Remarks on the Polarization Effects in the Positron Theory

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A simple method of obtaining the induced charge-density four vector on the basis of the subtraction formalism of the positron theory is given. Further, in the general case of time-dependent fields the result is calculated directly without use of the Lorentz invariance of the theory.

INTRODUCTION

It is the purpose of the following paper to show that the integrals which determine the additional polarization density δj_0 and the corresponding current density $\delta \mathbf{j}$, produced by an original charge density j_0 and a current density $\mathbf{j}$ according to the positron theory, can be evaluated in a simpler way than has been done previously.[1] We shall consequently restrict ourselves, however, to the approximation in which effects proportional to powers of the fine structure constant $\alpha = e^2/\hbar c$ higher than the first will be neglected. Then the connection between δj_0 and $\delta \mathbf{j}$ with the scalar and vector potential A_0 and $\mathbf{A}$ of the original field is a linear one and therefore there is no loss of generality if we use plane waves

$$\begin{aligned} A_0 &= a_0 \exp\,(i[(\mathbf{k}\cdot\mathbf{x}) - k_0 t]) + \text{conj.}, \\ \mathbf{A} &= \mathbf{a} \exp\,(i[(\mathbf{k}\cdot\mathbf{x}) - k_0 t]) + \text{conj.} \end{aligned} \quad (1)$$

for the potential field. In the following we employ $\hbar/mc$ as the unit of length, $\hbar/mc^2$ as the unit of time and the electron mass m as unit of mass.

For the sake of simplicity we shall treat explicitly the particular case in which $\mathbf{A}=0$ and we shall be interested only in the charge density j_0 not in the current density $\mathbf{j}$. According to Serber[1] it is then not difficult to treat also the general case characterized by (1). In the particular case

$$A_0 = a_0 \exp\,(i[(\mathbf{k}\cdot\mathbf{x}) - k_0 t]) + \text{conj.}, \quad \mathbf{A}=0 \quad (1a)$$

where according to Maxwell's equations one has

$$j_0 = (k^2/4\pi) A_0, \quad (2)$$

one gets according to Heisenberg[2] for the difference between the density matrix in the cases of the presence and of the absence of the external field in the approximation in question, after multiplication by the electron charge e and summation over the spin index, the expression

$$e \sum_{\rho=1}^{4} (\mathbf{x}', \rho\,|\,\delta R\,|\,\mathbf{x}'', \rho) = -\frac{\alpha}{\pi} F(\mathbf{k}, k_0, \mathbf{x}) j_0, \quad (3)$$

where
$$\mathbf{x} = \mathbf{x}' - \mathbf{x}'' \quad (4)$$

and F is given by

$$F(\mathbf{k}, k_0, \mathbf{x}) = \frac{1}{\pi k^2} \int \frac{\epsilon\epsilon' - (q^2 + 1 - k^2/4)}{\epsilon\epsilon'} \times \frac{\epsilon + \epsilon'}{(\epsilon+\epsilon')^2 - k_0^2} \exp\,(i(\mathbf{q}\cdot\mathbf{x})) d\mathbf{q} \quad (5)$$

with

$$\epsilon = [1 + (\mathbf{q}+\mathbf{k}/2)^2]^{\frac{1}{2}}, \quad \epsilon' = [1 + (\mathbf{q}-\mathbf{k}/2)^2]^{\frac{1}{2}}. \quad (6)$$

For large values of q, or what turns out to be the same, for small values of k one has the expansion

$$\epsilon \sim \epsilon' \sim (1+q^2)^{\frac{1}{2}}$$

and more exactly

$$\epsilon\epsilon' \sim 1 + q^2 + k^2/4 - \tfrac{1}{2}(\mathbf{q}\cdot\mathbf{k})^2 (1+q^2)^{-1},$$

so that we can separate F into two terms.

