THE SIGN OF THE NUCLEAR MAGNETIC MOMENT

From the spectroscopic methods of measuring nuclear spin the sign of the nuclear moment is obtained by noting which of the F states has the higher energy. To obtain this information with atomic beam methods we must know whether we are dealing with Fig. 2a or 2b. Owing to their symmetry it is impossible to obtain this information by deflection experiments alone. However if deflection is combined with the nonadiabatic transitions, sufficient data can be gathered to deduce the sign of the nuclear moment. However, in a two-field system the arrangement must utilize at least one weak field deflection. The experiment of Frisch and Segrè which used two strong field deflections does not yield sufficient data.

Example 1. Hydrogen

If we first deflect the hydrogen atoms in a weak field corresponding to about $x=0.4$ the deflection pattern has two components $m=\pm 1$, $F=1$ which are deflected more than the components $m=0$, $F=0$ and $m=0$, $F=1$. We can by means of a slit select atoms which are in one of these inner states. We select the atoms which are deflected toward the stronger field (positive over-all moment), pass them through the rotating field, and follow this by a strong analyzing field. If the state we have selected is the $F=0$ state then no transitions are possible and we obtain only one component. If the state is $F=1$, $m=0$, there will be transitions to the other levels with the nonadiabatic field properly chosen. Some of these levels have moments of opposite sign which the analyzing field can resolve into two components. If one component is obtained the proton moment is positive; if two are observed the moment is negative. A check is obtained by a similar observation of the side deflected into the weaker field, where the opposite situation prevails. The angle α should be $\pi/2$ for these transitions as is evident from Eq. (6). Many modifications of this procedure, but dependent on the same principle will readily suggest themselves.

Example 2. Potassium

Although the pattern as given in Fig. 2 for $I=3/2$ is more complicated than for $I=\frac{1}{2}$, a similar procedure can be applied. With a weak deflecting field we select the atoms from one of the levels by means of a slit on the strong field side and subject them to the rotating field. In the subsequent strong field analysis the line will be either single or double. If single the nuclear moment is positive, if double, negative. This is so because transition amongst the $F=I-\frac{1}{2}$ levels does not affect the strong field moment, but the $F=I+\frac{1}{2}$ levels always have the possibility of making transition which will result in a strong field moment of opposite sign. The treatment of any other case is along similar lines.

In conclusion the writer wishes to express his indebtedness to Professor E. Segrè for discussions on the details of the Frisch and Segrè experiment and to Messrs. Clark, Heller and Motz of the theoretical physics seminar for discussions on the details of the interpretation of Majorana's paper.

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Does the Alpha-Particle Possess Excited States?

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The kinetic energy operator for the internal motion of the alpha-particle can be expressed without cross derivatives in terms of a suitable set of internal coordinates. The usual methods when applied to a restricted class of Hamiltonian operators then yield sum rules from which are deduced upper limits to the excitation energies of the $2p$ levels. These upper limits involve certain diagonal matrix elements which are easily evaluated by using an approximate normal state wave function. Computations on three different nuclear models indicate definitely the existence of a singlet $2p$ level in the discrete eigenvalue spectrum of the alpha-particle if the range of the intranuclear forces exceeds 2.0×10^{-13} cm. A simple variation calculation gives excitation energies which fall slightly below the upper limit fixed by the sum rules.

RECENT experiments[1] on the artificial disintegration of light nuclei appear to establish the existence of excited states of the alpha-particle. It is therefore of interest to determine whether or not the current nuclear models allow the alpha-particle to possess excited states.

The straightforward procedure for the discussion of this problem involves the specification of a nuclear Hamiltonian and the use of the variation method in conjunction with suitable approximate wave functions to compute the excited state energies. These calculations are difficult and as yet not very satisfactory. Fortunately it is possible to supplement the straightforward procedure by a simple calculation based on certain sum rules.

Section I. The Sum Rules

Consider a nuclear Hamiltonian of the form

$$H=-\tfrac{1}{2}(\Delta_1+\Delta_2+\Delta_3+\Delta_4)+V(x_1, x_2, x_3, x_4) \quad (1)$$

with the eigenvalues E_0, E_1, $\cdots$. The sum rules are most simply expressed in terms of new coordinates

$$\begin{aligned}\xi_1&=x_1+x_2-x_3-x_4,\\ \xi_2&=x_1-x_2,\\ \xi_3&=x_3-x_4,\\ 4x&=x_1+x_2+x_3+x_4,\end{aligned} \quad (2)$$

and corresponding η, z components. The subscripts 1 and 2 refer to protons; 3 and 4 to neutrons. One consequence of the transformation (2) is the operator identity

