A New Relativity

Paper I. Fundamental Principles and Transformations Between Accelerated Systems

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A new approach to the relativity theory, suggested by the theory of E. A. Milne, is developed. This approach, like Milne's, dispenses with the concepts of measuring rods of undefinable rigidity and clocks of undefinable periodicity. A new category of equivalent relatively accelerated reference systems with Euclidean geometry and constant light-velocity is described, and the space-time transformation for such systems is developed. It is shown that in an effectively empty world Einstein's assumption of an invarient physical interval and an absolute four-dimensional space-time is in contradiction with the underlying principle of the relativity of motion, and therefore either the one or the other must be abandoned.

THE fundamental assumption underlying Einstein's theory of relativity is that the physical interval between two nearby events (the square of the element of measured distance minus the square of the product of the velocity of light by the element of measured time) is an invariant having the same value for all reference systems. This assumption is a natural inference derived from the Minkowskian complex four-dimensional space-time representation of the Lorentz transformation, and has led to cosmological predictions which have been verified by observation. Nevertheless, the author of the present paper believes that Einstein's postulate is too restricted to include all possible motions of material particles. In this paper he will present an alternative theory, and will give reasons for believing that it, rather than Einstein's theory, represents the proper formulation of relativity in an effectively empty world.

The present investigation was prompted by the perusal of a recent book by E. A. Milne,[1] to whom the writer wishes to make due acknowledgment. In this important work Milne offers an approach to the relativity theory which avoids the undefinable concepts of rigid measuring rods and periodic clocks. In spite of their great advantages, the writer believes that Milne's methods are faulty in certain respects, particularly his definition of equivalence, in that it implicitly involves synchronism as well, and his apparent belief that physical geometry is conventional. The fundamental principles propounded here, while suggested by Milne's treatment, differ from his in many essential particulars, and the space-time transformations for relatively accelerated reference systems are believed to be altogether new.

This contribution is divided into four parts. In Part 1 fundamental methods will be outlined and the principle of relativity will be stated in its general form for the effectively empty world in which we are interested; in Part 2 applications will be made to one-dimensional systems; in Part 3 the special theory for a three-dimensional space will be shown to follow immediately from the fundamental principles; and in Part 4 the new transformations for relatively accelerated three-dimensional reference systems will be developed and contrasted with Einstein's theory. It is the author's intention to follow this paper shortly by another in which the transformation of the electromagnetic field between accelerated systems will be developed and the necessary revision of the fifth or force equation of electromagnetic theory, which is demanded by the new relativity principle, will be obtained. In the conclusion to this paper some qualitative comments on the motion of an electron, which are expected to be developed quantitatively in the succeeding paper, will be presented.

PART I. FUNDAMENTAL PRINCIPLES

To emphasize the fact that a single observer's measurements are confined to the single point occupied by himself, we shall designate such an observer as a *particle-observer*. Each observer is supposed to possess a temporal intuition, that is

[1] E. A. Milne, *Relativity, Gravitation and World-Structure* (Clarendon Press, Oxford, 1935).

to say, if two events E_1 and E_2 occur at himself, he can judge without ambiguity whether E_2 takes place before E_1, simultaneously with E_1, or after E_1. We shall provide each particle-observer with a device for assigning numbers $\tau_1, \tau_2 \cdots$ to events occurring at himself in such a way that, if event E_2 occurs simultaneously with E_1, the numbers τ_2 and τ_1 assigned to the respective events are the same, whereas, if E_2 occurs after E_1, then $\tau_2 > \tau_1$, and *vice versa*. This device, which may be quite arbitrary in all other characteristics than the one specified, we shall call a *clock*, and we shall name τ the *local time* of the particle-observer under consideration.

Next we shall adopt certain conventions which will enable a particle-observer P to employ light-signals, timed by his clock, in such a way as to describe quantitatively the motion of any moving-element M. Let P dispatch a light-signal to M at time τ_1. On arrival at M the signal is immediately sent back to P, whom it reaches at time τ_3. Choosing an arbitrary constant c (a constant whose numerical value, once chosen, remains the same for subsequent repetitions of the experiment) we define the *distance* r_2 of M from P when the signal reaches M by

$$r_2 \equiv \tfrac{1}{2}c(\tau_3 - \tau_1), \tag{1}$$

and we define P's value of the *time* t_2 at which the signal reaches M by

$$t_2 \equiv \tfrac{1}{2}(\tau_3 + \tau_1). \tag{2}$$

Since $r_2/(t_2 - \tau_1) = r_2/(\tau_3 - t_2) = c$, the constant c represents the velocity of the light-signal in terms of the conventions adopted for measuring distance and time at a remote point.

It should be noted that both r_2 and t_2 are computed by P from coincidences occurring at himself. The first represents P's estimation of the distance of M and the second P's estimation of the time at M at which the signal reaches M. We shall call t_2 the *extended time* of P at M. If a second particle-observer is attached to M, the local time of the second observer when the signal reaches him may be quite different from P's extended time t_2, and his estimation of the distance of P may not agree at all with r_2. The notation used designates local time measured at a particle-observer by the Greek letter τ, the computed time at a distant point being denoted by the Italic letter t. The only measurements made are carried out at the particle-observer under consideration, and no yardstick of undefined rigidity nor clock of undefined periodicity is assumed.

Evidently each one of two particle-observers P and P' constitutes a moving-element in the experience of the other. Thus P, acting as observer, may describe the motion of P' relative to himself, or P', as observer, may describe the motion of P. We shall designate by letters without primes local times measured by P or quantities computed therefrom, and by corresponding letters with primes local times measured by P' or quantities computed from these times. We attribute to light-signals dispatched from one particle-observer to another the following property: *If two light-signals are sent from one particle-observer to another, the light-signal which is dispatched later from the one will be received later by the other.* This fundamental principle underlies all the theory to be developed. In effect it is equivalent to limiting our consideration to particle-observers with relative velocities less than the velocity of light.

Now suppose that a light-signal is dispatched from P toward P' at time τ_1 and is received by the latter at time τ_2'. Let a second light-signal be dispatched from P' toward P at a time τ_1' earlier than τ_2' and be received by P at a time τ_2 later than τ_1, the time τ_1' being so chosen that $\tau_2' - \tau_1' = \tau_2 - \tau_1$. Then we say that τ_1 and τ_1' are *corresponding times*. Evidently this condition can always be fulfilled, for if $\tau_2' - \tau_1' > \tau_2 - \tau_1$ the light-signal from P' can be replaced by one sent a little later, which will increase both τ_1' and τ_2 by virtue of the principle stated in the last paragraph, making $\tau_2' - \tau_1'$ smaller and $\tau_2 - \tau_1$ larger. The pair of light-signals under discussion are illustrated by the lower solid lines in Fig. 1, the time being plotted vertically and the separation of P and P' horizontally.

The statement that τ_1 and τ_1' are corresponding times does not necessarily imply that τ_2 and τ_2' are corresponding times also, for, if the signals received by P' and P at the times τ_2' and τ_2 are immediately returned and reach P and P' at the times τ_3 and τ_3' respectively, the fact that $\tau_2' - \tau_1' = \tau_2 - \tau_1$ does not of necessity lead to the

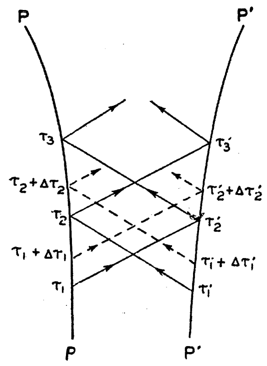

FIG. 1.

equality of $\tau_3'-\tau_2'$ and $\tau_3-\tau_2$. In conformity with our present notation we shall always designate corresponding times by identical subscripts.

