The Transformation of Reference Systems in the Page Relativity

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(Received March 16, 1936)

1

IN his paper, *A New Relativity*, Page[1] introduces reference systems of particle observers where the comparison of times and the measurement of distances are based solely on geometrical optics. Hence, by means of optical experiments, two such systems will be indistinguishable if suitable coordinates (x, y, z, t) can be introduced for the observers of each system such that light travels along straight lines with constant speed and the space coordinates (x, y, z) are Cartesian coordinates of a Euclidean space. These properties are preserved by Lorentz transformations. Page considered the question of the existence of such reference systems which are not Lorentz transforms of one another. He answered the question in the affirmative by constructing systems with elements having constant acceleration with respect to an original inertial system. He also proposed the problem of investigating other types of such systems and the present paper contains the determination of all these. It is further shown that the assumption of constant light velocity plus the existence of one reference system with straight light paths implies the straightness of light paths in all such systems.

2

Consider two reference systems S and S' of the above type. A correspondence is obtained between S and S' as follows. To each particle observer P and time t in S we associate the incident particle observer P' at the time t' in S'. We consider the four dimensional space-time manifolds (x, y, z, t) and (x', y', z', t') describing the observers in S and S'. The above correspondence may then be written

$$\begin{aligned} x' &= \phi_1(x, y, z, t), \quad & z' &= \phi_3(x, y, z, t), \\ y' &= \phi_2(x, y, z, t), \quad & t' &= \phi_4(x, y, z, t). \end{aligned} \tag{T}$$

We introduce the vector notation $\mathbf{x}=(x, y, z, t)=(x_1, x_2, x_3, x_4)$. The scalar product $\mathbf{a}\cdot\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3-c^2a_4b_4.$$

The light paths in S are of the form $\mathbf{x}=\mathbf{a}+\lambda\mathbf{b}$ where $\mathbf{a}$ is a vector from the origin to a point on the path, $\mathbf{b}$ is an "isotropic" vector, i.e., $\mathbf{b}\cdot\mathbf{b}=0$, and λ is a real parameter. Our problem is to completely characterize (T) under the hypothesis of constant light velocity, i.e., all light paths $\mathbf{a}+\lambda\mathbf{b}$, $\mathbf{b}\cdot\mathbf{b}=0$ in S are carried into isotropic curves in S', i.e., curves for which

$$d\mathbf{x}\cdot d\mathbf{x}=dx^2+dy^2+dz^2-c^2dt^2=0.$$

We may reduce the problem to known results by the theorem:

If all light paths $\mathbf{x}=\mathbf{a}+\lambda\mathbf{b}$ *in* S *are carried into isotropic curves by* (T), *then* (T) *leaves the differential equation* $dx^2+dy^2+dz^2-c^2dt^2=0$ *invariant.*

For since (T) carries $\mathbf{a}+\lambda\mathbf{b}$ into an isotropic curve for all $\mathbf{b}\cdot\mathbf{b}=0$, on substitution in (T) we see that

$$\sum_{i=1}^{3}\left(\sum_{j=1}^{4}\frac{\partial\phi_i}{\partial x_j}b_j\right)^2-c^2\left(\sum_{j=1}^{4}\frac{\partial\phi_i}{\partial x_j}b_j\right)^2=0. \tag{1}$$

But $d\mathbf{x}$ is an isotropic vector if $\mathbf{x}(\lambda)$ is an isotropic curve. Hence, replacing b_j in (1) by dx_j we obtain $d\mathbf{x}'\cdot d\mathbf{x}'=0$, or the transform $\mathbf{x}'(\lambda)$ of any isotropic curve $\mathbf{x}(\lambda)$ is isotropic.

The problem is thus reduced to that of determining all transformations which preserve $dx^2+dy^2+dz^2-c^2dt^2=0$. If we allow the coordinates to vary over a complex domain, t may be replaced by $(-1)^{\frac{1}{2}}(\tau/c)$ and the problem becomes that of determining the transformations which leave $dx^2+dy^2+dz^2+d\tau^2=0$ invariant. All such transformations were given by Lie[2] who showed that they are all conformal transformations and form a group depending on 15 parameters. He further showed that this group is generated by the orthogonal transformations which carry the expression $(x-x_1)^2+(x-x_2)^2+(x-x_3)^2+(x-x_4)^2$ into a constant multiple of itself, and the trans-

[1] Leigh Page, *A New Relativity*, Phys. Rev. **49**, 254 (1936).

[2] S. Lie, *Theorie der Transformationsgruppen* (Leipzig, 1893).

formations by reciprocal radii. The latter have, of course, singular points which are carried to infinity. In the following paragraph we consider properties of the analog of these reciprocal radii transformations.

