points out, the experimental results for nickel are best represented by $S=1/2$, leading to $\frac{2}{3}S(S+1)/S^2=2$. Thus we may expect the Curie energy to be somewhere between $\frac{2}{3}$ and twice the energy of magnetization at the absolute zero, with the probabilities favoring the latter value. The experimental Curie energy 440 cm^{-1} is 1.62 times the magnetization energy 271 cm^{-1} which we have found. The agreement seems entirely satisfactory, considering the nature of the approximations which we have made.

Electronic Specific Heat of Nickel at Low Temperature

It has been recently observed by Keesom[11] that the electronic contribution to the specific heat, which, being proportional to the temperature, outweighs the contribution of the lattice vibrations at low temperatures, is many times the normal amount in nickel. Both Keesom and Mott have suggested that this arises from the large concentration of energy levels in the neighborhood of the top of the Fermi distribution, on account of the 3*d* levels. We can easily test this hypothesis. Sommerfeld and Bethe (reference 2, p. 430) show that the specific heat per gram molecular weight on account of free electrons in a normal metal is $C_v=(2\pi^2/3)N(\zeta_0)k^2T$, where $N(E)dE$ represents the number of energy levels of one spin in dE, if we are dealing with a gram molecular weight, and ζ_0 is the maximum energy occupied at the absolute zero. Thus his N is half our distribution function f. For a ferromagnetic metal, however, at low temperatures all levels of positive spin are filled, so that they do not contribute to the specific heat, and we count only electrons of negative spin. This removes the factor 2 in the formula above. Furthermore, ζ_0 must be taken to be the maximum energy of electrons with negative spin, when the electrons of positive spin occupy the whole band. In this case, from the numerical data from which Fig. 1 is drawn, we find that $N(E)$ is 26.15 times Avogadro's number, if energy is expressed in atomic units. Substituting this value, and appropriate constants, we find $C_v=0.0011T$ calories per mole per degree. Keesom's experimental value is $0.001744T$. The agreement is not very good, but still it is correct in order of magnitude. To get better agreement, the peak in Fig. 1 at energy -0.45 atomic unit would have to be about half again as high as it is, presumably being correspondingly more narrow. This does not seem impossible, though it is unlikely. In any case, the high specific heat is definitely connected with the high peak in the 3*d* distribution curve of Fig. 1, verifying again this general feature of our curve.

[11] Keesom and Clark, Physica **2**, 513 (1935); W. H. Keesom, Proc. Roy. Soc. **A152**, 12 (1935); N. F. Mott, Proc. Roy. Soc. **A152**, 42 (1935). I am much indebted to Mr. R. H. Fowler for calling my attention to these references.

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A Note on Positron Theory and Proper Energies

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The origin of the infinite light-quantum proper energy which follows, according to Heisenberg, from the creation of matter by the field of the photon, is examined in some detail. We are led to investigate the inconsistencies which appear on the incorporation of the Dirac positron theory into the formalism of the quantum theory. These inconsistencies make it impossible to regard with confidence any predictions of the theory for which a consideration of the singularities of the density matrix is essential. It is shown that when the off-diagonal distance in the density matrix is taken different from zero, the proper energy of a light quantum and the electromagnetic energy of an electron are, in order e^2, finite, and that the off-diagonal distance plays in these calculations the part of a generalized electron radius.

In a field free vacuum, the infinite distribution of electrons in negative energy states envisaged in Dirac's theory of the positron clearly should contribute nothing to the expectation values of observables of the system (charge and

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current density, energy, electromagnetic fields). The contribution of this field free vacuum distribution has then to be subtracted from the expectation value of every observable. Since, however, the difference between the contributions of the vacuum distribution in the presence and absence of an electromagnetic field does not turn out to be finite,[1] the theory in this form cannot give sensible and determinate answers to questions involving the reaction of the electron distribution in varying fields. Dirac himself has now tried to modify the formalism in such a way that all expectation values would necessarily be finite.

