as already discussed elsewhere, there are three asymptotic directions while on the other there is but one. The value of γ_1 for

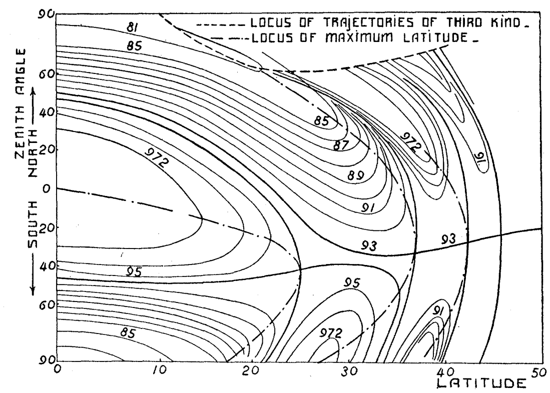

FIG. 5. The zenith angle in the meridian plane as a function of latitude.

which the first cusp C_0 lies on the line $\theta=0$ is somewhere near $\gamma_1=0.94$. The following cusps C_1, C_2, etc., lie practically on it for $\gamma_1=0.93$. Further, Störmer's results quoted above show that these values form a progression the limiting value of which is 0.9313.

The trajectories of the third kind are important because they mark the point where the earth's shadow begins to appear. It is usually possible, after examination of the family of asymptotic trajectories drawn by the differential analyzer, to decide with good approximation where a trajectory of the third kind appears and to interpolate among neighboring curves to find the angle at which it intersects the line $\theta=0$. The locus of the values of zenith angle and latitude for which a trajectory of the third kind appears, for different values of γ_1, is shown in Fig. 5. It should be carefully borne in mind, in interpreting this locus, that a region may be in shadow for a given value of γ_1 and not for others. Thus for instance a point in shadow for $\gamma_1=0.93$ is still in the region of light for $\gamma_1=0.81$ at $\lambda=20°$.

The relation connecting γ_1 and the energy r for points on the meridian plane at a geomagnetic latitude λ is, from (1)

$$r=(\cos^2 \lambda)/2\gamma_1. \tag{7}$$

By using this formula and the results of the previous analysis the least energies required for a particle to arrive at any given zenith angle in the meridian plane, at any point on the earth up to latitudes of 40°, may be readily found. It is simply necessary to read off the $\eta\lambda$ diagram

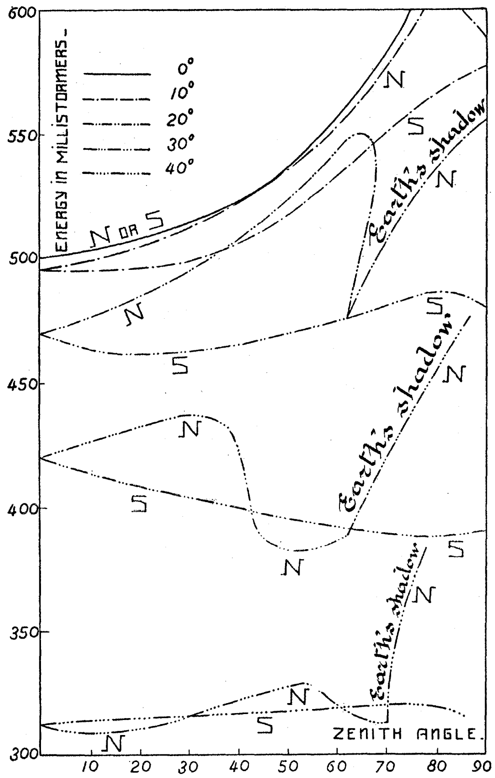

FIG. 6. North-south asymmetry.

(Fig. 5) the value of γ_1 corresponding to the given η and λ, interpolated if necessary. Typical curves relating the least energy of arrival with the zenith angle for latitudes 0°, 10°, 20°, 30° and 40° are reproduced in Fig. 6. It should be remembered that the energy (in millistörmer units) is measured with respect to the standard earth's radius (6370 km) and hence for observations at a given point on the earth account must be taken of the eccentricity of the earth's magnetic center.[14] Several important features are to be emphasized in connection with these curves. First it will be noted that, irrespective of the particles' sign, the energy region explored by the north-south asymmetry varies considerably with the latitude. The geomagnetic field thus provides for the separation of energies required for a spectral analysis of the corpuscular cosmic radiation. Experiments on this asymmetry thus furnish the most direct approach to the study of the energy spectrum. For this reason it is earnestly to be hoped that they will be continued particularly in the region between 35° and the geomagnetic equator, both at high and low altitudes above sea level. In this way valuable information will be obtained not only in view of the purpose mentioned above but also for the study of atmospheric absorption and scattering of particles of different energies.

Secondly it will be seen that the theory gives a full account of the minimum found by Johnson[1] in his Mexican experiments at a zenith angle of around 45°. It goes even further and, in a roughly quantitative way commensurate with present experimental precision, accounts for his whole result.

Lastly the effect of the earth's shadow should be carefully noted. It results in a large north-south asymmetry at high zenith angles for intermediate latitudes. For low latitudes the asymmetry near the horizon should therefore be reduced.

Acknowledgments

A piece of work as extensive as that briefly reported here could not have been brought to a conclusion without the cooperation of many. It is a pleasure to acknowledge our indebtedness to Dean Vannevar Bush who made the differential analyzer at the Massachusetts Institute of Technology available to us on several occasions and placed the very complete resources of his laboratory at our disposal. Professor Samuel H. Caldwell, in charge of the Research Laboratory, not only helped us with his knowledge of the intricacies of the differential analyzer, but also contributed an ingenious method of setting up our problem on the machine. His assistants headed on different occasions by Messrs. E. W. Kimbark, J. J. Jager and R. D. Taylor were responsible for its efficient operation during long hours of tedious work. Dr. L. Bouckaert and Mr. E. J. Schremp helped us with the elaborate numerical computations required. Mr. L. de Borman carried out a good deal of the reduction of the data. One of us is indebted to the Massachusetts Institute of Technology and the C.R.B. Educational Foundation for the opportunity of continuing these researches at the University of Louvain.

[14] M. S. Vallarta, Phys. Rev. **47**, 647 (1935).