Next consider a pair of corresponding times $\tau_1+\Delta\tau_1$ and $\tau_1'+\Delta\tau_1'$ defined by the pair of light-signals represented by broken lines on the figure, which are dispatched from P and P' at times respectively $\Delta\tau_1$ and $\Delta\tau_1'$ later than the signals sent at the corresponding times τ_1 and τ_1', and received at times $\Delta\tau_2$ and $\Delta\tau_2'$ later. As the times of dispatch correspond, $\Delta\tau_2'-\Delta\tau_1' =\Delta\tau_2-\Delta\tau_1$. Now, *if* $\Delta\tau_1'=\Delta\tau_1$, *and hence* $\Delta\tau_2' =\Delta\tau_2$, *whatever* $\Delta\tau_1$ *may be, we say that the clocks of* P' *and* P *are equivalent, or that the two particle-observers are equivalent.* If, in addition, the clocks of the two particle-observers are *set* so that $\tau_1'=\tau_1$, and therefore all corresponding times are identical, the clocks of the two observers are said to be *synchronized.* In future we shall deal only with particle-observers who are equivalent, and, when we are concerned with two particle-observers alone, we shall generally suppose their equivalent clocks to be synchronized.

It follows from the definition of equivalence that all pairs of corresponding times at two equivalent particle-observers differ by the same amount. Therefore, if τ_1' and τ_1 are corresponding times, τ_2' and τ_2 are also. Conversely, if all pairs of corresponding times differ by the same amount, the particle-observers are equivalent. If the clocks of two particle-observers are synchronous, corresponding times are identical. Hence synchronism implies equivalence, although equivalence may exist without synchronism.

Let P and P' (Fig. 1) be equivalent but not necessarily synchronous. Let the signals dispatched at the corresponding times τ_1 and τ_1' and received at the corresponding times τ_2 and τ_2' be immediately returned toward the particle-observers from whom they originated, reaching the latter at the corresponding times τ_3 and τ_3'. In this case we may say that the signals *interlock*, the signal dispatched from P at the time τ_1 being received by P' at the time τ_2' and immediately returned to P whom it reaches at the time τ_3. Evidently τ_2 is some function of τ_1, which could be obtained empirically by observing the values of τ_2 corresponding to different values of τ_1. Now, if τ_1 becomes τ_2, τ_2 becomes τ_3. So τ_3 must be the *same* function of τ_2 as τ_2 is of τ_1.

If we are given the law of motion of P' relative to P, that is, if we know r_2 as a function of P's extended time t_2 at P', we can express τ_3 as a function of τ_1 by (1) and (2). Let this functional relation be $\tau_3=F(\tau_1)$. Since we must have

$$\tau_3=f(\tau_2), \qquad \tau_2=f(\tau_1), \tag{3}$$

it follows that our problem is to find the function f such that

$$f(f(\tau_1))=F(\tau_1). \tag{4}$$

Not only are relations (3) *necessary* for equivalence; they are also *sufficient*. For all we need do is to assign the values τ_1+k, τ_2+k, $\tau_3+k \cdots$ to the times τ_1', τ_2', $\tau_3' \cdots$ at which the various signals in Fig. 1 are dispatched from P', where k is a constant. Then the clocks of the two particle-observers are equivalent. If $k=0$ they are synchronous as well.

An alternative condition for equivalence arises from the fact that Eqs. (3) imply that $d\tau_3/d\tau_2$ is the same function of τ_3 and τ_2 as $d\tau_2/d\tau_1$ is of τ_2 and τ_1. Let this function be denoted by g. Then

$$d\tau_3/d\tau_2=g(\tau_3,\tau_2), \qquad d\tau_2/d\tau_1=g(\tau_2,\tau_1). \tag{5}$$

But, from the equation of motion of P' relative to P, we get $d\tau_3/d\tau_1=G(\tau_3,\tau_1)$, where G is a known function. Consequently

$$g(\tau_3,\tau_2)g(\tau_2,\tau_1)=G(\tau_3,\tau_1). \tag{6}$$

If we can find a function g satisfying (6), we can then integrate (5) and determine the common constant of integration so as to satisfy (4), thus obtaining the relations (3). Consequently this condition, like the preceding one, is both necessary and sufficient for equivalence.

Consider two equivalent particle-observers P and P'. From (1) the distance of P' from P at time τ_2' at P' is

$$r_2 = \tfrac{1}{2}c(\tau_3 - \tau_1), \tag{7}$$

whereas the distance of P from P' at the corresponding time τ_2 is

$$r_2' = \tfrac{1}{2}c(\tau_3' - \tau_1'). \tag{8}$$

But $\tau_3' - \tau_2' = \tau_3 - \tau_2$ as the observers are equivalent. Hence $r_2' = r_2$, that is, *the two distances are the same at corresponding times*. As regards the velocity v_2 of P' relative to P, taken as positive if P' is receding from and negative if P' is approaching P, we find from (1) and (2),

$$v_2 \equiv \frac{dr_2}{dt_2} = c\frac{d\tau_3/d\tau_1 - 1}{d\tau_3/d\tau_1 + 1}, \tag{9}$$

whereas the velocity v_2' of P relative to P' is

$$v_2' \equiv \frac{dr_2'}{dt_2'} = c\frac{d\tau_3'/d\tau_1' - 1}{d\tau_3'/d\tau_1' + 1}. \tag{10}$$

As the two particle-observers are equivalent, $d\tau_3' = d\tau_3$ if $d\tau_1' = d\tau_1$. Therefore $v_2' = v_2$, that is, *the two velocities are the same at corresponding times*. Evidently the conclusions reached here hold also for accelerations or for higher derivatives with respect to the time.

As $d\tau_3/d\tau_1$ is necessarily positive, we see from (9) that v_2 can never have an absolute magnitude greater than c. For, as $d\tau_3/d\tau_1$ increases from 0 to ∞, v_2 increases monotonically from $-c$ to c. If we solve Eq. (9) for $d\tau_3/d\tau_1$ we get

$$d\tau_3/d\tau_1 = (1+\beta_2)/(1-\beta_2), \qquad \beta_2 \equiv v_2/c, \tag{11}$$

a relation we shall find useful later.

If P and P' are synchronous as well as equivalent, corresponding times are identical, and r', v', etc., are the *same* functions of the extended time t' of P' at P as r, v, etc., are of the extended time t of P at P'.

To pass from a pair of equivalent particle-observers to a group of such observers, it is necessary first to consider three. So let us introduce in addition to the two equivalent particle-observers P and P' a third particle-observer P''. The readings of the clocks of P and P' are fixed to within an arbitrary additive constant by the condition that they be equivalent. If P'' has a specified motion in terms of P's already assigned time scale such that (4) or (6) is satisfied, P'' can be furnished a clock equivalent to that of P. Indeed, if P'''s motion relative to P is of the same type as P''s, then the fact that P' is equivalent to P is sufficient to insure that P'' is also equivalent to P. The equivalence of both P' and P'' with P, however, does not necessarily imply equivalence with each other, which calls for separate investigation in each individual case.

In a space of more than one dimension, motion cannot be completely defined by reference to a single observer, for, in addition to motion along the line of sight, an angular motion about the observer may exist. We have recourse, then, to a *reference system*, which is defined as a dense assemblage of particle-observers filling all space, such that each particle-observer is synchronous with and at rest relative to every other particle-observer. Let P and P' be two equivalent particle-observers not relatively at rest with each of whom a reference system may be associated. If these two reference systems have the same geometry with respect to P and P', respectively, they are said to be *equivalent*. If, in addition, we may take as P and P' *any* pair of particle-observers in the two reference systems, the reference systems are *homogeneous*. In this case each particle-observer in the one reference system is equivalent to every particle-observer in the other. The Euclidean inertial systems of the special relativity theory are equivalent and homogeneous, but the Euclidean reference systems with constant relative accelerations, the discovery of which is reported in Part 4 of this paper, are equivalent but not homogeneous.

Insofar as electromagnetic theory is concerned we are interested in an effectively empty world. The philosophy underlying the relativity principle for such a world is that no preferred reference system exists in nature. Hence it is impossible to avoid the conclusion that *the laws of physics must be identical relative to all equivalent reference systems, that is, reference systems with the same geometry and the same constant light velocity associated with equivalent particle-observers*. This is our statement[2] of the principle of relativity for an empty world.