$$F(\mathbf{k}, k_0, \mathbf{x}) = F_0(\mathbf{k}, \mathbf{x}) + F_1(\mathbf{k}, k_0, \mathbf{x}) \quad (7)$$

with

$$F_0(\mathbf{k}, \mathbf{x}) = \frac{1}{4\pi} \int \left[1 - \frac{q^2 \cos^2 \vartheta}{1+q^2}\right] \frac{\exp\,(i(\mathbf{q}\cdot\mathbf{x}))}{(1+q^2)^{\frac{3}{2}}} d\mathbf{q}, \quad (8)$$

[1] W. Heisenberg, Zeits. f. Physik **90**, 209 (1934); R. Serber, Phys. Rev. **48**, 49 (1935); E. A. Uehling, Phys. Rev. **48**, 55 (1935).

[2] W. Heisenberg, reference 1, p. 221, Eq. (34) and p. 222 below.

ϑ being the angle between the direction of $\mathbf{q}$ and $\mathbf{x}$ and where F_1 is finite for $x=0$ and has there the value

$$f(\mathbf{k}, k_0) \equiv F_1(\mathbf{k}, k_0, 0) = \frac{1}{\pi k^2}\int\left\{\frac{\epsilon\epsilon' - (q^2+1-k^2/4)}{\epsilon\epsilon'}\,\frac{\epsilon+\epsilon'}{(\epsilon+\epsilon')^2-k_0^2} - \frac{k^2}{4}\left[1-\frac{q^2\cos^2\vartheta}{1+q^2}\right]\frac{1}{(1+q^2)^{\frac{3}{2}}}\right\}d\mathbf{q}. \quad (9)$$

While each of the parts in the latter formula decreases like $|\mathbf{q}|^{-3}$ for large values of $|\mathbf{q}|$ and therefore the corresponding parts of the integral are logarithmically divergent, the difference in the integrand decreases with the higher order $|\mathbf{q}|^{-5}$ so that the total integral is convergent. On the contrary the integral (8) which can be expressed by means of cylindrical functions is of the order $\log|\mathbf{x}|+$const. for small values of $\mathbf{x}$.

The formalism of the positron theory, which is accepted at present and which unfortunately is not yet substituted by a more satisfactory one, does not identify, however, the physical electric charge density with the value of the left side of (3) for $\mathbf{x}=0$ but introduces first properly chosen subtractive terms depending on $\mathbf{x}$ in such a way that the difference is finite for $\mathbf{x}=0$ ("subtraction-physics"). One remains in agreement with the more general prescription of Heisenberg if, in the case here considered and in the approximation in question (terms proportional to α), one identifies the subtraction terms with $F_0(\mathbf{k}, \mathbf{x})$ given by (8). Then one gets, according to (3),

$$\delta j_0 = -(\alpha/\pi)f(\mathbf{k}, k_0)j_0, \quad (10)$$

where the function f is given by (9).

The Evaluation of the Integral

While Serber evaluates the integral (9) only for the particular case of $k_0=0$ (time-independent fields) and uses the Lorentz invariance of the subtraction-formalism to get the result for the general case, we shall here directly compute the general integral (9). For this purpose we introduce, besides the azimuth φ around the coordinate axis parallel to $\mathbf{k}$ the variables v, w defined by

$$\tfrac{1}{2}(\epsilon-\epsilon')=v, \quad \tfrac{1}{2}(\epsilon+\epsilon')=w, \quad (11)$$

where ϵ and ϵ' are given by (6). From the latter equations one gets, by computing the functional determinant, for the volume element of the $\mathbf{q}$ space the simple expression

$$d\mathbf{q}=(2/k)\epsilon\epsilon' dv dw d\varphi = (1/\kappa)\epsilon\epsilon' dv dw d\varphi. \quad (12)$$

Here we introduced the convenient abbreviation

$$k/2=\kappa. \quad (13)$$

From (6) we get further

$$\tfrac{1}{2}(\epsilon^2+\epsilon'^2)=v^2+w^2=q^2+\kappa^2+1, \quad (14a)$$

$$\tfrac{1}{4}(\epsilon^2-\epsilon'^2)=vw=\tfrac{1}{2}(\mathbf{qk}). \quad (14b)$$