3

The analog of the transformation by reciprocal radii is

$$\mathbf{x}' = P_0(\mathbf{x}) = \mathbf{x}/(\mathbf{x}\cdot\mathbf{x}). \tag{2}$$

This transformation is valid for all elements (x, y, z, t) which are not on a light path through the origin. The effect of (2) on the direction $d\mathbf{x}/d\lambda$ of a tangent to a curve $\mathbf{x}=\mathbf{x}(\lambda)$ is given by

$$\frac{d\mathbf{x}'}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}}\right)$$

$$= \frac{1}{(\mathbf{x}\cdot\mathbf{x})^2}\left[(\mathbf{x}\cdot\mathbf{x})\frac{d\mathbf{x}}{d\lambda} - 2\mathbf{x}\left(\mathbf{x}\cdot\frac{d\mathbf{x}}{d\lambda}\right)\right]. \tag{3}$$

In particular, if $\mathbf{x}=\mathbf{x}(\lambda)$ is a light path not through the origin we have $\mathbf{x}(\lambda)=\mathbf{a}+\lambda\mathbf{b}$, $\mathbf{b}\cdot\mathbf{b}=0$. Hence $d\mathbf{x}/d\lambda = b$ and

$$\frac{d\mathbf{x}'}{d\lambda} = \frac{1}{(\mathbf{x}\cdot\mathbf{x})^2}[(\mathbf{x}\cdot\mathbf{x})\mathbf{b} - 2\mathbf{x}(\mathbf{x}\cdot\mathbf{b})]$$

$$= \frac{1}{(\mathbf{x}\cdot\mathbf{x})^2}[(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - 2\mathbf{a}(\mathbf{a}\cdot\mathbf{b})].$$

Since the vector factor on the right is independent of λ the direction of the tangent is a constant and we have the theorem:

The image of a light path is a straight line.

The speed of an element described by the four-dimensional curve $\mathbf{x}=\mathbf{x}(\lambda)$ is c if and only if $(d\mathbf{x}/d\lambda)\cdot(d\mathbf{x}/d\lambda)=0$. That the velocity of light is constant follows simply. However it is convenient to calculate $(d\mathbf{x}'/d\lambda)\cdot(d\mathbf{x}'/d\lambda)$. We obtain

$$\frac{d\mathbf{x}'}{d\lambda}\cdot\frac{d\mathbf{x}'}{d\lambda} = \frac{1}{(\mathbf{x}\cdot\mathbf{x})^4}\left[(\mathbf{x}\cdot\mathbf{x})\frac{d\mathbf{x}}{d\lambda} - 2\mathbf{x}\left(\mathbf{x}\cdot\frac{d\mathbf{x}}{d\lambda}\right)\right]$$

$$\cdot\left[(\mathbf{x}\cdot\mathbf{x})\frac{d\mathbf{x}}{d\lambda} - 2\mathbf{x}\left(\mathbf{x}\cdot\frac{d\mathbf{x}}{d\lambda}\right)\right] = \frac{1}{(\mathbf{x}\cdot\mathbf{x})^4}\left(\frac{d\mathbf{x}}{d\lambda}\cdot\frac{d\mathbf{x}}{d\lambda}\right). \tag{4}$$

As an immediate consequence of (4) we conclude not only that the velocity of light is constant and that isotropic lines are preserved but the validity of the theorem:

The quotient of two physical intervals $(d\mathbf{x}\cdot d\mathbf{x})/(d_1\mathbf{x}\cdot d_1\mathbf{x})$ is invariant.

Lie asserts that the transformations P_0 together with the Lorenz transformations generate all the required transformations. We shall describe this result more explicitly in the next paragraph.

4

In this paragraph we give the results of Lie in a form which is directly applicable. We combine P_0 with the translations

$$\mathbf{x}' = \frac{\mathbf{x}-\mathbf{a}}{(\mathbf{x}-\mathbf{a})\cdot(\mathbf{x}-\mathbf{a})} = P_a(\mathbf{x}),$$

which are valid for all elements not on a light path from $\mathbf{a}$, and obtain

$$C_a(\mathbf{x}) = P_a P_0(\mathbf{x})$$

$$= \left[\frac{\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} - \mathbf{a}\right] \Big/ \left[\left(\frac{\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} - \mathbf{a}\right)\cdot\left(\frac{\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} - \mathbf{a}\right)\right].$$

By a simple calculation it is seen that

$$C_a(\mathbf{x}) = \frac{\mathbf{x} - \mathbf{a}(\mathbf{x}\cdot\mathbf{x})}{1 + (\mathbf{a}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{x}) - 2(\mathbf{a}\cdot\mathbf{x})}.$$

Hence C_0 is the identity. The singular points for these transformations, i.e., the points for which the denominator is zero, form a hyperplane if $\mathbf{a}\cdot\mathbf{a}=0$, otherwise a quadratic surface. It can be shown that these singular manifolds are different if $\mathbf{a}$ and $\mathbf{b}$ are distinct. Let us suppose that $\mathbf{x}'=L(\mathbf{x})$ is a Lorentz transformation or proportional to one. It is known that: (1) $L(\mathbf{x})$ has no singular points, (2) straight lines are preserved, (3) $\mathbf{x}\cdot\mathbf{x}$ is preserved except for a constant factor. Hence all transformations LC_a preserve the differential equation $d\mathbf{x}\cdot d\mathbf{x} = dx^2+dy^2+dz^2-c^2dt^2=0$. Furthermore we have the following theorem:

If $L_1C_a = L_2C_b$ then $L_1 = L_2$ and $C_a = C_b$.

For the singular manifolds of L_1C_a and L_2C_b must be identical. Since L_1 and L_2 are regular this implies that $C_a = C_b$ and hence $L_1 = L_2$. As a consequence we see that the transformations LC_a form a fifteen parameter manifold, hence they contain all the required transformations sufficiently near to identity.