In particular all equivalent reference systems with Euclidean geometry and the same constant light velocity must be physically indistinguishable. The inertial systems of the special relativity constitute such a category, which possesses the incidental property that the physical interval $dx^2+dy^2+dz^2-c^2dt^2$ is an invariant. If this were the only group of equivalent reference systems with Euclidean geometries and constant light velocities, there would be no need for a restatement of the principle of relativity. The significance of the present contribution lies in the discovery of a new category of reference systems with Euclidean geometries and constant light velocities which have constant relative accelerations (in the relativity sense) and for which the physical interval is *not* an invariant. In all probability there are many other such categories as yet unsuspected.

Part 2. One-Dimensional Reference Systems

As a space of one dimension has no geometry, it is much simpler to treat than a space of three dimensions. Consequently we shall confine our attention in this Part to equivalent particle-observers and equivalent reference systems in relative motion in a space of one dimension. First we shall consider a one-dimensional reference system.

Reference system

Let P and P' be two synchronous particle-observers such that P' is at rest relative to P. Then the distance r_2 of P' from P at the time t_2 is not a function of t_2. Hence $r_2=r$ (a constant). Putting $\frac{1}{2}c(\tau_3-\tau_1)$ for r_2, $\tau_3-\tau_1=2r/c$, and Eqs. (3) are

$$\tau_3=\tau_2+r/c, \qquad \tau_2=\tau_1+r/c. \tag{12}$$

Since the distance r_2' of P from P' as computed by P' must be the same function of t_2' as r_2 is of t_2, $r_2'=r_2=r$ and P is at rest relative to P'. Moreover, as the clocks of the two particle-observers are synchronous, $\tau_c'=\tau_c$, the identical subscripts indicating corresponding times. As

$$t_2\equiv\tfrac{1}{2}(\tau_3+\tau_1)=\tau_2=\tau_2' \tag{13}$$

from (12), the *extended time of the one particle-observer coincides with the local time of the other.*

If we introduce a third particle-observer P'' at rest relative to P it is easily proved that P'' is at rest relative to P' and may be synchronized simultaneously with P and P'. Furthermore the addition law of distances is readily obtained. This law states that the *distance of P'' from P as calculated by P is equal to the distance of P' from P as calculated by P plus the distance of P'' from P' as calculated by P'.*

As the local time of an event at P' or P'' is identical with P's extended time of the event, there is no need of distinguishing between the local time of an event at one of the particle-observers and the extended time of the occurrence of that event in the experience of one of the other particle-observers. We may time distant events at P' or P'' by means of the extended time t of P, secure in the knowledge that the local time of the event is the same as P's extended time of the event. Furthermore, as the distance between P' and P'' as computed by either of them is the same as the excess of the distance of P'' from P over that of P' from P as calculated by P we may introduce a coordinate system with P as origin and employ only distances as computed by P. The aggregate of these distances we shall call the *extended space* of P.

It follows from the above that we can adjoin to *any* particle-observer P a dense linear assemblage of particle-observers P', P'', P''' $\cdots$ at rest relative to P and synchronous with him. Each one of these particle-observers is at rest relative to every other, and synchronous with every other. The aggregate of particle-observers, therefore, forms a reference system. As all time and space measurements made in the extended time and space of P are identical with those made in the local time of the particle-observer concerned, we may refer without ambiguity to the extended time and space of P as the time and space of the reference system.

[2] The principle of relativity was stated in almost identical terms in the author's *Introduction to Electrodynamics*, published in 1922, but the full significance of the statement was not realized at that time.

Constant relative velocity

Next consider two equivalent particle-observers P and P' the second of which has a constant velocity relative to the first. Then $r_2=v(t_2-t_0)$ where t_0 is a constant. In accord with the conventions adopted in Part 1, r_2 is essentially positive, and the velocity v is equal to a positive constant for $t_2>t_0$ (particles separating) and to the same constant with the opposite sign for $t_2<t_0$ (particles approaching). Putting $\frac{1}{2}c(\tau_3-\tau_1)$ for r_2 and $\frac{1}{2}(\tau_3+\tau_1)$ for t_2, this equation becomes

$$(\tau_3-t_0)/(\tau_1-t_0)=(1+\beta)/(1-\beta), \quad \beta\equiv v/c, \quad (14)$$

from which it follows that Eqs. (3) take the form

$$\frac{\tau_3-t_0}{\tau_2-t_0}=\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \quad \frac{\tau_2-t_0}{\tau_1-t_0}=\left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \quad (15)$$

which may be written as

$$\frac{\tau_2-t_0}{(1-\beta^2)^{\frac{1}{2}}}=\tau_3-t_0-\frac{r_2}{c}=\tau_1-t_0+\frac{r_2}{c}. \quad (16)$$

Hence P's extended time of the event τ_2' is given by

$$t_2-t_0=\tfrac{1}{2}(\tau_3+\tau_1)-t_0=\frac{\tau_2-t_0}{(1-\beta^2)^{\frac{1}{2}}}=\frac{\tau_2'-t_0'}{(1-\beta^2)^{\frac{1}{2}}}, \quad (17)$$

where $t_0'=t_0$ if P and P' are synchronous.

In addition to the two equivalent particle-observers P and P' moving with constant relative velocity we shall now introduce a third particle-observer P'' moving with constant velocity $v_{P''}$ relative to P and therefore equivalent to P. We shall show that the velocity $v_{P''}'$ of P'' relative to P' is constant and therefore that P'' is equivalent to P' as well as to P. Also we shall obtain the addition law of velocity. In order to make our notation consistent throughout, we shall designate here the velocity of P' relative to P by $v_{P'}$, and that of P relative to P' by v_P'. As shown earlier, $v_P'=v_{P'}$.

Consider the interlocking signals $\tau_1\rightarrow\tau_3''$ and $\tau_3''\rightarrow\tau_5$ of Fig. 2. By (11) we have

$$d\tau_5/d\tau_1=(1+\beta_{P''})/(1-\beta_{P''}), \quad (18)$$

and

$$d\tau_4'/d\tau_2'=(1+\beta_{P''}')/(1-\beta_{P''}'), \quad (19)$$

where $\beta_{P''}\equiv v_{P''}/c$ is constant by hypothesis.

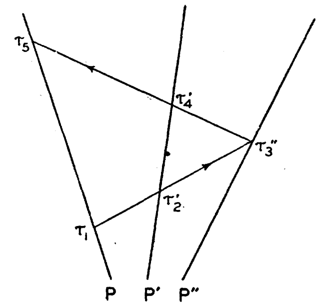

FIG. 2.

Now as P and P' are equivalent,

$$\frac{d\tau_2'}{d\tau_1}=\frac{d\tau_2}{d\tau_1}=\left(\frac{1+\beta_{P'}}{1-\beta_{P'}}\right)^{\frac{1}{2}}, \quad \frac{d\tau_5}{d\tau_4'}=\frac{d\tau_5}{d\tau_4}=\left(\frac{1+\beta_{P'}}{1-\beta_{P'}}\right)^{\frac{1}{2}},$$

from (15). But

$$\frac{d\tau_4'}{d\tau_2'}=\frac{d\tau_4'}{d\tau_5}\frac{d\tau_5}{d\tau_1}\frac{d\tau_1}{d\tau_2'}=\frac{1+\beta_{P''}}{1-\beta_{P''}}\,\frac{1-\beta_{P'}}{1+\beta_{P'}}.$$

Comparing with (19) we have

$$\frac{1-\beta_{P''}}{1+\beta_{P''}}=\frac{1-\beta_{P'}}{1+\beta_{P'}}\,\frac{1-\beta_{P''}'}{1+\beta_{P''}'}, \quad (20)$$

which shows that $\beta_{P''}'$ or $v_{P''}'$ is constant. Hence P'' is equivalent to P' as well as to P. It is easily shown that the same clock which makes him equivalent to P makes him equivalent to P'.