From these equations there follows the inequalities

$$v^2w^2 \leqslant \kappa^2(w^2-\kappa^2-1+v^2), \quad v^2 \leqslant \kappa^2\frac{w^2-\kappa^2-1}{w^2-\kappa^2}.$$

We get the whole $\mathbf{q}$ space if we first for a given value of w integrate over the surface $w=$const., where v runs through the interval

$$-\kappa\left(\frac{w^2-\kappa^2-1}{w^2-\kappa^2}\right)^{\frac{1}{2}} \leqslant v \leqslant \kappa\left(\frac{w^2-\kappa^2-1}{w^2-\kappa^2}\right)^{\frac{1}{2}}, \quad (15)$$

and after that w runs from $(\kappa^2+1)^{\frac{1}{2}}$ to ∞.

$$(\kappa^2+1)^{\frac{1}{2}} \leqslant w \leqslant \infty, \quad (16)$$

the point $v=0$, $w=(\kappa^2+1)^{\frac{1}{2}}$ corresponding to the origin $\mathbf{q}=0$.

Before we definitely compute the integral (9), we remark that it would not be convenient to use the variables v and w for the second (subtractive) part of (9) also. To avoid this we use the circumstance that for large values of W the surface $w=$const.$=W$ is nearly a sphere. More exactly we see from (14a) that the least and the largest values of q on this surface corresponding to $v=0$ and $v_{\max}$, given by

$$q_1^2=W^2-\kappa^2-1,$$

$$q_2^2=(W^2-\kappa^2-1)(1+\kappa^2/(W^2-\kappa^2))$$

have a quotient q_2/q_1 converging to unity as W approaches infinity. Now we should first integrate both parts of (9) over a volume limited by the same surface $w=$const. and then go to the limit $W\rightarrow\infty$. Instead of that we shall extend the subtractive integral only over the sphere inscribed in the surface in question with the radius $q_1=(W^2-\kappa^2-1)^{\frac{1}{2}}$. The error caused by this procedure is certainly smaller than the subtractive integral extended over the spherical shell with the radii q_1 and q_2 given above, because its integrand does not change sign in the domain in question; and even the latter integral converges to zero for $W\rightarrow\infty$ because the subtractive integral diverges only logarithmically and $\lim q_2/q_1=1$.

Performing the integration over the azimuth and introducing spherical polar coordinates in the subtractive integral, we now get from (9)

$$f(k, k_0)=\lim_{W\rightarrow\infty}[J_1(W)-J_2(W)], \qquad (17)$$

where

$$J_1(W)=\frac{1}{2\kappa^3}\int_{(k^2+1)^{\frac{1}{2}}}^{W}\frac{wdw}{w^2-k_0^2/4}\int_{-\kappa z}^{\kappa z}(\kappa^2-v^2)dv \qquad (18)$$

and z is an abbreviation for

$$z=((w^2-\kappa^2-1)/(w^2-\kappa^2))^{\frac{1}{2}}; \qquad (19)$$

and

$$J_2(W)=\int_0^{(W^2-\kappa^2-1)^{\frac{1}{2}}}\left[1-\frac{q^2/3}{1+q^2}\right]\frac{q^2dq}{(1+q^2)^{\frac{3}{2}}}. \qquad (20)$$

Performing the integration over v in (18) and introducing z given by (19) instead of w as integration variable, one gets

$$w^2=\frac{1+\kappa^2(1-z^2)}{1-z^2}, \quad wdw=\frac{zdz}{(1-z^2)^2},$$

$$J_1(Z)=\int_0^Z\frac{z^2-z^4/3}{1-z^2}\,\frac{dz}{1+(\kappa^2-k_0^2/4)(1-z^2)},$$

where Z is connected with the upper limit W in (18) analogous to (19).

$$Z=((W^2-\kappa^2-1)/(W^2-\kappa^2))^{\frac{1}{2}}, \qquad (19a)$$

Z tending to unity when $W\rightarrow\infty$.