Dirac's treatment of the problem[2] is based on a study of the density matrix

$$(x''k''\,|\,\mathfrak{R}\,|\,x'k')=\sum_{\text{occ}}\psi_n(x''k'')\psi_n{}^{+}(x'k').$$

Here $\psi_n(x''k'')$ is a one-electron wave function at a point $x_0''=ct''$, x_1'', x_2'', x_3'', and k'' is the spin variable. The summation is to be extended over all occupied states. Dirac has shown, by integration of the equation of motion for $\mathfrak{R}$, taking the ψ_n as solutions of the ordinary Dirac equation, that $\mathfrak{R}$ is finite when the length of the four-vector $\mathbf{x}=\mathbf{x}'-\mathbf{x}''$ is different from zero, i.e., the density matrix is singular only when $x_\lambda x^\lambda=0$. The worst singularities in $\mathfrak{R}$ can be eliminated by symmetricizing the density matrix between positron and electron, that is, by replacing $\mathfrak{R}$ by

$$(x''k''\,|\,R\,|\,x'k')=\tfrac{1}{2}\Big[\sum_{\text{occ}}\psi_n(x''k'')\psi_n{}^{+}(x'k')-\sum_{\text{unocc}}\psi_n(x''k'')\psi_n{}^{+}(x'k')\Big].$$

Explicit expressions for the singular terms, S, which remain in R have been given by Heisenberg.[3] These singular terms may be deleted by replacing R by $r=R-S$. It is of course possible also to subtract finite terms. Heisenberg has included such a term in S; it is chosen to renormalize the polarization of vacuum for slowly varying and weak fields to zero. The expectation value of an operator F at a point $\mathbf{X}=\frac{1}{2}(\mathbf{x}'+\mathbf{x}'')$ is now given by

$$\bar{F}(X)=\lim_{x\to 0}\sum_{k}(X-\tfrac{1}{2}x,\,k\,|\,Fr\,|\,X+\tfrac{1}{2}x,\,k),$$

and is guaranteed to be finite and determinate.

The electromagnetic field produced by the electron distribution is taken into account by including in the total current which appears in the Maxwell field equations the expectation value of the current calculated from the density matrix,

$$j_\lambda(X)=-e\lim_{x\to 0}\sum_{kk'}\alpha_{\lambda kk'}(X-\tfrac{1}{2}x,\,k'\,|\,r\,|\,X+\tfrac{1}{2}x,\,k).$$

Here $j_0=\rho$, $\alpha_0=1$. Heisenberg has shown that, despite the subtraction of the singular terms, the conservation law for the current still holds in the limit $\mathbf{x}\to 0$, as do also conservation laws for the energy and momentum of the entire system, material field plus electromagnetic field.

The passage from the density matrix treatment, based on the correspondence principle, to a quantized field theory can be made by regarding the equations of motion (the Dirac equation and the Maxwell equations) as matrix equations in the variables ψ, ψ^+, E_i, A_i, which satisfy the usual commutation laws. The consistency of such a procedure depends upon the compatibility of commutation laws and equations of motion: these equations must, namely, be such that the values of the commutators remain unchanged in the course of time. This condition is not satisfied by the system of equations with subtractative terms. Thus, with $\dot{A}_i=-E_i$ and $\mathbf{E}=\text{curl}\,\mathbf{H}+4\pi e\boldsymbol{\alpha}(R-S)$, $(d/dt)[E_i(X),\,A_i(X')]=-4\pi e[\alpha_i S(X),\,A_i(X')]\neq 0$. In fact the time rate of change of this commutator is singular.

A second scheme has also been proposed by Heisenberg. This consists in taking as a Hamiltonian function the energy expression of the density-matrix treatment, now regarded as a function of the q-numbers ψ, ψ^+, E_i, A_i. One takes as canonical conjugates ψ and ψ^+, A_i and E_i, as in the usual quantum electrodynamics. One sets the scalar potential $A_0=0$, and $x_0=0$. The Hamiltonian function is then[4]

[1] P. A. M. Dirac, Solvay Congress, 1933; W. H. Furry and J. R. Oppenheimer, Phys. Rev. **45**, 245 (1934); R. Peierls, Proc. Roy. Soc. **A146**, 420 (1934).

[2] P. A. M. Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934).

[3] W. Heisenberg, Zeits. f. Physik **90**, 209 (1934).

