f increases linearly with electron energy, one obtains the surprisingly good curve C shown in Fig. 6. This agreement is probably partially fortuitous, and a more rigorous development must eventually involve allowances for the effects of varying cross section, scattering angle, and inelasticity of impact in detail. The general character of the problem, however, seems to be indicated by the above considerations, and furnishes further evidence for the inelasticity of electron impact in molecular gases.

These experiments are being continued to determine electron mobilities in other gases. The authors desire to thank the Research Committee of Stanford University for a grant which has made these investigations possible.

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Note on the Quantum-Mechanical Theory of Measurement

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In recent notes by Einstein, Podolsky and Rosen and by Bohr, attention has been called to the fact that certain results of quantum mechanics are not to be reconciled with the assumption that a system has independently real properties as soon as it is free from mechanical interference. We here investigate in general, and in abstract terms, the extent of this disagreement. When suitably formulated, such an assumption gives to certain types of questions the same answers as does quantum mechanics; this is true of the formulas usually given in discussions of the theory of measurement. There exists, however, a general class of cases in which contradictions occur. That such contradictions are not restricted to the abstract mathematical theory, but can be realized in the commonest physical terms, is shown by the working out of an example.

1. Introduction

Some time ago there appeared a paper by Einstein, Podolsky and Rosen[1] entitled "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" The writers concluded that the answer must be negative, on the ground that quantum mechanics forbids simultaneous measurement of two noncommuting variables even when both variables simultaneously possess "physical reality," in the sense that either might be measured "without in any way disturbing the system." Recently Bohr[2] has upheld the view that quantum-mechanical description of nature can be considered complete, by demonstrating how the restrictions on simultaneous measurement which it imposes are inherent in the character and use of the measuring instruments. These measuring instruments must always be included as part of the physical situation from which our experience is obtained, and by doing this one sees that quantum mechanics provides a complete and peculiarly apt interpretation of experience.

Bohr has again clearly called attention to this circumstance, and has remarked that one must be careful not to suppose that a system is an independent seat of "real" attributes simply because it has ceased to interact dynamically with other systems. The paper of Einstein, Podolsky and Rosen has shown the sort of situations in which this characteristic of quantum mechanics may become especially prominent. This indicates an extension of the usual discussions of the theory of measurement.[3] In the present note a discussion more comprehensive in this respect will be summarized, and some further consideration will be given to the possibility of illustrating the point in question in concrete physical terms.

We shall have to make use of the concepts and results presented in von Neumann's rigorous and

[1] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. **47**, 777 (1935). Referred to as EPR.

[2] N. Bohr, Phys. Rev. **48**, 696 (1935).

[3] Cf. W. Heisenberg, *The Physical Principles of the Quantum Theory*, particularly pp. 55ff.; J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, Chap. VI; W. Pauli, *Handbuch der Physik*, Vol. 24, No. 1, pp. 143ff., p. 165.

detailed discussion[3] of the theory of measurement by means of an instrument. Since the mathematical language of von Neumann's work is not that most current among physicists, it is desirable first to explain the meaning of these concepts in more usual terms. A similar rephrasing of the proofs of important results is omitted in the interests of brevity, and the results are simply stated.

2. Possible Types of Statistical Information about a System

Our statistical information about a system may always be expressed by giving the expectation values of all observables.[4] Now the expectation value of an arbitrary observable F, for a state whose wave function is φ, is

$$\bar{F}=(\varphi,\, F\varphi). \tag{1}$$

If we do not know the state of the system, but know that w_i (with $\sum_i w_i=1$) are the respective probabilities of its being in states whose wave functions are φ_i, then we must assign as the expectation value of F the weighted average of its expectation values for the states[5] φ_i. Thus

$$\bar{F}=\sum_i w_i(\varphi_i,\, F\varphi_i). \tag{2}$$

This formula for $\bar{F}$ is the appropriate one when our system is one of an ensemble[6] of systems of which numbers proportional to w_i are in the states φ_i. It must not be confused with any such formula as

$$\bar{F}=(\sum_i (w_i)^{\frac{1}{2}}\varphi_i,\ F\sum_i (w_i)^{\frac{1}{2}}\varphi_i),$$

which corresponds to the system's having a wave function which is a linear combination of the φ_i. This last formula is of the type of (1), while (2) is an altogether different type.