In (20) we substitute

$$q/(1+q^2)^{\frac{1}{2}}=z, \quad dq/(1+q^2)^{\frac{3}{2}}=dz$$

and then the upper limit of the integral (20) in the new variable z coincides exactly with Z given by (19a) and we get

$$J_2(W)=\int_0^Z\frac{z^2-z^4/3}{1-z^2}dz. \qquad (20a)$$

The integrand of the difference J_1-J_2 is finite for $z=1$ and we can now easily perform the $\lim Z\rightarrow 1$ (corresponding to $\lim W\rightarrow\infty$) and get the final result

$$f(k, k_0)=-\int_0^1\frac{(\kappa^2-k_0^2/4)(z^2-z^4/3)}{1+(\kappa^2-k_0^2/4)(1-z^2)}dz. \qquad (21)$$

The most important fact is that f does not depend on k and k_0 separately but on the simple combination

$$L=\kappa^2-k_0^2/4=(k^2-k_0^2)/4, \qquad (22)$$

$$f(k, k_0)=f(L)=-L\int_0^1\frac{z^2-z^4/3}{1+L(1-z^2)}dz. \qquad (21a)$$

This result is a consequence of the Lorentz invariance of the formalism and was assumed by Serber with this argument without direct proof.[3]

For small values of L one can neglect the term proportional to L in the denominator and get at once

$$f(L)\cong -4L/15 \quad \text{for} \quad |L|\ll 1 \qquad (23)$$

in accordance with Heisenberg's original result. Further one gets from (21a) for $f(L)$ a power series in L which converges for $|L|<1$.

The exact evaluation of (21a), which is elementary, leads to the following result:

$$f(L)=-\frac{1}{3}\left\{-\frac{5}{3}+\frac{1}{L}+\frac{(L+1)(2L-1)}{L}\varphi(L)\right\}, \qquad (24)$$

$$\text{where} \quad \varphi(L)=\int_0^1 dz[1+L(1-z^2)]^{-1}. \qquad (25)$$

One has

$$\varphi[L]=[L(1+L)]^{-\frac{1}{2}}\log[(1+L)^{\frac{1}{2}}+L^{\frac{1}{2}}] \quad \text{for} \quad L>0, \qquad (26a)$$

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[3] In the particular case $k_0=0$ Serber gives for $f(k^2)$ (our function is connected with his function χ by $f=\chi/\pi k^2$) the formula (comp. his Eq. (7)),

$$f=-\frac{1}{2}\int_0^1(1-z^2)\log[1+(k^2/4)(1-z^2)]dz.$$

Using $(1-z^2)dz=d(z-z^3/3)$ and integrating by parts, one gets at once from it our expression (21) for $k_0=0$.

$$\varphi(L)=[|L|(1-|L|)]^{-\frac{1}{2}}\arcsin|L|^{\frac{1}{2}}\quad\text{for}\quad -1<L<0,\tag{26b}$$

$$\varphi(L)=-[|L|(|L|-1)]^{-\frac{1}{2}}\log[(|L|-1)^{\frac{1}{2}}+|L|^{\frac{1}{2}}]\quad\text{for}\quad L<-1.\tag{26c}$$

In the latter case the denominator in the integrand in (21a) and (25) becomes zero at one point of the integration path and one has there to take the principal value. For large positive values of L one gets the asymptotic expression

$$f(L)\cong\frac{5}{9}-\frac{2}{3}\log(2L^{\frac{1}{2}})\quad\text{for}\quad L\gg 1.^{4}\tag{27}$$