[4] We shall employ rational units, measuring length in terms of the Compton wave-length $\hbar/mc$, time in terms of $\hbar/mc^2$, and mass in terms of the electronic mass m. In these units $e^2=\alpha$.

$$H=\tfrac{1}{2}\int d\mathbf{X}\sum_{k'k''}[\alpha_{jk'k''}\{i\partial/\partial x_j+\tfrac{1}{2}e[A_j(X+\tfrac{1}{2}x)+A_j(X-\tfrac{1}{2}x)]\}+\beta_{k'k''}][\psi^+(X+\tfrac{1}{2}x,\,k')\psi(X-\tfrac{1}{2}x,\,k'')$$

$$-\psi(X-\tfrac{1}{2}x,\,k'')\psi^+(X+\tfrac{1}{2}x,\,k')]+\int d\mathbf{X}\sum_{k'}i\partial/\partial x_0(X-\tfrac{1}{2}x,\,k'|S|X+\tfrac{1}{2}x,\,k')+H_{EM},\quad(1)$$

where $H_{EM}=(1/8\pi)\int d\mathbf{X}(E^2+H^2)$. This Hamiltonian function can be formally treated as Hermitian, provided one understands that in taking the transpose of a matrix one is also to reverse the direction of $\mathbf{x}$. The interaction between electromagnetic field and matter is to be supposed a small perturbation and the usual perturbation theory, expanding in powers of the electronic charge e, is to be applied. In the result of any calculation one is to pass to the limit $\mathbf{x}\to 0$.

However, the Hamiltonian function carries with it its own equations of motion, and these are not the same as the equations of motion of the simple density matrix treatment. Thus the Hamiltonian equation for ψ is

$$i\dot{\psi}(X)=[\alpha_j(-i\partial/\partial X_j+\tfrac{1}{2}e\{A_j(X)+A_j(X-x)\})+\beta]\psi(X-x),\quad(2)$$

which differs from the ordinary Dirac equation through the appearance of $\mathbf{x}$ on the right-hand side. Since the Heisenberg subtractative terms were determined on the supposition that ψ obeys the Dirac equation, there is no warrant for believing them correct when, instead, ψ obeys (2). The Hamiltonian equations for the electromagnetic field differ from the Maxwell equations even more radically, for the subtractative terms introduced into (1) are explicit functions of E and A, and hence give rise to new terms in the equations of motion. Thus instead of the ordinary equation $-\dot{A}_i=E_i$, one finds, aside from terms of order x,

$$-\dot{A}_i=E_i-(e^2/3\pi)[x_ix_jE_j/x_\lambda x^\lambda-E_i\log(|x_\lambda x^\lambda|/c_1)].\quad(3)$$

The equations for E_i can be written

$$-\dot{E}_i+\mathrm{curl}_i\,\mathbf{H}=4\pi[j_i(x)-e^2S_i(j)]+e^2\,(\text{singular terms}).\quad(4)$$

Here $\mathbf{j}(x)$ represents the contributions of the terms in (1) which are linear in e, and $-e^2\mathbf{S}(j)$ contains the subtractative terms necessary to cancel the singularities which appear in the expectation value of $\mathbf{j}(x)$ in order e^2. The appearance of singular terms in (3) and (4) is of course fatal to the theory, since E_i and A_i will evidently be singular as $\mathbf{x}\to 0$ and no provision has been made to remove such singularities. Moreover, in the derivation of the subtractative terms explicit use is made of the relation $-\dot{A}_i=E_i$. One would be just as badly off if one attempted to regard the subtractative terms in (1) as not being explicit functions of dynamical variables, since then the subtractative terms for the current would be lacking in the Hamiltonian equations for the field.

Heisenberg has applied the Hamiltonian function (1) to the calculation of the proper energy of a photon in order e^2 but he did not obtain a finite result. The reason for this is easily seen. Since H is a function of $\mathbf{x}$ the transformation matrix which diagonalizes it will also depend on $\mathbf{x}$. The transformed Hamiltonian function is $\bar{H}(x)=S^{-1}(x)H(x)S(x)$. However Heisenberg set $\mathbf{x}=0$ in the transformation matrix, taking $\bar{H}(x)=S^{-1}(0)H(x)S(0)$, presumably in order to avoid any possible consequences of the change from the Dirac equation to (2). (2) of course reduces to the Dirac equation when $\mathbf{x}=0$. This precaution would be unnecessary if $S(0)$ led to a finite result, since in that circumstance it would be quite immaterial whether one used $S(0)$ or $S(x)$. Actually, if $S(0)$ is used, one term appears in $\bar{H}$, namely, $S^{-1}(0)H_{EM}S(0)$, which is completely independent of $\mathbf{x}$, and this term diverges.[5] Since the off-