Eq. (20) is the *addition law of velocity* obtained by Einstein in 1905. It may be put in the more usual form

$$v_{P''}=(v_{P'}+v_{P''}')/(1+v_{P'}v_{P''}'/c^2). \quad (21)$$

Let us adjoin reference systems S and S', respectively, to two synchronous particle-observers P and P' moving with constant relative velocity v. It can be shown very simply that each particle-observer in S is equivalent to every particle-observer in S', and hence that the two reference systems are homogeneous as well as equivalent. Taking P and P' as origins of axes fixed in their respective reference systems and making $\tau=\tau'=0$ when P' passes P, we can obtain the relations between P's and P''s specifications of the position and time of the event Q (Fig. 3) by means of the light-signals indicated in the figure. From (15)

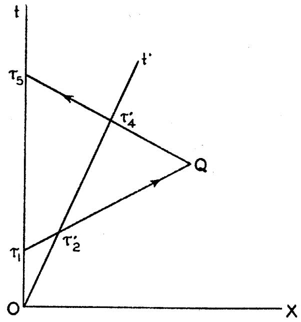

FIG. 3.

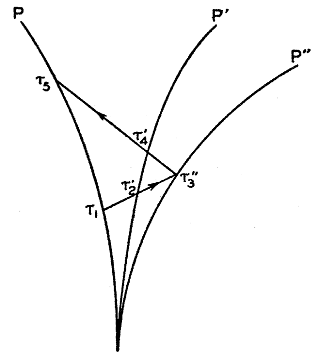

FIG. 4.

$$(1-\beta)^{\frac{1}{2}}\tau_5=(1+\beta)^{\frac{1}{2}}\tau_4', \quad (1-\beta)^{\frac{1}{2}}\tau_2'=(1+\beta)^{\frac{1}{2}}\tau_1,$$

as $t_0=0$ and $\tau_c'=\tau_c$, and

$$\begin{aligned} t' &\equiv \tfrac{1}{2}(\tau_4'+\tau_2') \\ &= \frac{1}{(1-\beta^2)^{\frac{1}{2}}}\{\tfrac{1}{2}(\tau_5+\tau_1)-\tfrac{1}{2}\beta(\tau_5-\tau_1)\} = \frac{t-(\beta/c)x}{(1-\beta^2)^{\frac{1}{2}}}, \\ x' &\equiv \tfrac{1}{2}c(\tau_4'-\tau_2') \\ &= \frac{1}{(1-\beta^2)^{\frac{1}{2}}}\{\tfrac{1}{2}c(\tau_5-\tau_1)-\tfrac{1}{2}v(\tau_5+\tau_1)\} = \frac{x-vt}{(1-\beta^2)^{\frac{1}{2}}}. \end{aligned} \tag{22}$$

These constitute the Lorentz transformation of the special relativity theory for one dimension.

Constant relative acceleration

We shall now investigate the properties of the linear reference systems adjoined to two synchronous particle-observers P and P' which have a constant relative acceleration ϕ (in the relativity sense). The differential equation of motion of P' relative to P is

$$d^2r_2/dt_2^2=(1-v_2^2/c^2)^{\frac{3}{2}}\phi,$$

the integral of which is

$$1+\phi r_2/c^2=(1+\phi^2t_2^2/c^2)^{\frac{1}{2}}, \tag{23}$$

if the particle-observers meet at rest at time zero. Expressing r_2 and t_2 in terms of τ_1 and τ_3 as usual we find

$$1/\tau_1-1/\tau_3=\phi/c, \tag{24}$$

and Eqs. (3) become

$$1/\tau_2-1/\tau_3=\phi/2c, \quad 1/\tau_1-1/\tau_2=\phi/2c. \tag{25}$$

First we shall obtain the addition law of acceleration by introducing a third particle-observer P'' who has a constant acceleration $\phi_{P''}$ relative to P, and who meets P without passing at the same time that P' does. Then P'' as well as P' is equivalent to P. We shall make P'' as well as P' synchronous with P, with $\tau=\tau'=\tau''=0$ at the instant of meeting. For uniformity of notation we shall here denote the constant acceleration of P' relative to P by $\phi_{P'}$. Considering the interlocking signals $\tau_1\rightarrow\tau_3''$ and $\tau_3''\rightarrow\tau_5$ of Fig. 4 we have from (24)

$$1/\tau_1-1/\tau_5=\phi_{P''}/c,$$

and from (25)

$$1/\tau_1-1/\tau_2'=\phi_{P'}/2c, \quad 1/\tau_4'-1/\tau_5=\phi_{P'}/2c,$$

as $\tau_c'=\tau_c$ since P' and P are synchronous. By combining,

$$1/\tau_2'-1/\tau_4'=(\phi_{P''}-\phi_{P'})/c,$$

which shows that P'' has the constant acceleration $\phi_{P''}'=\phi_{P''}-\phi_{P'}$ relative to P' and therefore is equivalent to P'. It can be shown very simply that the same clock which makes P'' synchronous with P makes him synchronous with P'. The *addition law of acceleration*, is, then,

$$\phi_{P''}=\phi_{P'}+\phi_{P''}'. \tag{26}$$

We can adjoin to each of the synchronous particle-observers P and P' moving with constant relative acceleration ϕ a dense linear assemblage of synchronous particle-observers relatively at rest. The two reference systems S and S' so formed are equivalent but not homogeneous. To find the space-time transformations we shall take X and X' axes in the direction of

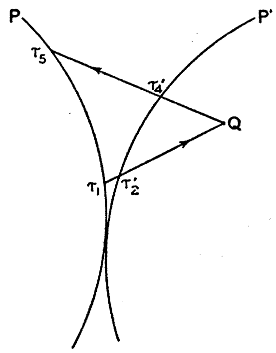

FIG. 5.

the acceleration of P' relative to P with P and P' as origins of S and S', respectively, and consider the signals (Fig. 5) necessary to specify the event Q. From (25)

$$\tau_4'=\tau_4=\frac{\tau_5}{1+(\phi/2c)\tau_5},\quad \tau_2'=\tau_2=\frac{\tau_1}{1-(\phi/2c)\tau_1},$$

as $\tau_c'=\tau_c$. So P''s extended time of the event Q is

$$t'\equiv\tfrac{1}{2}(\tau_4'+\tau_2')$$
$$=\frac{\tfrac{1}{2}(\tau_5+\tau_1)}{\left\{1+\dfrac{\phi}{2c}\left(\dfrac{\tau_5-\tau_1}{2}\right)\right\}^2-\dfrac{\phi^2}{4c^2}\left(\dfrac{\tau_5+\tau_1}{2}\right)^2}$$
$$=\frac{t}{\left(1+\dfrac{\phi x}{2c^2}\right)^2-\dfrac{\phi^2t^2}{4c^2}},\tag{27}$$

and P''s estimation of the distance x' of Q is

$$x'\equiv\tfrac{1}{2}c(\tau_4'-\tau_2')$$
$$=\frac{\tfrac{1}{2}c(\tau_5-\tau_1)+\dfrac{\phi}{2}\left(\dfrac{\tau_5-\tau_1}{2}\right)^2-\dfrac{\phi}{2}\left(\dfrac{\tau_5+\tau_1}{2}\right)^2}{\left\{1+\dfrac{\phi}{2c}\left(\dfrac{\tau_5-\tau_1}{2}\right)\right\}^2-\dfrac{\phi^2}{4c^2}\left(\dfrac{\tau_5+\tau_1}{2}\right)^2}$$
$$=\frac{x\left(1+\dfrac{\phi x}{2c^2}\right)-\dfrac{\phi t^2}{2}}{\left(1+\dfrac{\phi x}{2c^2}\right)^2-\dfrac{\phi^2t^2}{4c^2}}.\tag{28}$$

Note that S' extends only from $-2c^2/\phi+(c^2t^2)^{\frac{1}{2}}$ to ∞.

