TABLE III. *Ranges and energies of particles.*

Range Singly Charged Particle	Range Heavy Particle	Energies of Particles of These Ranges E_{H^1}	E_{H^2}	$E_{C^{13}}$	$E_{C^{14}}$	$E_{H^2}+E_{C^{13}}$	$E_{H^1}+E_{C^{14}}$
Neutron Source: $Be+{}_1H^2$							
2.42 cm	0.12 cm	1.04	1.4	0.2	0.3	1.6	1.3
>1.8	0.32	>0.87	>1.1	1.3	1.4	>2.4	>2.3
>3.4	0.20	>1.28	>1.6	0.6	0.6	>2.2	>1.9
>2.1	0.22	>0.95	>1.2	0.7	0.7	>1.9	>1.7
1.30	0.11	0.68	0.8	0.2	0.2	1.0	0.9
2.26	0.09	1.00	1.3	0.2	0.2	1.5	1.2
>1.4	0.18	>0.73	>0.8	0.5	0.5	>1.3	>1.2
0.25	0.35	0.26	0.2	1.6	1.7	1.8	2.0

From these data we cannot determine the maximum energy which appears in such forks, but we can say that at least 2 MEV appears when neutrons with energies up to 4.42 MEV are used. According to calculations from Bethe's masses, reaction (2) is endothermic by 4.7 MEV. Thus we must turn to reaction (3) to explain the singly charged particles.

$$_7N^{14}+{}_0n^1\rightarrow{}_6C^{14}+{}_1H^1. \quad (3)$$

The C^{14} would probably be radioactive, going into N^{14} with the emission of an electron. However, such a radioactive C^{14} has not been observed. The upper limit of the mass of C^{14}, assuming that 2.0 MEV of energy appears in reaction (3) when 4.42 MEV neutrons are used, is 2.8 MEV more than that of N^{14}. Thus the maximum energy of the beta-particles from C^{14} is less than 2.8 MEV.

We have found one trident which we have attributed to the disintegration of nitrogen according to reaction (4). Fig. 2B shows a stereoscopic pair of photographs of the single disintegration of this type which we observed when we bombarded nitrogen with the high energy neutrons from $Li+H^2$. The energy appearing in the three forks is 4.8 MEV and the calculated energy of the neutron which produced the disintegration is 12.9 MEV. From a calculation of the masses involved, this reaction is endothermic by 7.0 MEV, which is consistent with the observed data.

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The Energy Distribution of Neutrons Slowed by Elastic Impacts

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The problem of finding the distribution in energy of particles of mass m, initially of the same energy, which have made n impacts with particles of mass M all initially at rest, is solved. It is supposed the impacts are elastic and the distribution in angle isotropic in a coordinate system in which the center of mass is at rest. If x is the ratio of the energy after n impacts to the initial energy then the chance that x lie in dx at x is $(\log 1/x)^{n-1}/(n-1)!$ for $m=M$. For unequal masses the expression is more complicated but easy to calculate. The results have some interest in connection with the slowing of neutrons by elastic impacts with other nuclei, especially with hydrogen nuclei.

IN this note we work out the energy distribution of neutrons which, starting with the same initial energy, have made n impacts with other nuclei all initially at rest. We suppose the impacts are elastic and the scattering isotropic in a coordinate system in which the center of mass is at rest. The result is of some interest in connection with current researches on "slow" neutrons. The work grew out of a desire to understand a statement due to Fermi[1] that "It is easily shown that an impact of a neutron against a proton reduces, on the average, the neutron energy by a factor $1/e$."

Let the nuclei of the medium be all at rest and of mass M while the incident neutron is of mass m and energy, E_0. Then by a simple application of the conservation laws it is found that the neutron energy after an impact is given by $E_1=E_0(1-\alpha x)$

where $\alpha=4mM/(m+M)^2$ and $\cos\varphi=1-2x$, φ being the angle of scattering of the neutron in

[1] Amaldi, D'Agostino, Fermi, Pontecorvo, Rasetti and Segrè, Proc. Roy. Soc. **A149**, 522 (1935). See p. 524.

a coordinate system in which the center of mass of m and M is at rest. For elastic spheres on classical mechanics or for short range forces of any type in quantum mechanics the chance that φ lie between φ and $\varphi+d\varphi$ is proportional to $d(\cos\varphi)$ so we see that x may take each value between 0 and 1 with equal probability.