$$\Delta_1+\Delta_2+\Delta_3+\Delta_4\equiv 4\Delta_{\xi_1}+2\Delta_{\xi_2}+2\Delta_{\xi_3}+\tfrac{1}{4}\Delta_x, \quad (3)$$

which is useful because it expresses the kinetic energy operator for the internal motion in terms of relative coordinates and yet contains no cross terms. The usual methods yield the sum rules

$$\sum_n (E_n-E_m)\,|(n|\xi_1|m)|^2=2, \quad (4)$$

$$\sum_n (E_n-E_m)\,|(n|\xi_l|m)|^2=1, \quad (l=2, 3), \quad (5)$$

with the understanding that the schematic summation over all states includes an integration over the continuous spectrum. The inequality

$$(E_1-E_0)\sum_n |(n|\xi_1|0)|^2<2 \quad (6)$$

follows at once from Eq. (4) if E_1 is the eigenvalue of the state for which the excitation energy is as small as possible subject to the condition that the matrix element $(1|\xi_1|0)$ does not vanish (clearly the $2p$ singlet state). There exists a level E_2 (singlet $3d$ or $2s$) which is related to E_1 in the same way that E_1 is related to E_0. Then

$$(E_2-E_1)\sum_{n\neq 0} |(n|\xi_1|1)|^2 < 2+(E_1-E_0)\,|(1|\xi_1|0)|^2. \quad (7)$$

$$\text{But} \quad \sum_n |(n|\xi_1|0)|^2=(0|\xi_1^2|0), \quad (8)$$

$$\sum_{n\neq 0} |(n|\xi_1|1)|^2=(1|\xi_1^2|1)-|(1|\xi_1|0)|^2. \quad (9)$$

Consequently

$$E_1-E_0<2/(0|\xi_1^2|0), \quad (10)$$

$$E_2-E_1<4/[(1|\xi_1^2|1)-|(1|\xi_1|0)|^2]. \quad (11)$$

Similar relations follow, of course, from Eq. (5) for levels E_1' (triplet $2p$), E_2' (singlet $2s$) having properties with respect to the matrix elements of ξ_2 identical with those of E_1, E_2 with respect to the matrix elements of ξ_1. Nonvanishing matrix elements connecting singlet and triplet states occur because the coordinate ξ_2 itself has the symmetry property of a triplet state wave function and the spin coordinates and spin wave functions do not appear in the problem.

The presence in the Hamiltonian of operators which interchange the spin coordinates of unlike particles introduces a dependence on spin orientation which destroys the possibility of obtaining states with a definite multiplicity. In this case Eqs. (4) and (5) are rigorously true only if the complete space-spin wave functions are used in constructing the matrix elements. However, when the spin dependence is small the interaction operator can be replaced by an effective ordinary potential function which is the same for all states of a given symmetry type with respect to interchange of the space coordinates of like particles.[2] The inequalities (10) and (11) will then remain valid if the matrix elements are constructed from solutions of the modified spin free wave equation containing the proper effective ordinary potential. Levels belonging to

[1] H. R. Crane, L. A. Delsasso, W. A. Fowler and C. C. Lauritsen, Phys. Rev. **48**, 100, 102, 125 (1935).

[2] Feenberg and Knipp, Phys. Rev. **48**, 906 (1935), Eqs. (18–22). To obtain the case discussed above replace the Majorana exchange operator by unity and the Heisenberg exchange operator by P^MP^H. *Added in proof:* see also J. H. Bartlett, Phys. Rev. **49**, 102 (1936).

the symmetry type of the normal state are not changed in any way by the substitution of saturation type potentials for ordinary potentials in the operator representing the interaction between like particles.[3]

The Eqs. (4) and (5) are not valid for Hamiltonians containing neutron-proton potentials of the Majorana type because the coordinates ξ_1, ξ_2, ξ_3 do not commute with the Majorana exchange operators. In this case a familiar argument leads to the relations

$$4(m|\partial/\partial\xi_1|n)=(E_n-E_m)-(m|\xi_1|n)(m|\xi_1V-V\xi_1|n), \quad (12)$$

$$2(m|\partial/\partial\xi_l|n)=(E_n-E_m)-(m|\xi_l|n)(m|\xi_lV-V\xi_l|n), \quad (l=2,\ 3), \quad (13)$$

and the sum rules

$$2\sum_m(E_m-E_n)|(n|\xi_1|m)|^2 = 4-(n|\xi_1^2V+V\xi_1^2-2\xi_1V\xi_1|n), \quad (14)$$

$$2\sum_m(E_m-E_n)|(n|\xi_l|m)|^2 = 2-(n|\xi_l^2V+V\xi_l^2-2\xi_lV\xi_l|n), \quad (l=2,\ 3). \quad (15)$$