An alternative way of expressing our statistical information is to give the probability that measurement of an arbitrary observable F will give as result an arbitrary one of its eigenvalues, say δ. When the system is in the state φ, this probability is

$$|(\varphi,\, \chi_\delta)|^2, \tag{1'}$$

where χ_δ is the eigenfunction of F corresponding to the eigenvalue δ. When we know only that w_i are the probabilities of the system's being in the states φ_i, the probability in question is

$$\sum_i w_i|(\varphi_i,\, \chi_\delta)|^2. \tag{2'}$$

Formula (2′) is not the same as any special case of (1′) such as

$$|(\sum_i (w_i)^{\frac{1}{2}}\varphi_i,\, \chi_\delta)|^2.$$

It differs generically from (1′) as (2) does from (1).

When such equations as (1), (1′) hold, we say that the system is in the "pure state" whose wave function is φ. The situation represented by Eqs. (2), (2′) is called a "mixture" of the states φ_i with the weights w_i. It can be shown[7] that the most general type of statistical information about a system is represented by a mixture. A pure state is a special case, with only one nonvanishing w_i. The term "mixture" is usually reserved for cases in which there is more than one nonvanishing w_i. It must again be emphasized that a mixture in this sense is essentially different from any pure state whatever.

3. Reduction of Wave Packets

Let $\Psi(x_1, x_2)$ be the wave function for two systems I and II which have at some previous time interacted and have now ceased to interact. One can show[8] that there always exists an expansion, which is in general unique, in the form

$$\Psi(x_1, x_2)=\sum_k (w_k)^{\frac{1}{2}}\varphi_{\lambda_k}(x_1)\,\xi_{\rho_k}(x_2), \tag{3}$$

where the φ_{λ_k} are eigenfunctions of an observable L corresponding to eigenvalues λ_k, and the ξ_{ρ_k} are eigenfunctions of an observable R corre-

[4] v. Neumann, reference 3, p. 163. We confine ourselves throughout to the case of discrete spectra. (φ, ψ) means the Hermitian inner product, $\int\varphi^*\psi d\tau$. The same letters, $A, B \cdots$, are used to denote observables and the corresponding operators.

[5] For convenience we shall often refer to "the state ψ" instead of "the state whose wave function is ψ." All functions are normalized unless otherwise stated.

[6] The usefulness of this concept has recently been remarked upon by Kemble (Phys. Rev. **47**, 973 (1935)).

[7] v. Neumann, reference 3, pp. 167–168.

[8] v. Neumann, reference 3, pp. 225 ff. In the following arguments we use the word "observable" to mean "complete set of commuting observables," in the sense of Dirac, *Principles of Quantum Mechanics*, §17 (1st edition). In like fashion a set of eigenvalues of such a set of observables is referred to as an "eigenvalue" of the "observable."

sponding to eigenvalues ρ_k. The λ_k are all distinct, and so are the ρ_k. It can be shown[9] that, so far as system II alone is concerned, the statistical information available when (3) is the wave function of the combined systems is represented by a mixture of the states ξ_{ρ_k} with the weights w_k. A similar result holds, of course, for system I. A measurement of L and R on the total system can never give any other value than ρ_i for R to correspond to the value λ_i of L. Thus a measurement of L suffices to predict the value of R and the state of system II, which, *after*[9] such a measurement, is always in one of the pure states ξ_{ρ_k}. Eq. (3) shows that the coupling between the systems has been such as to make system I a suitable instrument for measuring the observable R on system II, the quantity L serving as a "pointer reading."

Now the conclusions to be drawn on the basis of these developments are just those we should expect if we ascribed "real" characteristics to system II as soon as it ceased to interact with system I; this will be shown explicitly later. The contradictions we wish to investigate can be brought out only by going beyond the considerations given in connection with Eq. (3). We may either look for particular cases in which the expansion (3) is not unique—e.g., the example given by EPR and the one we shall give in Section 5—or develop a way of interpreting expansions of a less special type.