If we put

$$\xi\equiv 1+\phi x/2c^2,\qquad T\equiv\phi t/2c,$$
$$\xi'\equiv 1-\phi x'/2c^2,\qquad T'\equiv\phi t'/2c,$$

which amounts to taking a new origin in S at $-2c^2/\phi$, and a new origin in S' at $2c^2/\phi$ combined with a change in sense of the axis, then the space-time transformation assumes the simpler form

$$\begin{aligned}T'&=T/(\xi^2-T^2),\quad & T&=T'/(\xi'^2-T'^2),\\ \xi'&=\xi/(\xi^2-T^2),\quad & \xi&=\xi'/(\xi'^2-T'^2).\end{aligned}\tag{29}$$

This transformation gives

$$(\xi^2-T^2)(\xi'^2-T'^2)=1,\tag{30}$$

and yields the invariants

$$T'/\xi'=T/\xi,\tag{31}$$

and
$$\frac{dx'^2-c^2dt'^2}{\xi'^2-T'^2}=\frac{dx^2-c^2dt^2}{\xi^2-T^2}.\tag{32}$$

It is seen from (32) even in this one-dimensional case that the physical interval $dx'^2-c^2dt'^2$ in S' is *not* in general equal to the physical interval $dx^2-c^2dt^2$ in S. Hence the rather firm foundations on which the present theory rests are quite incompatible with the fundamental postulate on which Einstein's theory is based.

The relation between the velocity $V'=dx'/dt'$ of a moving point relative to S' and its velocity $V=dx/dt$ relative to S is given by

$$\left(1-\frac{T'}{\xi'}\right)\left(\frac{1-V'/c}{1+V'/c}\right)^{\frac{1}{2}}=\left(1+\frac{T}{\xi}\right)\left(\frac{1-V/c}{1+V/c}\right)^{\frac{1}{2}}.\tag{33}$$

In particular, the velocity v relative to S of a point fixed in S' is given by

$$\beta\equiv v/c=2\xi T/(\xi^2+T^2).\tag{34}$$

When $t=0$, then, all points in S' are simultaneously at rest in S. The acceleration relative to S of a particle fixed in S' is

$$f\equiv dv/dt=(1-\beta^2)^{\frac{3}{2}}\xi'\phi.\tag{35}$$

A particle-observer in S', therefore, has the constant relativity acceleration $\xi'\phi$ relative to S. In terms of the coordinate measure of S, this is an acceleration

$$\phi_\xi = \xi\phi/(\xi^2 - T^2) \tag{36}$$

for a particle-observer in S' at ξ at time t. At the instant $t=0$ when all particle observers in S' are at rest relative to S, this reduces to $\phi_\xi = \phi/\xi$.

It is of interest to note that even when the two reference systems S and S' are relatively at rest time and space measurements do not agree except at the common point occupied by P and P'. For

$$dt'/dt = dx'/dx = 1/(1+\phi x/2c^2) = 1 - \phi x'/2c^2.$$

Hence it is obvious that S and S', while equivalent, are not homogeneous, and that the physical interval cannot be invariant.

Part 3. Equivalent Three-Dimensional Reference Systems with Constant Relative Velocities

Although we have seen that a linear assemblage of synchronous particle-observers relatively at rest can be associated with any particle-observer in a one-dimensional space, it does not follow that a dense three-dimensional assemblage of particle-observers can be associated with an arbitrary particle-observer P in such a way that each particle-observer in the group is at rest relative to and synchronous with every other. An analysis of the problem shows that at most we can associate three particle-observers satisfying these conditions with an arbitrary observer P, such that no three of the four observers lie on the same light ray. We cannot, then, associate a three-dimensional reference system with an arbitrary particle-observer. The existence of a three-dimensional reference system in nature is a matter which must be investigated empirically. Furthermore, if such a reference system is found, its space geometry is also a matter for experimental investigation. For we have adopted a definite convention for the measurement of distance, and we can use it, once we have found a reference system, to determine experimentally the ratio of the circumference to the diameter of a circle, etc. The geometry of a reference system is, therefore, not conventional, but a matter to be investigated empirically by means of the procedure adopted for measuring distance.

We take it therefore, as a matter of experimental knowledge, that there exists, at least in the limited region occupied by the solar system, *one* reference system with effectively Euclidean geometry and constant light velocity c. We need not be disturbed by the fact that the deflection of a ray of light passing near the limb of the sun, or the solar red shift, may indicate a slight departure from the ideal conditions assumed, for these effects are so small that the departures represented by them are negligible insofar as the electromagnetic theory in which we are interested is concerned.

Once the existence of a single reference system S with Euclidean geometry and constant light velocity is established, it becomes possible to prove or disprove the existence of other equivalent reference systems with Euclidean geometry and the same constant light velocity. In this part we shall outline the methods necessary to show that any dense three-dimensional assemblage of particle-observers all of whom are moving with the same constant velocity $\mathbf{v}$ relative to S constitute a reference system S' equivalent to S, and therefore having the same Euclidean geometry and the same constant light velocity as S. As the transformation between S' and S is merely the Lorentz transformation of the special relativity theory, the discussion, which is presented solely for the purpose of developing the methods to be employed in Part 4, will be compressed as much as possible.

Let P be particle-observer in S chosen as origin of a set of axes X, Y, Z, and let P' be an equivalent particle-observer moving with constant velocity $\mathbf{v}$ relative to S along a line parallel to the X axis and distant h from it. Then the equation of motion of P' relative to P is

$$r_2{}^2 = h^2 + v^2(t_2 - t_0)^2,$$

where t_0 is a constant, and, if we put

$$a^2 \equiv h^2(1-\beta^2)/v^2,$$

$$\frac{(a^2+(\tau_3-t_0)^2)^{\frac{1}{2}}+\tau_3-t_0}{(a^2+(\tau_1-t_0)^2)^{\frac{1}{2}}+\tau_1-t_0} = \frac{1+\beta}{1-\beta}. \tag{37}$$

Hence (3) becomes

$$\begin{aligned} \frac{(a^2+(\tau_3-t_0)^2)^{\frac{1}{2}}+\tau_3-t_0}{(a^2+(\tau_2-t_0)^2)^{\frac{1}{2}}+\tau_2-t_0} &= \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \\ \frac{(a^2+(\tau_2-t_0)^2)^{\frac{1}{2}}+\tau_2-t_0}{(a^2+(\tau_1-t_0)^2)^{\frac{1}{2}}+\tau_1-t_0} &= \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}, \end{aligned} \tag{38}$$

which can be written in the more convenient form

$$\frac{\tau_2 - t_0}{(1-\beta^2)^{\frac{1}{2}}} = \tau_3 - t_0 - \frac{r_2}{c} = \tau_1 - t_0 + \frac{r_2}{c}. \quad (39)$$

Eq. (39) is identical with (16) for the one-dimensional case. As P may be *any* particle-observer in S, it follows that each particle-observer in S' is equivalent to every particle-observer in S.

Now suppose that P' is moving along the X axis of S. As proved in the discussion of the one-dimensional case we can associate with P' a dense linear assemblage of particle-observers distributed along this axis all of which are at rest relative to P' and synchronous with him. All of these particle-observers have the same constant velocity $\mathbf{v}$ relative to S, and, in view of what we have just proved, are equivalent to every particle-observer in S. We will take their locus for the X' axis of S'. The Lorentz transformation (22) applies, then, to the linear assemblages of particle-observers constituting the X and X' axes of S and S', respectively.