Now if $$V=-\Sigma f(r_{\nu_i\pi_j})P^{M}{}_{\nu_i\pi_j}-f_\pi(r_{12})-f_\nu(r_{34}), \quad (16)$$

these last equations reduce to

$$\sum_m(E_m-E_n)|(n|\xi_1|m)|^2 = 2+2(n|\Sigma x^2{}_{\nu_i\pi_j}f(r_{\nu_i\pi_j})P^{M}{}_{\nu_i\pi_j}|n), \quad (17)$$

$$2\sum_m(E_m-E_n)|(n|\xi_l|m)|^2 = 2+(n|\Sigma x^2{}_{\nu_i\pi_j}f(r_{\nu_i\pi_j})P^{M}{}_{\nu_i\pi_j}|n), \quad (l=2,\ 3). \quad (18)$$

The matrix element on the right in Eqs. (17) and (18) is positive and has the order of magnitude unity when n designates the normal state. Consequently the upper limit on the excitation energy which can be deduced from these equations will generally lie too far up to be useful.

II. Applications of the Sum Rules

The function

$$\psi_0=N\exp\left[-(\nu/2)(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2)-(\mu/2)(r_{12}^2+r_{34}^2)\right] \quad (19)$$

is perhaps the simplest approximate wave function for the normal state. Using ψ_0 we obtain

$$(0|\xi_1^2|0)\sim\tfrac{1}{2}\nu, \quad (0|\xi_2^2|0)\sim\tfrac{1}{2}(\nu+\mu). \quad (20)$$

It is important to know the direction in which these matrix elements are in error. Since the approximate wave function vanishes too rapidly for large separations of the particles the immediate inference is that the matrix elements (20) are too small. The possibility that for very small separations the correct wave function may be larger than ψ_0 cannot be excluded, but this effect, if it occurs, can hardly be important enough to reverse the sign of the error. The conclusion seems warranted that the use of the approximate matrix element $(0|\xi_1^2|0)$ in (10) actually strengthens the inequality. Thus Eq. (10) may be replaced by

$$E_1-E_0<4\nu. \quad (21)$$

[3] Reference 2, italics below Eq. (3).

Table I. ν, μ and A as functions of α.

α	$1/\alpha^{\frac{1}{2}}$ (cm)	A I	A II	A III	A IV	ν I	ν II	ν III	ν IV	μ I	μ II
10	2.8×10^{-13}	65	65	44	83	10.0	8.9	8.1	8.9	6.8	6.2
20	2.0×10^{-13}	100	100	68	127	15.1	13.6	12.5	12.0	10.4	9.5
30	1.6×10^{-13}	132	132	90	167	19.7	17.7	16.2	14.5	13.5	12.4

The parameters ν, μ and the potentials required to fit the binding energy of the alpha-particle are determined by the equations[4]

$$E_0(\nu,\mu)\equiv\int\cdots\int\psi_0H\psi_0d\tau=-56mc^2, \quad (22)$$

$$(\partial/\partial\nu)E_0(\nu,\mu)=(\partial/\partial\mu)E_0(\nu,\mu)=0. \quad (23)$$

These equations have been solved for the following nuclear models:

I. The interactions occur between unlike particles only and are given by the potentials $Ae^{-\alpha r^2}$,

II. Same as model I with Majorana potentials $Ae^{-\alpha r^2}P^M$,

III. Equal interaction potentials, $Ae^{-\alpha r^2}$, between all pairs of particles,

IV. Same as model III with the potentials $Ae^{-(3\alpha)^{\frac{1}{2}}r}$.

The energy function $E_0(\nu,\mu)$ has already been found for the first three models.[5] For the fourth model an elementary calculation yields the result

$$E_0(\nu,\nu)=27\alpha/8\sigma^2-6A\{(1+2\sigma^2)e^{\sigma^2}\times(1-2\pi^{-\frac{1}{2}}\textstyle\int_0^\sigma e^{-x^2}dx)-2\pi^{-\frac{1}{2}}\sigma\}, \quad (24)$$

$$8\nu\sigma^2=3\alpha.$$

Table I lists values[6] of ν, μ and A computed from Eqs. (22) and (23). The results for Model II are included because it is thought that they may prove useful in other calculations.

Stability against spontaneous disintegration

[4] The value $56mc^2$ is obtained by adding $2mc^2$ to the experimental binding energy to correct for the neglect of the coulomb repulsion between protons.