[5] It would be of no aid to introduce $\mathbf{x}$ into E^2+H^2 in a way analogous to that in which it is introduced into $\psi^+\psi$. If the propagation vector of the photon is $\mathbf{k}$, such a modification would introduce into H_{EM} a factor depending on $\mathbf{k}\cdot\mathbf{x}$. Such a factor does not help, since by the conservation laws $\mathbf{k}$ is equal to the sum of the momenta of the pair virtually created by the perturbation, whereas the singularities arise from integration over the difference of the momenta.

diagonal matrix elements of $\bar{\bar{H}}$, as well as the diagonal elements, approach no limit as $\mathbf{x}\to 0$, the artifice of setting $\mathbf{x}=0$ in the transformation matrix is clearly not permissible, nor can any conclusions regarding the characteristic values of H be drawn from such a calculation.

If one carries out the calculation using $S(x)$ no difficulties are encountered as long as $\mathbf{x}\neq 0$, since now every term in $\bar{\bar{H}}$ involves $\mathbf{x}$. The photon proper energy in order e^2 is found to be

$$W=-(\alpha h\nu/3\pi)(z/R^2-2\log \tfrac{1}{2}CR+\mathrm{O}(R)), \qquad (\log C=0.577\cdots) \quad (5)$$

where z is the component of $\mathbf{x}$ in the direction of the electric vector of the photon, and R is the length of $\mathbf{x}$. But in the limit $\mathbf{x}\to 0$, the photon proper energy becomes infinite.

The entire contribution to (5) comes from H_{EM}, in fact (5) is just the expectation value of $H_{EM}-h\nu$. This seems paradoxical, since the unwanted singular terms in (3) and (4) do not affect the expectation value of H_{EM} in order e^2, while from the remaining terms all singularities have presumably been removed. That the singular terms in (3) and (4) play no part may be seen most easily by inspection of the ordinary perturbation theory formula for H_{EM}. If we write $H=H^{(0)}+eH^{(1)}+e^2H^{(2)}$ we see that, since H_{EM} is diagonal in the original representation, its expectation value in order e^2 is completely independent of $H^{(2)}$. However, the singular terms in (3) and (4) can be regarded as coming entirely from $H^{(2)}$.

Heisenberg has ascribed the photon proper energy difficulties to the point nature of the photon, that is, to the possibility of large fluctuations in the quantized electromagnetic field. That this is not the true explanation can be seen by thinking of the calculation as carried out by direct integration of the equations of motion in powers of e. One then sees that in order e^2 the energy contains no terms higher in degree than quadratic in the unperturbed value of the electromagnetic field variables, $b_{k\lambda}$ and $b_{k\lambda}^{+}$. For the photon proper energy, we require the terms in the energy proportional to $M_{k\lambda}$, the number of photons. Fluctuations in the field consequently contribute nothing: one must have terms at least quartic in $b_{k\lambda}$ and $b_{k\lambda}^{+}$ before fluctuation terms proportional to $M_{k\lambda}$ can appear.

The real source of the trouble lies in the fact that a Hamiltonian perturbation calculation in powers of e is equivalent to integration of the equations of motion under the supposition that e is a slowly varying function of the time, which increases from an initial value zero to its final value, e. But in the calculation of the subtractative terms it is explicitly assumed that e is independent of time: all time dependences are to be included in the vector potential $\mathbf{A}$. If it were not for the subtractative terms it would be quite immaterial, in computing the current, which one supposed to vary, e or the electromagnetic fields, since only the produce $e\mathbf{A}$ appears in the Dirac equation. However the subtractative terms in the current also involve $e\dot{\mathbf{A}}$, and hence will by no means be the same in the two cases. Suppose that the perturbation is applied over a time T. During this time transient terms of order $1/T$ will appear in the expression for the induced current, before inclusion of the subtractative terms. If one supposes $\mathbf{A}$ to vary with time, transient fields will also be present, which will contribute additional subtractative terms which at every instant just cancel the transient terms in the uncorrected current. However if e is varied these additional subtractative terms will not appear. The transient terms which in consequence remain in the current vanish in the limit $T\to\infty$, but since they also act for a time T, their contribution to the electromagnetic fields will be of order unity as $T\to\infty$. If one calculates the contribution of these transient terms to H_{EM}, one obtains just the result (5). The photon proper energy calculation illustrates a further difficulty involved in the passage to a Hamiltonian theory: even though it is possible to give a rule by which non-singular solutions of the equations of motion may be obtained, these solutions will not, in general, correspond to stationary states of the Hamiltonian system.