The second alternative brings us directly to the general method of "reducing the wave packet." This procedure is commonly known and accepted among physicists, and is applied by EPR; but the writer has been unable to find in the literature an explicit description of its application to the present case. Such a description is briefly as follows: If M is any observable of system I, and ψ_μ its eigenfunction corresponding to the eigenvalue μ, then we can express $\Psi(x_1, x_2)$ as a series in the orthogonal functions $\psi_\mu(x_1)$, with coefficients which are functions[10] of x_2:

$$\Psi(x_1, x_2) = \sum_\mu \psi_\mu(x_1)\zeta_\mu(x_2), \tag{4}$$

where $$\zeta_{\mu'}(x_2) = \int \psi_{\mu'}^*(x_1)\Psi(x_1, x_2)dx_1. \tag{5}$$

The statistical information which one will have about system II after[9] a measurement of M on system I has given the value μ' may be obtained by the following process: We suppose a large number of measurements made on combined systems I+II prepared so that their wave functions are given by (4), each of these measurements consisting in a determination of the values of M for system I and some observable F for system II. We then obtain the relative numbers of times the different values δ are found for F, counting only those measurements in which the value μ' is found for M. These relative numbers are by definition proportional to the quantities $|(\Psi(x_1, x_2), \psi_{\mu'}(x_1)\chi_\delta(x_2))|^2$ (cf. Eq. (1′)), and these quantities are, by (5), just equal to $|(\zeta_{\mu'}, \chi_\delta)|^2$. Since this is true for all observables F, we see that after a measurement on system I has given the value μ' for M, system II is in the pure state with wave function given—apart from normalization—by (5).

4. Probability Calculations and Their Results

We are now ready to discuss in detail the degrees of agreement and disagreement between the results of quantum-mechanical calculations and those to be expected on the assumption that a system once freed from dynamical interference can be regarded as possessing independently real properties. For we can give a definite form to this assumption, and base on it a method for answering all questions which can be asked about the probabilities of finding different results by measurements on system II. This we call Method A:

Assumption and method A. We assume that during the interaction of the two systems each system made a transition to a definite state, in which it now is, system I being in one of the states φ_{λ_k} and system II in one of the states ξ_{ρ_k}. These transitions are not causally determined, and there is no way of finding out *which* transitions occurred, except by making a suitable measurement. In the absence of measurements we know

[9] i.e., *immediately* after. The wave functions used are in general not stationary solutions of the wave equation, but in our discussion we can abstract from the time, because our statistical information about a system at one time can be calculated from that at another time according to a definite differential equation (v. Neumann, reference 3, p. 186).

[10] Unnormalized, and in general not orthogonal to each other.

only that the probabilities of the different transitions are respectively w_k, and that if system I is in the state φ_{λ_i} system II is in the state ξ_{ρ_i}. This provides a sufficient basis for making all needed calculations of probabilities, the methods being those of ordinary probability theory.

Method B. We shall compare with the results of Method A those obtained by quantum-mechanical calculations, using the facts explained in connection with Eqs. (3) and (4), (5).

There are four types of questions for which answers may be required. The notation used in discussing them is the same as that previously described; in particular, the reader is reminded that the observables L and R have a special significance through their connection with the expansion (3), while M and S are arbitrary observables. The questions and their answers are as follows:

(a) If S, having eigenvalues σ and eigenfunctions η_σ is measured on system II without any measurements having been made on system I, what is the probability of obtaining the result σ'? Both methods give the same result

$$\sum_k w_k |(\xi_{\rho_k}, \eta_{\sigma'})|^2, \tag{6}$$

as is at once evident. (Cf. remarks following Eq. (3).)

(b) If L has been measured on system I and the value λ_i obtained, what is the probability of finding the value σ' for S in II? Both methods at once give the answer

$$|(\xi_{\rho_i}, \eta_{\sigma'})|^2. \tag{7}$$

When $S=R$ and $\sigma'=\rho'$, we get in particular the value

$$|(\xi_{\rho_i}, \xi_{\rho'})|^2 = \delta_{\rho_i\rho'}, \tag{7$'$}$$

so that a definite result is predicted. The possibility of such definite predictions was taken by EPR as a "criterion of the physical reality" of the observable R; it is, *par excellence*, the bit of evidence which might incline one to believe Assumption A to be true.

(c) If M has been measured on system I and the value μ' obtained, what is the probability of finding the value ρ_i for R in II? Some calculations are required for this case, but are omitted, being analogous to those which will be given in detail for (d). Both methods give the result

$$w_i|(\varphi_{\lambda_i}, \psi_{\mu'})|^2 / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2]. \tag{8}$$

(d) If M has been measured on system I and the value μ' obtained, what is the probability of finding the value σ' for S in II?