As far as the general case is concerned which we have mentioned in the beginning, where a vector potential given by (1) is also present, one can again check Serber's general result

$$\begin{aligned}\delta j_0&=-(\alpha/\pi)f(L)j_0,\\ \delta\mathbf{j}&=-(\alpha/\pi)f(L)\mathbf{j},\end{aligned}\tag{28}$$

where f is again given by (21a), with the method here considered. Only the subtractive integrals due to the singularities, become more complicated in this case.[5]

THE FUNCTION $U(r)$

We shall finally add some remarks concerning the function $U(r)$ defined by

$$\delta j_\lambda(\mathbf{x}')=(\alpha/4\pi^2)\int U(r)\Delta j_\lambda(\mathbf{x}'')d\mathbf{x}'';\quad r=|\mathbf{x}'-\mathbf{x}''|.\tag{29}$$

This formula is valid for the case of the time independent j_λ. According to Uehling[6] this function also determines the interaction energy $V(r)$ of two particles with charges $Z'e$ and $Z''e$ separated by a distance r.

$$V(r)=Z'Z''e^2[1/r-(\alpha/\pi)U(r)].$$

From the definition (29) and (28) it follows that

$$U(r)=\frac{1}{2\pi^2}\int\frac{f(k^2)}{k^2}\exp(-i(\mathbf{k}\cdot\mathbf{x}))d\mathbf{k}=\frac{2}{\pi}\int\frac{f(k^2)}{k^2}\frac{\sin kr}{r}kdk.\tag{30}$$

If one now introduces the integral (21) (with $k_0=0$) for f the integral over k can be evaluated in an elementary manner and the remaining integral over z can, by the substitution $z=(q^2-1)^{\frac{1}{2}}/q$ and by partial integration, be reduced to the well-known integrals for the Bessel functions of the second kind,

$$K_0(x)=\int_1^\infty\frac{e^{-qx}}{(q^2-1)^{\frac{1}{2}}}dq,$$

$$K_1(x)=x\int_1^\infty e^{-qx}(q^2-1)^{\frac{1}{2}}dq$$

and an indefinite integral of K_0

$$B(x)=\int_x^\infty K_0(y)dy=\int_1^\infty\frac{e^{-qx}}{q(q^2-1)^{\frac{1}{2}}}dq.$$

The final result is

$$U(r)=-\frac{1}{3r}\left\{2\left(\frac{r^2}{3}+1\right)K_0(2r)-\frac{2r}{3}(2r^2+5)K_1(2r)+r\left(\frac{4r^2}{3}+3\right)B(2r)\right\}.$$

From this expression one can obtain easily the asymptotic forms for $U(r)$ for small and large values of r. These have been given by Uehling.[7]

[4] In the case $k_0=0$ one has $L=k^2/4$ and the formulas (24) and (26a) lead to the result

$$f=-\frac{1}{3}\left\{-\frac{5}{3}+\frac{4}{k^2}+2\frac{(k^2+4)^{\frac{1}{2}}(k^2-2)}{k^3}\log\frac{(k^2+4)^{\frac{1}{2}}+k}{2}\right\}.$$

This result is contained indirectly in Uehling's Eq. (22), p. 61, reference 1. Indeed, in order to calculate the scattering of two particles surrounded by a continuous charge distribution according to the Born approximation, one has to make the Fourier expansion of the electric density and to identify k with the difference of the wave-vector for the incident and the scattered beam of the particle (measured in the unit $mc/\hbar$).

[5] The fact that the polarization effects here considered do vanish for $L=0$ even if k and k_0 are different from zero, has the consequence that Heisenberg's conclusion about a correction of the relative order α to the Klein-Nishina formula for the scattering of light by free electrons (comp. Heisenberg, reference 1, p. 223) cannot be maintained. Indeed the effect considered by him vanishes because for the intensity of the scattered light only the terms with $L=0$ give any contribution. A definite answer to the question of the corrections to the Klein-Nishina formula in the approximation here considered needs however a more detailed investigation.

[6] Reference 1, Eq. (21).

[7] Uehling, reference 1, Eqs. (9) and (10).