After n collisions the energy will be E_n where

$$E_n=E_0(1-\alpha x_1)(1-\alpha x_2)\cdots(1-\alpha x_n).$$

Let $x=E_n/E_0$. We have to find the probability that x lie in dx at x being given that each x_i has equal chance of having any value between zero and unity. Hence x may vary between $(1-\alpha)^n$ and unity. Write

$$(1-\alpha x_i)=e^{-u_i} \quad\text{and}\quad u=\sum u_i \quad\text{so}\quad x=e^{-u}.$$

Each u_i varies between 0 and $a=\log(1-\alpha)^{-1}$. The chance of a given set of u_i values is

$$dx_1dx_2\cdots dx_n=e^{-u}du_1du_2\cdots du_n.$$

The chance that the sum u, lie in du at u is therefore the factor e^{-u} multiplied by the chance that u lie in du at u assuming each u_i to be equally likely to have any value between 0 and a. The evaluation of this turns out to be a very old problem, apparently first considered by Laplace,[2] and the result is

$$f_n(u)du=\frac{1}{a^n(n-1)!}\left\{u^{n-1}-\binom{n}{1}(u-a)^{n-1}+\binom{n}{2}(u-2a)^{n-1}-\cdots+(-1)^{n-1}\binom{n}{n-1}\times[u-(n-1)a]^{n-1}\right\}$$

with the understanding that the term involving $(x-ka)^{n-1}$ is to be assigned the value zero for $x<ka$. Thus the distribution function has a discontinuous $(n-1)$st derivative with discontinuities at the places $x=ka$ with $k=0, 1, 2\cdots n$. A modern derivation of the result has been given by Rietz.[3]

Before we looked it up in the library we had worked out a solution of the problem which is enough different from those we have seen to make it worth communicating briefly. The first step is to consider the recursion relation connecting the distribution function for $n+1$ with that for n. Let $u=\sum_{i=1}^{n}u_i$ and $v=u+u_{n+1}$. Let $I_k^n(u)$ be the expression for $f_n(u)$ in the range $(k-1)a<u<ka$. The chance of u being in du at u and u_{n+1} in du_{n+1} at u_{n+1} is then $I_k^n(u)dudu_{n+1}$. Changing the variables to v and u and summing over the values of u which lead to a fixed value of v we find the recursion relation

$$I_k^{n+1}(v)=\int_{v-a}^{(k-1)a}I^n{}_{k-1}(u)du+\int_{(k-1)a}^{v}I_k^n(u)du$$

with the understanding that $I_0^n(u)\equiv 0$.

Now let us consider the geometrical situation. We have to find the $(n-1)$ dimensional measure of the intersection of the hypercube $0\leq u_i\leq a$ with the two hyperplanes $\sum u_i=u$ and $\sum u_i=u+du$. Plainly it will be proportional to $u^{n-1}du$ for $0<u<a$ by a dimensional argument. As u becomes a little greater than a the hyperplane passes n corners and so the expression $u^{n-1}du$ has to be corrected by subtracting off $n(u-a)^{n-1}du$ to allow for the part of the hyperplane that is outside the hypercube. When u becomes just greater than $2a$ the hyperplane passes $\binom{n}{2}$ corners so correction by $\binom{n}{2}(u-2a)^{n-1}du$ is necessary. That the correction has to be added this time is readily seen by inspection in the three-dimensional case where the figure is easily visualized. This argument suggests the general form for $I_k^n(u)$ and it is easily verified that it satisfies the recursion relation and is therefore correct when properly normalized so that its integral from 0 to na will give the correct volume, a^n.

For $u>na$, $I_n^n(u)$ is a sum of terms in u^{n-1}, $(u-a)^{n-1}\cdots$ which is easily seen to vanish identically. In fact this polynomial may be written as $[1-e^{-D}]^nu^{n-1}/(n-1)!$ which must vanish because $[1-e^{-D}]^n$ expanded in D contains only D^n and higher powers. Here D is the differentiation operator. The ratio of the coefficients of $(u-ka)^{n-1}$ is fixed because the n

[2] Laplace, *Théorie Analytique des Probabilités* (1820), pp. 257–263.

[3] Rietz, Proc. Int. Math. Congress, Toronto **2**, 795 (1924). The problem is also discussed by H. P. Lawther, Jr., Annals of Math. Statistics **4**, 241 (1933) whose Fig. 1 graphs the distribution function for $n=1, 2, 4, 8, 16, 32$.