Method A: If the measurements are carried out on a large number of similarly prepared pairs of systems, the fraction giving the value μ' for M is $\sum_k w_k|(\varphi_{\lambda_k}, \psi_{\mu'})|^2$. The fraction giving this value and having system II in state $\xi_{\rho k}$ is $w_k|(\varphi_{\lambda_k}, \psi_{\mu'})|^2$. Then the fraction giving the values μ' for M and σ' for S is $\sum_k w_k|(\varphi_{\lambda_k}, \psi_{\mu'})|^2 \times |(\xi_{\rho_k}, \eta_{\sigma'})|^2$. Dividing this by the fraction giving the value μ' for M, we find as the required *a posteriori* probability,

$$[\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2 |(\xi_{\rho_k}, \eta_{\sigma'})|^2] / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2]. \tag{9A}$$

Method B: The wave function from which we must calculate this probability is, by (5), (3):

$$\int \psi_{\mu'}^*(x_1)\Psi(x_1, x_2)dx_1 = \sum_k (w_k)^{\frac{1}{2}} (\psi_{\mu'}, \varphi_{\lambda_k}) \xi_{\rho_k}(x_2).$$

On normalizing this function and taking the square of its inner product with $\eta_{\sigma'}$ one gets for the required probability

$$[|\sum_k (w_k)^{\frac{1}{2}} (\varphi_{\lambda_k}, \psi_{\mu'})(\xi_{\rho_k}, \eta_{\sigma'})|^2] / [\sum_k w_k |(\varphi_{\lambda_k}, \psi_{\mu'})|^2], \tag{9B}$$

where the denominator comes from normalization.

The difference between (9A) and (9B) comes from the well-known phenomenon of "interference" between probability amplitudes. The absence of such an effect in case (a) is usually stressed in discussions of the theory, since it shows plainly the effect which the mere attaching of an instrument must in general have on the behavior of a system. Since case (d) is not mentioned, it is possible for a reader to form the impression that the theory is consistent with Assumption A.[11]

The formal discrepancy between (9A) and (9B) is a consequence of the fact that, according to the remarks following (4) and (5), after a measurement of M on system I has been made system II is in a pure state, which is in general not one of the $\xi_{\rho k}$. Now no possible manipulation of the w_k will produce from the statistics of the mixture those of any pure state other than one of the $\xi_{\rho k}$. Thus not only is Method A inconsistent with Method B, but also there is no conceivable modification of Method A which could produce consistency between Assumption A and Method B.

The contradiction here, like that between quantum mechanics and the classical doctrine of causality, indicates a radical change in concept rather than a mere change in the details of a mechanism. The idea which is found to be untenable may, roughly, be said to be that of the independent existence of two entities, the state of system II and one's knowledge of its state, only the latter being affected by measurements made on system I. Quantum theory shows that this is not an adequate concept of the relation between subject and object.

5. A Physical Example

The inconsistency of quantum mechanics with the point of view which finds its definite formulation in Assumption A has been demonstrated mathematically. One may still wish to inquire whether it can be realized in a concrete physical example; for we may certainly suppose that not all mathematical operators, even though subject to the proper formal requirements,[12] correspond to experimentally measurable quantities.

An example has been outlined in mathematical form by EPR. As Bohr has remarked, and as is evident from the mathematics, the physical realization of this example involves certain difficulties, in particular the necessity of abstracting altogether from the time in circumstances in which this is not really permissible. We shall now outline an example in physical terms, in which full account is taken of the time-dependence of the quantities involved.

In order to give a physical example of the point in question, it would suffice to describe any case in which one system (I) is used as a measuring instrument in observing another system (II), and in which, after interaction has ceased, some other observable besides the one suited to serve as a "pointer reading" can be measured on system I. The resulting inferences about system II would in the main fall under case (d), in which the contradiction between quantum mechanics and Assumption A is evident. A variety of such examples could doubtless be given.[13] There is, however, a particularly neat and striking special type of example, to which EPR have directed attention; and the one we shall describe is of this type. Before describing it in detail, we shall indicate the nature of this sort of example and its connection with the argument of the preceding section.

The characteristic feature of such an example is obtained by choosing a case in which the expansion (3) is not unique.[14] An assumption in the form of Assumption A can be stated corresponding to each of two expansions of type (3). Each of these assumptions is consistent with a number of quantum-mechanical results of the forms (6), (7), (8), and particularly of the form (7′); thus both of them are to be accepted as

[11] Cf. Pauli, reference 3, p. 89. The remarks there given are entirely correct, but liable to be misleading to an unwary reader with a predilection for Assumption A. The same is to some extent true of the remarks in v. Neumann, reference 3, p. 232, and Heisenberg, reference 3, pp. 59–62.