Next consider an event P_t'' occurring at the point x, y, 0 at the time t. If the coordinates of P' relative to S at time 0 were x_1, 0, 0, they are x_1+vt, 0, 0 at time t. To determine the distance of the event P_t'' and the time of its occurrence in the experience of P', we must send a light-signal from P' so as to reach P_t'' at time t, and then send it back to P'. Let the light signal leave P' at time t_1 when P' is at a distance r_1 from P_t'', and arrive back at P' at time t_3 when P' is at a distance r_3 from P_t''. Then

$$r_1^2 = \{x - x_1 - v(t - r_1/c)\}^2 + y^2,$$
$$r_3^2 = \{x - x_1 - v(t + r_3/c)\}^2 + y^2.$$

We solve these equations for r_1 and r_3 and then determine t_1 and t_3. By means of the Lorentz transformations for the linear reference systems constituting the X and X' axes of S and S', respectively, we find the local times τ_1' and τ_3' at P' of the departure and return of the signal. Then from (1) and (2) we have

$$t' \equiv \tfrac{1}{2}(\tau_3' + \tau_1') = \frac{t - (\beta/c)x}{(1-\beta^2)^{\frac{1}{2}}}, \quad (40)$$

$$r' \equiv \tfrac{1}{2}c(\tau_3' - \tau_1') = \left[\frac{(x - x_1 - vt)^2}{1-\beta^2} + y^2\right]^{\frac{1}{2}}. \quad (41)$$

Now suppose that P'' represents a particle-observer moving relative to S with the same velocity $\mathbf{v}$ as P'. Then $x = x_2 + vt$, $y = y_2$, where x_2 and y_2 are constants. In this case (41) becomes

$$r' = \left[\frac{(x_2 - x_1)^2}{1-\beta^2} + y_2^2\right]^{\frac{1}{2}}, \quad (42)$$

showing that r' does not change with the time. Hence all particle-observers moving relative to S with the same constant velocity $\mathbf{v}$ as P' are at rest relative to P'. To be synchronous with P' their local times must be equal to P''s extended time t'. We observe from (40) that they are then synchronous each with each. Finally (42) represents the Pythagorean theorem for S'. From it we see that we can construct a Euclidean mesh in S', and therefore that the geometry of S' is Euclidean. Consequently S and S' are equivalent homogeneous reference systems with Euclidean geometries. The Lorentz space-time transformation for the three-dimensional case under discussion is obtained immediately from (40) and (41).

Part 4. Equivalent Three-Dimensional Reference Systems with Constant Relative Accelerations

We have shown that a dense assemblage of particle-observers all moving with the same constant velocity $\mathbf{v}$ relative to a given Euclidean reference system S may be synchronized each with each so as to constitute an equivalent reference system. This category of reference systems, however, does not comprise all reference systems equivalent to S which have Euclidean geometry and equal constant light velocity. We shall now show that a three-dimensional reference system S', equivalent to S, may be adjoined to a particle-observer P' moving with constant relativity acceleration ϕ relative to S, and that the geometry of S' is Euclidean.

Take as origin of S the synchronous particle-observer P with whom P' coincides when momentarily at rest in S and orient the X axis in the direction of the acceleration of P' relative to S. As proved in Part 2, we can adjoin to P' a linear reference system extending along the X axis of S from $-2c^2/\phi + (c^2t^2)^{\frac{1}{2}}$ to ∞, the particle-observers constituting this reference system having the constant accelerations relative to S specified by (36). The transformation (29) applies to the linear assemblages of particle-observers constituting the X and X' axes of S and S', respectively.

Let Q' be a particle-observer in the linear reference system adjoined to P' at a distance x' from P', which serves as the origin of the X' axis in S'. We shall denote by x_0 the coordinate of Q' in S when $t=t'=0$. Then the constant acceleration ϕ_1 of Q' relative to S is given by

$$\phi/\phi_1=1+\phi x_0/2c^2 \tag{43}$$

in accord with (36).

Next consider an event Q_t'' occurring at $x, y, 0$ at the time t. We dispatch a light signal from Q' at a time t_1 so chosen that the signal will reach Q_t'' at time t, and then send the signal back to Q' whom it reaches at time t_3. Let r_1 be the distance of Q_t'' from the position x_1 of Q' at time t_1, and r_3 the distance of Q_t'' from the position x_3 of Q' at time t_3. Then

$$\begin{aligned} 1+\frac{\phi_1}{c^2}(x_1-x_0)&=\left[1+\frac{\phi_1^2}{c^2}\left(t-\frac{r_1}{c}\right)^2\right]^{\frac{1}{2}},\\ 1+\frac{\phi_1}{c^2}(x_3-x_0)&=\left[1+\frac{\phi_1^2}{c^2}\left(t+\frac{r_3}{c}\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{44}$$

where $r_1^2=(x-x_1)^2+y^2$, $r_3^2=(x-x_3)^2+y^2$. (45)

From these equations we must find r_1 and r_3, and then $t_1=t-r_1/c$ and $t_3=t+r_3/c$ as functions of x, y, x_0, t. If we put $\rho\cos\theta\equiv 1+\phi x/2c^2$, $\rho\sin\theta\equiv\phi y/2c^2$, $\xi_0\equiv 1+\phi x_0/2c^2$, $T\equiv\phi t/2c$, after expressing ϕ_1 in terms of ϕ by (43), we find as the result of a laborious algebraic calculation

$$\frac{\phi t_1}{2c}=\frac{T\{\xi_0^2+2(\rho^2-T^2-\xi_0\rho\cos\theta)\}-(2\rho\cos\theta-\xi_0)[(\rho^2-T^2)^2-2(\rho^2-T^2)\xi_0\rho\cos\theta+\xi_0^2\rho^2]^{\frac{1}{2}}}{(2\rho\cos\theta-\xi_0)^2-4T^2},$$

$$\frac{\phi t_3}{2c}=\frac{T\{\xi_0^2+2(\rho^2-T^2-\xi_0\rho\cos\theta)\}+(2\rho\cos\theta-\xi_0)[(\rho^2-T^2)^2-2(\rho^2-T^2)\xi_0\rho\cos\theta+\xi_0^2\rho^2]^{\frac{1}{2}}}{(2\rho\cos\theta-\xi_0)^2-4T^2}.$$

To find the local times τ_1' and τ_3' at Q' corresponding to t_1 and t_3 we have from (29)

$$\phi\tau'/2c=-c/\phi t+(c^2/\phi^2t^2+1/\xi_0^2)^{\frac{1}{2}}.$$

Thus we find

$$\frac{\phi\tau_1'}{2c}=\frac{\xi_0T-[(\rho^2-T^2)^2-2(\rho^2-T^2)\xi_0\rho\cos\theta+\xi_0^2\rho^2]^{\frac{1}{2}}}{\xi_0(\rho^2-T^2)},$$

$$\frac{\phi\tau_3'}{2c}=\frac{\xi_0T+[(\rho^2-T^2)^2-2(\rho^2-T^2)\xi_0\rho\cos\theta+\xi_0^2\rho^2]^{\frac{1}{2}}}{\xi_0(\rho^2-T^2)}.$$

Finally, we get for Q''s time t' and distance r' of the event Q_t'',

$$\phi t'/2c\equiv(\phi/4c)(\tau_3'+\tau_1')=T/(\rho^2-T^2), \tag{46}$$

$$\frac{\phi r'}{2c^2}\equiv\frac{\phi}{4c}(\tau_3'-\tau_1')=\frac{[(\rho^2-T^2)^2-2(\rho^2-T^2)\xi_0\rho\cos\theta+\xi_0^2\rho^2]^{\frac{1}{2}}}{\xi_0(\rho^2-T^2)}. \tag{47}$$

Now suppose that Q'' is a particle-observer moving along a radial line in the XY plane of S drawn from the point O at $-2c^2/\phi$, 0, 0 at an angle θ with the direction of P''s or Q''s motion, as shown in Fig. 6. Let Q'' be at rest in S at the same instant that P' and Q' are, and let Q'' have a constant acceleration ϕ_2 which is the same function of his distance from O as that of P' or Q' is. Then, if we denote by r the distance of Q'' from an origin in S on the line OQ'' at the same distance from O as P,

$$\phi/\phi_2=1+\phi r_0/2c^2, \tag{48}$$

where r_0 is the value of r at $t=0$, and

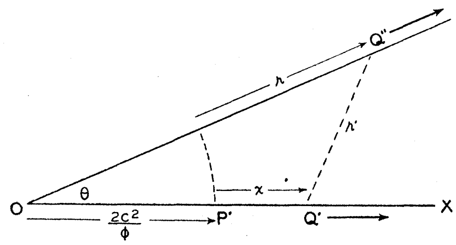

FIG. 6.