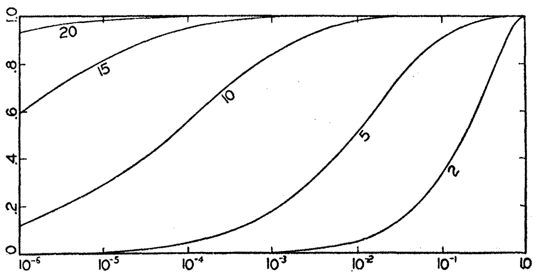

FIG. 1. Showing the probability as ordinate that a neutron have a fraction of its initial energy less than or equal to the abscissa after a number of impacts with protons that is marked on each curve.

independent ratios must satisfy $n+1$ conditions. It is thus not even necessary to verify the recurrence relation for $I_k{}^n(u)$.

The case of most interest experimentally is that corresponding to the slowing of neutrons by protons. Here $m=M$ and $\alpha=1$ so $a=\infty$ and the complications due to passing corners of the hypercube do not arise. All finite values of u are now less than a so only $I_1{}^n(u)$ plays a role. The distribution function for u is therefore $e^{-u}u^{n-1}du/(n-1)!$ or for x,

$$F_n(x)dx=(\log 1/x)^{n-1}dx/(n-1)!$$

Thus the probability that x have a value between 0 and ξ is given by

$$P_n(\xi)=\int_0^\xi F_n(x)dx.$$

The integral involved here is an incomplete gamma function for which very complete tables exist.[4] We have prepared Fig. 1 in which $P_n(\xi)$ is plotted against a logarithmic scale of ξ for several values of n. There is not much point in extending the curves below $\xi=10^{-6}$ since, for ordinary initial energies of neutrons, a value of ξ much less than this brings the neutron energies down to thermal values where the assumption that the protons are initially at rest is no longer valid. The curves give a good idea of the way in which the energy is rapidly reduced by a moderate number of collisions.

Although one can calculate the average of any power of x from the distribution function, it is simpler to do it from the expression for x as a product of the n factors $(1-\alpha x_i)$. Then we have

$$\overline{x^s}=\left[\int_0^1(1-\alpha x_i)^s dx_i\right]^n=\left[\frac{1-(1-\alpha)^{s+1}}{\alpha(s+1)}\right]^n.$$

In particular for the case of equal masses, $\alpha=1$, and the ordinary arithmetic mean, $s=1$ we have $\bar{x}=2^{-n}$. The statement of Fermi referred to above is verified for the logarithmic mean, $\exp\overline{(\log x)}$.

The function $f_n(u)$ can be represented around its maximum approximately by $e^{-6(u-na/2)^2/na^2}$ as may be found by considering $\overline{x^2}-\bar{x}^2$ according to the above formulas or by empirical fitting of numerical computations. The latter indicate that 5.7 gives a somewhat better approximation than 6.

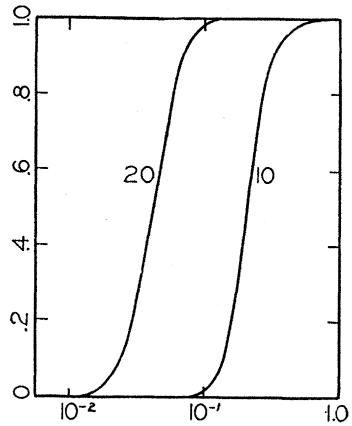

FIG. 2. Same as Fig. 1 except that it is for collisions with carbon nuclei instead of with protons.

Calculation with the distribution function for unequal masses is quite simple. In Fig. 2 we give, as an illustrative example, the integrated energy distribution of neutrons which have made 10 and 20 collisions with carbon nuclei ($M=12$). Comparison of Figs. 1 and 2 affords a striking indication of how little the carbon nuclei in paraffin contribute to the slowing down of the neutrons.

The above discussion is not intended to give the distribution of neutrons slowed down by passing through a given thickness of paraffin—some of the emerging neutrons obviously perform more collisions than others. The distribution function considered here is nevertheless useful for approximate estimates when most neutrons can be considered to have performed the same number of collisions.

[4] Karl Pearson, ed. *Tables of the Incomplete Gamma Function* (London, H. M. Stationery Office, 1922).