[12] v. Neumann, reference 3, pp. 75 ff.

[13] e.g., the example we shall discuss could be stated in this way. In the construction and discussion of such examples a difficulty arises owing to the fact that in experimental practice the observables used are almost always incomplete (cf. note 8), and that the theory of measurements of incomplete observables is not altogether free from ambiguity (cf. v. Neumann, reference 3, pp. 184–185). For this reason it is much easier and more satisfactory to treat our example—which itself involves the use of incomplete observables—in the way we have here used.

[14] The condition for this is that the w_k be not all distinct (cf. v. Neumann, reference 3, p. 232 and p. 175). In the example of EPR, the w_k are all equal and *all* states of system II are included. This means that there are an infinite number of different expansions of the form (3), and that any measurement made is as likely to give any one result as any other. Measurement in quantum mechanics has in general a twofold aspect: it gives information (of a statistical nature) about the properties of the state of the system *before* the measurement, and it enables us to predict the state of the system *after* the measurement. In order that it may serve the first purpose, care must be used in choosing a suitable coupling of object and instrument. In the example of EPR, the coupling has been so violent that all trace of the original state of system II is lost, so that the word "prediction" is the only correct one to apply to one's conclusions about system II.

true, according to the general attitude underlying Assumption A. Now each assumption asserts that system II has made a transition to a definite (though unknown) state, and that it is a state for which a certain observable has a well-defined value. In the case chosen for the example the two requirements thus imposed on the state of system II cannot be fulfilled simultaneously by any quantum-mechanical state, because of the limitations imposed by the uncertainty principle. The point of view expressed in Assumption A is accordingly found to conflict with quantum mechanics.

With this outline of the argument in mind, we proceed to the detailed discussion of the example. In this we shall use the language of experimental physics rather than that of the mathematical formalism.

We consider the determination of the position of a heavy particle of mass M, say a proton, by the use of a "microscope." This microscope is designed for use, not with γ-rays, but with light charged particles of mass m, say electrons. If we shine soft light in from the side of the "barrel" and let it be scattered from an electron which is on its way to the "lens," a transverse component[15] of the electron's momentum can be calculated from the Doppler effect on the scattered light. Under suitable circumstances this will enable us to infer the momentum[15] of the proton quite accurately. On the other hand, we have only to let the electron travel undisturbed to the photographic plate in order to be able to infer with considerable accuracy the corresponding coordinate[15] of the proton. The "Assumption A" which corresponds to the first experiment is: During the collision between the electron and the proton, the proton made a transition into some state with well-defined momentum; *which* state this is can be determined by a measurement made on the electron. Corresponding to the second experiment one has an "Assumption A" which reads exactly the same except that the word "momentum" is replaced by the word "coordinate." Since one is still free to choose which experiment is to be performed *after* the electron has been scattered into the microscope[16] and ceased to interact with the proton, these assumptions must both be true at once, if one accepts the point of view on which they are based. But we shall see that their simultaneous truth can be in conflict with the uncertainty principle.

Before deriving the actual expressions for the uncertainties in the two alternative predictions, let us consider briefly their physical origin. In principle the Doppler effect experiment can be made with arbitrary precision (cf. latter part of note 16). The uncertainty Δp may accordingly be regarded as fixed by the original uncertainties in the momenta of the particles; to make it small, we have only to prepare them properly beforehand. Now the use of particles so prepared will in some degree limit the accuracy of our prediction of x. This comes about through the fact that the predictions we want must refer to a definite time, and that the prediction of x read directly from the photographic plate refers to the moment of the collision: this is not precisely known, because our long wave trains take a finite time to pass over each other. In making a prediction for a definite moment, which we choose to be that at which the wave trains cease to overlap, we must allow for the distance the proton may have moved since the collision. This leads to the existence of a lower limit for the product $\Delta p\Delta x$; but since the electron, with its small mass, moves rapidly across the region where the proton may be found, whereas the proton's motion after the collision is comparatively sluggish, this limit will be found to be not h, but about $(m/M)h$.

[15] i.e., transverse to the axis of the microscope. This component of the momentum will be called simply the "momentum," and the corresponding coordinate will be called the "coordinate" x.