[5] Feenberg, Phys. Rev. **47**, 850 (1935), Eqs. (17) and (21); reference 2, Eq. (9) (replace $A_{\nu\pi}$ and A_ν by A and set $p=1$).

[6] Unit of energy $-mc^2=510{,}000$ electron volts; unit of length $-(\hbar^2/mm_pc^2)^{\frac{1}{2}}=8.97\times10^{-13}$ cm.

into H^3 and H^1 is assured if the excitation energy lies below $40mc^2$ (since the binding energy of H^3 is about $16mc^2$). This condition is satisfied for $1/\alpha^{\frac{1}{2}}>2.8\times10^{-13}$ cm. However the quantity 4ν is certainly considerably larger than the excitation energy. If it exceeds the excitation energy by at least one-third, which does not seem unlikely, the stability condition is satisfied for $1/\alpha^{\frac{1}{2}}>2.0\times10^{-13}$ cm. Calculations on the binding energies of the hydrogen and helium isotopes with error function potentials[2] determine a radius of action 2.2×10^{-13} cm. Thus the existence of a singlet $2p$ level is definitely indicated. Since $2(\nu+\mu)\leqq4\nu$, the triplet $2p$ level will be stable also if the forces between like particles are of the ordinary type (without exchange operators).

The sum rules (4) and (17) can be used to fix an upper limit to the half-breadth of the singlet $2p$ level. The average lifetime of this level is

$$\Delta t=1\Big/\left\{\frac{4\omega^3}{3\hbar c^3}\frac{e^2\hbar^2}{mm_p c^2}|(1|\tfrac{1}{2}\xi_1|0)|^2\right\}. \quad (25)$$

(ω is the radiation frequency expressed in radians per second). By Eq. (4) or Eq. (17)

$$\hbar\omega|(1|\xi_1|0)|^2<2mc^2 \quad (26)$$

and therefore

$$\Delta t>3.8\times10^5mc^2/\hbar\omega^2. \quad (27)$$

The half-breadth ΔE satisfies the relation

$$\Delta E\Delta t\sim\hbar \quad (28)$$

which together with Eq. (27) implies

$$\Delta E<2.6\times10^{-6}(\hbar\omega)^2/mc^2. \quad (29)$$

The energy of the hardest gamma-ray observed in disintegration experiments is $16\times10^{+6}$ electron volts or $32mc^2$. With this value for $\hbar\omega$ we obtain

$$\Delta E<1300 \text{ electron volts}. \quad (30)$$

Section III. Approximate Wave Functions and Matrix Elements for $2p$ States

The wave function

$$\psi_{2p}=N\xi_1\exp\left[-(\tau/2)(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)\right] \quad (31)$$

Table II. *On the singlet 2p state (Model III).*

α	τ	$-E_1$	$(1\|\xi_1^2\|1)$	$(1\|\xi_1\|0)^2$	$(0\|\xi_1^2\|0)$
10	6.5	24	0.231	0.057	0.061
20	9.0	11	0.167	0.035	0.040
30	10.0	−1	0.150	0.023	0.031

yields a value for the singlet $2p$ energy level determined by the equations

$$E_1(\tau)=\int\cdots\int\psi_{2p}H\psi_{2p}dr, \quad (32)$$

$$(\partial/\partial\tau)E_1(\tau)=0. \quad (33)$$

By using model III,

$$E_1(\sigma)=5.5\alpha\sigma-A(\sigma/(\sigma+1))^{\frac{3}{2}}\{6-2/(\sigma+1)\}, \quad (34)$$

$$2\tau=\alpha\sigma.$$

Exactly the same energy equation is found with the triplet wave function

$$\psi_{2p}=N\xi_2\exp\left[-(\tau/2)(r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2)\right]. \quad (35)$$

Neither of these approximate wave functions is suitable for the investigation of a model with Majorana type forces because ξ_1 and ξ_2 are not invariant under the interchange of the coordinates of a pair of unlike particles.

The approximate matrix elements are

$$(1|\xi_1^2|1)\sim3/2\tau, \quad (36)$$

$$(1|\xi_1|0)^2\sim[2(\nu\tau)^{\frac{1}{2}}/(\nu+\tau)]^{11}/2v. \quad (37)$$

Numerical values for τ, E, and the matrix elements are listed in Table II.

The table shows that the simple variation calculation yields a slight improvement in the determination of the excited state energy level over the upper limit set by the inequality (21). Obviously a considerably more complicated calculation would be required for an accurate determination of the excitation energy. The near equality of $(1|\xi_1|0)^2$ and $(0|\xi_1^2|0)$ is noteworthy. No useful information can be obtained from the inequality (11) since it puts the upper limit on E_2 far out in the continuous spectrum.