$$1+(\phi_2/c^2)(r-r_0)=(1+\phi_2^2t^2/c^2)^{\frac{1}{2}}. \quad (49)$$

Eliminating ϕ_2 from (49) by (48) and putting ρ_0 for the value of ρ at $t=0$, we find

$$\rho^2-T^2=\rho\rho_0. \quad (50)$$

Combining (50) with (47) *we find that we can eliminate both* ρ *and* T, getting,

$$\frac{\phi r'}{2c^2}=\frac{[\rho_0^2-2\xi_0\rho_0\cos\theta+\xi_0^2]^{\frac{1}{2}}}{\xi_0\rho_0}. \quad (51)$$

Therefore the distance r' *of* Q'' *from* Q' *as measured by* Q' *does not change as the motion progresses. The aggregate of uniformly accelerated linear reference systems radiating from the point* O *forms a three-dimensional reference system* S' *each particle-observer of which is permanently at rest relative to every other.* Moreover, comparison of (46) with (29) shows that Q'''s extended time at Q'' is identical with Q'''s local time. *Hence each particle-observer in* S' *is synchronous with every other.*

It remains to show that the geometry of S' is Euclidean. Since the particle-observers in S' form a rigid aggregate in their own distance measure, it is sufficient to investigate their geometry at the instant $t=t'=0$ when S' is at rest relative to S. As the projection of S' on S is symmetric about the X axis of S, we can simplify the analysis by first investigating the geometry of a section of S' lying in the XY plane of S.

First we shall find the projections on S of all straight lines in S'. Put $R_0\equiv 2c^2/\phi+r_0$, $X_0\equiv 2c^2/\phi+x_0$. Then R_0 and X_0 are the distances of Q'' and Q' from the singular point O at $t=0$, and the Pythagorean theorem (51) becomes

$$r'=\frac{4c^4}{\phi^2}\frac{[R_0^2-2R_0X_0\cos\theta+X_0^2]^{\frac{1}{2}}}{R_0X_0}. \quad (52)$$

If Q' and Q'' are neighboring points, this gives for the element of distance

$$dr'=\frac{4c^4}{\phi^2}\frac{[dR_0^2+R_0^2d\theta^2]^{\frac{1}{2}}}{R_0^2}. \quad (53)$$

A straight line is defined by $\delta\int dr'=0$ between fixed limits. Minimizing the integral by the usual methods we get

$$d^2R_0/d\theta^2+R_0=0, \quad (54)$$

the complete integral of which is

$$(R_0\cos\theta-A)^2+(R_0\sin\theta-B)^2=A^2+B^2, \quad (55)$$

where A and B are constants of integration. *Hence all circles through* O *in* S *are straight lines in* S' *at* $t'=0$. *In particular all straight lines in* S *radiating from the point* O *are straight lines in* S'. The point O in S is the point at infinity of S', and therefore there are an infinite number of straight lines joining O to any point in S'.

Next we shall find the projections on S of all spheres in S'. The equation in S of a sphere about Q' in S' with radius r' is the relation between R_0 and θ given by (52) when X_0 and r' are held constant. Rearranging this equation,

$$\left\{R_0\cos\theta-\frac{X_0}{1-(\phi^2r'/4c^4)^2X_0^2}\right\}^2+R_0^2\sin^2\theta=\frac{(\phi^2r'/4c^4)^2X_0^4}{\{1-(\phi^2r'/4c^4)^2X_0^2\}^2}, \quad (56)$$

which is a sphere about a point on the OX axis distant

$$\frac{(\phi^2r'/4c^4)^2X_0^3}{1-(\phi^2r'/4c^4)^2X_0^2} \quad (57)$$

from Q'. The projections on S of the radii of this sphere in S' are, of course, circles through O and Q'.

Now we shall show that angles are preserved in passing from S' to S. First we express the Eq. (56) of a sphere in S' about Q' as center in terms of polar coordinates p, χ with Q' as origin and OX as polar axis, getting,

$$p^2=(\phi^2r'/4c^4)^2X_0^2(p^2+2X_0p\cos\chi+X_0^2). \quad (58)$$

To obtain an element of the circumference in the XY plane of S we must differentiate (58) holding r' and X_0 constant, and substitute in

(53), which becomes

$$dr' = \frac{4c^4}{\phi^2} \frac{(dp^2 + p^2 d\chi^2)^{\frac{1}{2}}}{p^2 + 2X_0 p \cos\chi + X_0^2}, \qquad (59)$$

in our new coordinates. Dividing dr' by the radius r' of the sphere we find the angle $d\chi'$ subtended in S'. Thus:

$$d\chi' = \frac{d\chi}{1 + (p/X_0)\cos\chi}. \qquad (60)$$

As p approaches zero, whatever X_0 and χ may be, $d\chi'$ approaches $d\chi$. *Hence angles are invariant for the transformation from S' to S.*

It is easily shown that the circles through Q' and O in S, which are the projections on S of straight lines radiating from Q' in S', intersect orthogonally the projections (56) on S of all spheres in S' with Q' as center.

We are ready now to construct a Euclidean mesh in S'. First we show that all circles in S tangent to OX at O are the projections of straight lines in S' parallel to the X' axis. The equations of such circles are

$$R_0^2 \cos^2\theta + (R_0 \sin\theta - B)^2 = B^2, \qquad (61)$$

and the perpendiculars to the X' axis in S' are the circles

$$(R_0 \cos\theta - A)^2 + R_0^2 \sin^2\theta = A^2. \qquad (62)$$

These two families of circles intersect orthogonally, showing that in S' (61) is perpendicular to a straight line at right angles to the X' axis.

All that remains is to calculate the lengths of the sides of a rectangle in S' bounded by the straight lines (61) and (62). If Q_1' is a point in S' on the perpendicular to the X' axis through Q', then the coordinates R_0, θ of Q_1' must satisfy the relation $R_0 = X_0 \cos\theta$, where X_0, O are the coordinates of Q'. Hence the distance of Q_1' from Q', obtained from (52), is

$$r' = \frac{4c^4}{\phi^2} \frac{\sin\theta}{R_0} = \frac{4c^4}{\phi^2} \frac{\tan\theta}{X_0}. \qquad (63)$$

This is identical with (61), showing that all straight lines in S' represented by (61) are equidistant from the X' axis. Incidentally, for a given X_0, $r' = \infty$ when $\theta = \pi/2$.

Next consider two perpendiculars in S' to the X' axis passing through the points whose coordinates are X_1, O and X_2, O, respectively. The equations of their projections in S, obtained from (62), are $R_0 = X_1 \cos\theta$ and $R_0 = X_2 \cos\theta$. Let Q_1' and Q_2' be the intersections of these curves with (61). Then the coordinates R_1, θ_1 of Q_1' and R_2, θ_2 of Q_2' satisfy the relations

$$R_1 = X_1 \cos\theta_1 = 2B \sin\theta_1,$$
$$R_2 = X_2 \cos\theta_2 = 2B \sin\theta_2.$$

The distance of Q_2' from Q_1' in S' is, in accord with the Pythagorean theorem (52),

$$\begin{aligned} r' &= \frac{4c^4}{\phi^2} \frac{[R_2^2 - 2R_2R_1\cos(\theta_2 - \theta_1) + R_1^2]^{\frac{1}{2}}}{R_2R_1} \\ &= \frac{4c^4}{\phi^2} \frac{X_2 - X_1}{X_2X_1}, \end{aligned} \qquad (64)$$

which is independent of B.

We have now constructed a Euclidean mesh in any plane of S passing through the X' axis. Lastly we have to consider measurements involving a change in azimuth ψ about the X axis. Let Q_1' and Q_2' be two points with coordinates R_0, θ, ψ and R_0, θ, $\psi + d\psi$, respectively. The angle subtended at O by radii vectors to Q_1' and Q_2' is $\sin\theta d\psi$. Hence (53) gives for the distance between them in S',

$$dr' = \frac{4c^4}{\phi^2} \frac{\sin\theta d\psi}{R_0}. \qquad (65)$$

Their common perpendicular distance in S' from the X' axis is specified by (63). Dividing (65) by (63) we obtain for the difference in azimuth in S

$$d\psi' = dr'/r' = d\psi. \qquad (66)$$

So, if we measure ψ' and ψ from coincident planes through the X axis, we have $\psi' = \psi$.