[16] Of course the electron need not have been scattered into the microscope; in any given case we can only go on with the experiment in good faith, hoping that it will have been so scattered. The experiment can fail; and from the fact that it has succeeded when one choice was made, quantum mechanics of course offers no such inference as that it "would have" succeeded if the other choice had been made. But under the proper circumstances, such an inference does follow from the point of view of Assumption A. The circumstances required are, that in the Doppler effect experiment enough quanta must be sent in so that "if the electron is in the barrel," several are sure to be scattered back into a suitably small solid angle. The quanta must be extremely soft, so that scattering a great many of them will not change the electron's momentum too much; and their energy must be measured very accurately. This means that the electron is in the barrel a long time; and it turns out that, to secure the accuracy represented by Eq. (14), the length of the microscope must be much greater than $(M/m)^{11/3}\Delta x$, while the lateral distance to the source and analyzer of the light must be much greater than $(M/m)^{13/3}\Delta x$. These requirements make in principle no difficulty.

To show this, we shall discuss the possible sources of uncertainty in the prediction of x. These are: (1) The finite resolving power of the microscope; (2) inaccuracy of focus; (3) allowance for the proton's motion between the moment of the collision and the moment to which the predictions refer.

(*1*) *Finite resolving power.* This gives an uncertainty

$$\Delta_1 x \sim (\lambda/\epsilon) \sim (h/p\epsilon), \tag{10}$$

where ϵ is the numerical aperture and λ, p are wave-length and momentum of the electron.

(*2*) *Inaccuracy of focus.* Let the electron[17] be sent in through a slit of width s. At the far side of the field of view the half-width of the beam will have become, through diffraction, about $(s/2)+(L\lambda/2s)$, where L is the breadth of the field of view. By proper choice of s, this expression takes on its minimum value, $(\lambda L)^{\frac{1}{2}}$. Now L must be equal to $(h/\Delta p)$, the whole length of the wave train; otherwise the restriction of the field of view would cause enough diffraction of the scattered waves to spoil the significance of the contemplated Doppler effect experiment. Thus we get for the uncertainty in x from this source

$$\Delta_2 x \sim \epsilon(\lambda h/\Delta p)^{\frac{1}{2}} = \epsilon h/(p\Delta p)^{\frac{1}{2}}. \tag{11}$$

(*3*) *Allowance for proton's motion.* The time available for this motion is of the order of magnitude

$$\Delta t \sim (h/\Delta p)/(p/m),$$

which is the time required for the electron to travel the length of such a wave train as must be used. To make the proton's velocity after the collision as small as possible, we can send the two particles into the field of view with equal and opposite momenta. Then if the electron were scattered exactly at a right angle, the resulting x component of the proton's velocity would be zero. But on account of the finite aperture we must take it to be roughly

$$v_x' \sim \epsilon(p/M).$$

We then get

$$\Delta_3 x \sim v_x' \Delta t \sim \epsilon(m/M)(h/\Delta p). \tag{12}$$

Thus we have finally

$$\Delta p \Delta x \sim \Delta p(\Delta_1 x + \Delta_2 x + \Delta_3 x)$$
$$\sim h\{(\Delta p/\epsilon p) + \epsilon(\Delta p/p)^{\frac{1}{2}} + \epsilon(m/M)\}. \tag{13}$$

By making Δp extremely small compared to p, we can make

$$\Delta p \Delta x \sim (m/M)h. \tag{14}$$

By taking ϵ also to be small, we can in principle make $\Delta p \Delta x$ *arbitrarily* small; also we could in principle dispense with the advantage we obtained by using the disparity in mass of two known particles.

As explained in the preliminary discussion of the example, the comparison of a result such as (14) with the uncertainty principle shows that Assumption A is inconsistent with quantum mechanics.

Concluding Remark

Both by mathematical arguments and by discussion of a conceptual experiment we have seen that the assumption that a system when free from mechanical interference necessarily has independently real properties is contradicted by quantum mechanics. This conclusion means that a system and the means used to observe it are to be regarded as related in a more subtle and intimate way than was assumed in classical theory. It does not mean that quantum mechanics is not to be regarded as a satisfactory way of correlating and describing experience; it does illustrate the difficulty, often remarked upon by Bohr, which is inherent in the problem of the distinction between subject and object.

[17] Only one beam need be narrow. Under (3) we shall see that it is expedient to make the wave-lengths of electron and proton originally equal, so that $\Delta_2 x$ will be the same whichever beam is limited. By admitting the proton through a slit much broader than s, we can assure that the electron escapes unless it is scattered through a fairly large angle, thus avoiding the prevalence of spurious effects.