The efficacy of the subtractative terms is in any event severely limited by the fact that they have been determined only in the approximation in which the density-matrix treatment is applicable, and so cannot be expected to give, in general, finite answers for problems in which fluctuations are involved.

The possibility of constructing a Hamiltonian theory, along the lines laid down by Heisenberg,

which would avoid the difficulties discussed above is very doubtful. Certainly no simple modification of the original scheme, such as the introduction of new subtractative terms in the Hamiltonian function, will give equations of motion and expectation values for energy which are both free of singularities. On the other hand formal arguments alone hardly provide an adequate basis for a more radical revision.

In the absence of any consistent theory, the simplest procedure which suggests itself is to dispense entirely with subtractative terms, to carry out all calculations with $\mathbf{x}$ different from zero, and in the result to pick out the term independent of $\mathbf{x}$ as the only one of significance. This scheme has the flaw that a classification of terms as depending on $\mathbf{x}$ or not depending on $\mathbf{x}$ is essentially ambiguous. It is of course possible to pick out terms which are singular or which vanish as $\mathbf{x}\rightarrow 0$, but since, unfortunately, $(x^2+y^2+z^2)/R^2=1$, an unambiguous separation into "determinate" and "indeterminate" terms is not possible. In this connection, it should be remembered that after the introduction of $\mathbf{x}$, an operator F is in general no longer either Lorentz covariant or gauge invariant. This could lead to no inconsistencies provided the lack of the proper transformation properties manifested itself only in terms which were recognizably x-dependent. However, we have not succeeded in formulating a criterion for the seperation of determinate and indeterminate terms which, in any given problem, can be relied upon to lead to Lorentz covariant and gauge invariant results. Calculations made in this manner accordingly cannot lead to unequivocal results. But it may be hoped that they will give some indication of the sort of corrections which must be applied to present theory, and of the order of magnitude of the correction terms.

We shall introduce $\mathbf{x}$ into the theory in a somewhat different way than did Dirac and Heisenberg. The operators of the usual field theory will be left altogether unaltered, except for their symmetrization in electron and positron $(\psi^+(X)\psi(X)\rightarrow\frac{1}{2}[\psi^+(X)\psi(X)-\psi(X)\psi^+(X)])$. The commutation laws however, will be taken to involve $\mathbf{x}$:

$$[\psi^+(X),\ \psi(X')]^+=\delta(X-X'+x).$$

This formalism would correspond to the impossibility of making field measurements without mutual disturbance at two arbitrarily near points. It thus suggests a not unreasonable motive for the introduction of the off-diagonal distance $\mathbf{x}$.

As in the usual quantum electrodynamics, the wave equation associated with the Hamiltonian function is to be augmented by the auxiliary relation $\operatorname{div}\mathbf{E}-4\pi\rho=0$. This relation is, in fact, not compatible with the equations of motion, since the conservation law for the current holds only when $\mathbf{x}=0$. The inconsistency of the auxiliary relation with the equations of motion is only to be expected, for, with $\mathbf{x}\neq 0$, the Hamiltonian function is no longer gauge invariant. The same inconsistency appears if $\mathbf{x}$ is introduced into ψ and ψ^+ instead of into the commutation relations. (Even in the density matrix treatment, it will be remembered, the conservation laws fail for $\mathbf{x}\neq 0$). Since the covariance of an inconsistent set of equations is in any event illusory, we have been content to introduce $\mathbf{x}$ into the theory in a patently noncovariant manner. As we shall see, this lack of covariance is much the same as that which appears in classical electrodynamics when one attempts to introduce the radius of the electron.

The formalism just described finds an interesting application in the calculation of proper energies of photon and electron in order e^2. The photon proper energy is found to be (in c.g.s. units)

$$W=-\frac{\alpha h\nu}{2\pi}\left\{\left(\frac{mc^2}{h\nu}\right)^2\left[8\frac{z^2}{R^4}+\left(\frac{4}{3}\right)\frac{x^2z^2}{R^4}-2\frac{z^2}{R^2}\right]+\frac{1}{3}\frac{x^2}{R^2}+O(R)\right\},$$

where x is the component of $\mathbf{x}$ in the direction of propogation of the photon, z is the component in the direction of the photon's electric vector, and R is the length of $\mathbf{x}$. This result may reasonably be interpreted as indicating a null photon proper energy in order e^2.