We have completed the proof that S' is a Euclidean reference system with constant light velocity c, constituted of synchronous particle-observers relatively at rest. We shall now assemble the space-time transformation between S and S'. First we note from (63) that the distance in S', measured along the perpendicular to the X' axis, of points on the radial line OQ'' of Fig. 6 becomes less and less as R_0 increases. From (29) we can write (63) in the form

$$r' = X' \tan \theta \tag{67}$$

at $t = t' = 0$, if X' is the distance of Q' (Fig. 6) from an origin O' at a distance $2c^2/\phi$ to the right of the former origin P'. All radial lines diverging from O in S converge at O' in S'. Just as O is the point at infinity of S', so O' is the point at infinity of S.

Now, taking O' as origin of a set of spherical coordinates R', θ', ψ' in S', where the polar angle θ' is measured from the *negative* direction of the X axis, we have at once from (67) that $\theta' = \theta$. We have already shown that $\psi' = \psi$, although, if we wish to make both sets of coordinates right-handed we must measure ψ' in the opposite sense to ψ and write $\psi' = -\psi$. The relations between R', R, t' and t are given by (29). So, if we put

$$\rho \equiv \phi R/2c^2, \quad \rho' \equiv \phi R'/2c^2, \quad T \equiv \phi t/2c, \quad T' = \phi t'/2c,$$

we have for the complete space-time transformation

$$\begin{aligned} T' &= T/(\rho^2 - T^2), & T &= T'/(\rho'^2 - T'^2), \\ \rho' &= \rho/(\rho^2 - T^2), & \rho &= \rho'/(\rho'^2 - T'^2), \\ \theta' &= \theta, & \theta &= \theta', \\ \psi' &= -\psi, & \psi &= -\psi'. \end{aligned} \tag{68}$$

The differential invariant of this transformation is

$$\begin{aligned} &[1/(\rho^2 - T^2)]\{d\rho^2 + (\rho^2 - T^2)(d\theta^2 + \sin^2 \theta d\psi^2) - dT^2\} \\ &\quad = [1/(\rho'^2 - T'^2)]\{d\rho'^2 \\ &\quad + (\rho'^2 - T'^2)(d\theta'^2 + \sin^2 \theta' d\psi'^2) - dT'^2\}. \end{aligned} \tag{69}$$

The physical interval $dR^2 + R^2(d\theta^2 + \sin^2 \theta d\psi^2) - c^2dt^2$ between two nearby events as measured in S is *not* equal to the physical interval $dR'^2 + R'^2(d\theta'^2 + \sin^2 \theta' d\psi'^2) - c^2dt'^2$ between the same two events as measured in S'.

As the physical interval is not an invariant the present theory is incompatible with Einstein's. But the present theory is based solely on the assumptions that in an effectively empty region (1) there exists at least one Euclidean reference system with constant light velocity, and (2) all equivalent Euclidean reference systems with constant light velocity are physically indistinguishable as regards the formulation of the laws of nature. The first assumption is generally admitted to represent the result of measurement; it is difficult to see how the second can be denied without denying the philosophy underlying the whole idea of the relativity of motion. For equivalent particle-observers and equivalent reference systems have been defined in such a manner that two such particle-observers or two such reference-systems stand in precisely the same relation to the underlying constant light velocity. In fact the space-time of a reference system has been constructed, not from yardsticks of undefinable rigidity and clocks of undefinable periodicity, but from the concept of a universal constant light velocity. Hence the conclusion seems inevitable that the fundamental assumption of an invariable physical interval, which underlies Einstein's relativity, is untenable. *Either* the postulate of an absolute four-dimensional space-time, *or* the postulate of the relativity of motion in an effectively empty world, must be abandoned.

Let us suppose that no external electromagnetic field is present in S at the time $t = 0$ at which the relatively accelerated reference system S' is at rest relative to S. Then, clearly, no external electromagnetic field is present in S'. Consider an electron at rest in both S' and S at this instant. Presumably an electron will remain at rest in a Euclidean reference system with constant light velocity in the absence of an external field. Therefore it appears as if the electron under consideration has a choice as to whether it shall remain at rest in S, S' or another of the infinitely many equivalent reference systems with constant acceleration relative to S. Actually, however, no indeterminacy exists. For the *angular distribution* relative to S of the charge of an electron of finite dimensions is different according as the electron is permanently at rest in S or S'. What determines this angular distribution the present theory does not indicate. But a given angular distribution specifies the reference system in which the electron remains at rest.

Consider an electron permanently at rest in S'. Then, relative to S, the electron, starting from rest in a field-free space, moves away with constant acceleration and ever increasing velocity. Although no work is performed by external forces, the kinetic energy of the electron continually increases. No violation of the conservation principle relative to S is involved, however. For, as the velocity of the electron grows, its linear dimensions relative to S increase, and its mass relative to S decreases. We have, then, a conversion of mass into energy. As the velocity of the electron approaches that of light, this process of conversion approaches completion. We have here a possible method (although probably not precisely that occurring in nature) of conversion of matter into energy. The converse transformation takes place during a retardation.

It is hoped to deal with these matters quantitatively in a succeeding communication. In addition it would seem desirable to investigate equivalent reference systems having other types of motion, particularly relative rotation, in the hope of finding a rational detailed description of atomic structure.

Further consideration leads to the suspicion that it was not necessary to give a detailed proof of the fact that the geometry of the reference system S' considered in this Part is Euclidean. For the particle-observer P' to whom the reference system S' is adjoined is in exactly the same situation with respect to light-signals as is the particle-observer P to whom the reference system S is adjoined. Therefore, as our geometry is one based on light-signals, the geometry of a reference system adjoined to P' must be identical with that of a reference system adjoined to P. If one is Euclidean, the other must be also. Nevertheless, most of the analysis presented would be required to find the space-time transformation between S and S'.

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Two-Centers Solution of the Gravitational Field Equations, and the Need for a Reformed Theory of Matter

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As has been shown by Levi-Civita[1] and as the reader may ascertain more directly in a perfectly straightforward way, the field equations outside of matter, $R_{ik}=0$, are satisfied by the axially symmetrical line-element

$$ds^2=e^{2\nu}dx_4^2-e^{-2\nu}[e^{2\lambda}(dx_1^2+dx_2^2)+x_1^2dx_3^2], \quad (1)$$

where ν and λ, functions of x_1, x_2 only, satisfy, respectively, the Laplace equation

$$\nabla^2\nu=\frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1\frac{\partial \nu}{\partial x_1}\right)+\frac{\partial^2\nu}{\partial x_2^2}=0 \quad (2)$$

and the condition (equivalent to two partial differential equations)

$$d\lambda=x_1\left[\left(\frac{\partial\nu}{\partial x_1}\right)^2-\left(\frac{\partial\nu}{\partial x_2}\right)^2\right]dx_1+2x_1\frac{\partial\nu}{\partial x_1}\frac{\partial\nu}{\partial x_2}dx_2, \quad (3)$$

where $d\lambda=(\partial\lambda/\partial x_1)dx_1+(\partial\lambda/\partial x_2)dx_2$ is a total differential, namely, in virtue of (2).

Since (2) is linear and homogeneous, the superposition of any integrals is again an integral of that equation.

The object of this paper is to derive a solution of (2), (3) corresponding to two mass centers A, B, a field, that is, which has singularities at A and B only, and not (as in R. Bach's and H. Weyl's physically trivial solution[2]) along the straight segment joining these two points.

I may mention that I have constructed such a solution (a stationary one) in December, 1933 and have then communicated it to Einstein, pointing out, rather emphatically, that this is a case of a perfectly rigorous solution of his field equations and yet utterly inadmissible physi-

[1] Levi-Civita, Rend. Ac. Linc., Note VIII, Rome (1919).

[2] R. Bach and H. Weyl, Math. Zeits. **XII**, 134, Berlin 1922; see especially page 141 *et seq.*