If we carry out the elimination of the electrostatic field variables in the usual way, overlooking, of course, the inconsistency of the auxiliary relation, we obtain in the energy a term

$$\tfrac{1}{4}\sum_{nmst}(a_n{}^+a_m-a_ma_n{}^+)(a_s{}^+a_t-a_ta_s{}^+)Q_{nm;\,st}, \tag{6}$$

where

$$Q_{nm;\,st}=\tfrac{1}{2}e^2\int d\mathbf{X}\int d\mathbf{X}'\frac{v_n{}^+(X)v_m(X)v_s{}^+(X')v_t(X')}{|\mathbf{X}-\mathbf{X}'|}\exp\,-\tfrac{1}{2}i(\mathbf{k}_n+\mathbf{k}_m+\mathbf{k}_s+\mathbf{k}_t)\cdot\mathbf{x},$$

and ψ and ψ^+ have been expanded in electronic plane wave functions v_n, of momentum $\mathbf{k}_n$: $\psi=\sum_n a_nv_n$ $\exp-\tfrac{1}{2}i\mathbf{k}_n\cdot\mathbf{x}$, $\psi^+=\sum_n a_n{}^+v_n{}^+\exp-\tfrac{1}{2}i\mathbf{k}_n\cdot\mathbf{x}$. The factors $\exp-\tfrac{1}{2}i\mathbf{k}_n\cdot\mathbf{x}$ are inserted so that the a_n and $a_n{}^+$ will obey the usual commutation relations. The diagonal matrix elements of (6) give the ordinary Coulomb and exchange terms, and in addition, terms which represent the electrostatic proper energies of electrons and positrons. The proper energy of an electron in a state r is

$$W_{ES}=\sum_{r'}Q_{rr';\,r'r}-\sum_{\rho}Q_{r\rho;\,\rho r}. \tag{7}$$

Here r and r' refer to positive energy states, ρ to negative energy states. Evaluating (7) in a Lorentz frame in which the electron is at rest, we find

$$W_{ES}=(e^2/16\pi^3)\int d\mathbf{r}/|\mathbf{r}+\mathbf{x}|\int d\mathbf{k}(1+k^2)^{-\frac{1}{2}}\exp\,(i\mathbf{k}\cdot\mathbf{r})=(e^2/4\pi^2)\int d\mathbf{r}K_1(r)/r|\mathbf{r}+\mathbf{x}|,$$

where $K_1(r)$ is the Bessel function defined in Whittaker and Watson, *Modern Analysis*, §17.71. The angular integration gives

$$W_{ES}=(e^2/\pi)\left[R^{-1}\int_0^R rK_1(r)dr+\int_R^\infty K_1(r)dr\right],$$

which can be reduced, by use of the recurrence relation $K_1(r)=(d/dr)K_0(r)$, to

$$W_{ES}=(e^2/\pi R)\int_0^R K_0(r)dr=-(2e^2/\pi)[\log\tfrac{1}{2}CR-1+O(R)].$$

The electromagnetic proper energy in order e^2 can also be shown to be finite when $\mathbf{x}\neq 0$. Thus the artifice introduced to eliminate the singularities due to the sea of negative energy electrons has a very natural extension to the treatment of the electronic proper energy difficulties.

It will be observed that the filling of the negative energy states results in the summations over positive and negative states appearing in the electrostatic proper energy with opposite sign. If the negative energy states were left empty, the proper energy would instead be $\sum_{r'}Q_{rr';\,r'r}$ $+\sum_{\rho}Q_{r\rho;\,\rho r}$. This gives $W_{ES}=e^2/2R$, which is just the electrostatic energy of an electron of radius R. The off-diagonal distance $\mathbf{x}$ thus appears to find a classical analogue in the electron radius.

Our method of calculation, too, is strongly reminiscent of that employed in classical electrodynamics, which can likewise be described as the introduction of a parameter, the electron radius, and the final abandonment of all terms except those independent of this parameter. The aim, of course, is to separate, insofar as possible, those terms which obviously depend on the particular model employed or on the way in which calculations are carried out, from those which seem to be independent of these factors, and which may be hoped therefore to have some significance, despite the unquestionable inadequacy of the theory.

I wish to express my deepest thanks to Professor J. R. Oppenheimer, to whom I am indebted for many illuminating discussions of this subject.