Coulomb Wave Functions in Repulsive Fields

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Quantitative discussion of nuclear reactions due to bombardment with charged particles requires the knowledge of wave functions in repulsive inverse square fields of force. It is customary to approximate these functions by formulas of the type due to Wentzel, Kramers and Brillouin (referred to as WKB). Errors of unknown amount are frequently introduced by such approximations. Formulas necessary for the *exact* calculation of the needed functions are derived and discussed in the present paper. Results of numerical calculations with estimated total errors are tabulated in a companion article in the *Journal of Terrestrial Magnetism and Atmospheric Electricity*. The proton energy range covered is from 0 to 2 MEV for Li and from 0 to 8 MEV for *C*. For the partial wave with 0 angular momentum the range of radii in proton collisions is covered for Li from 0 to 10^{-12} cm and for *C* from 0 to 0.5×10^{-12} cm. For the partial wave with angular momentum $\hbar$, the range of radii extends to 4×10^{-12} cm in proton collisions with Li and to 2×10^{-12} cm in proton collisions with *C*. The tables are applicable to other reactions as well. Phase shifts necessary for the theory of anomalous nuclear scattering are readily calculated by means of them. The calculation of the regular function and its derivative by means of the tables is easier than that of the irregular solution. The tables are therefore supplemented by graphs in the present paper. By means of these it is possible to calculate the irregular function quickly even though less accurately than by means of the tables. The regular and irregular functions are given, respectively, by $F_L=C_L\rho^{L+1}\Phi_L$, $G_L=D_L\rho^{-L}\Theta_L$. Here the angular momentum of the partial wave is $L\hbar$, C_L and D_L depend only on the energy, ρ is $2\pi r/\Lambda$ where r is the radius and Λ is the de Broglie wave-length. It is found empirically that for low energies the quantities Φ_L, Θ_L depend only on the radius and not on the energy. This fact is useful in applications and it is explained analytically by means of an apparently new expansion of the confluent hypergeometric function into series of Bessel functions. The successive terms of the series are arranged in ascending powers of the energy and these expansions furnish an independent way of calculating the functions. They are useful for low energies. The exact solutions are compared numerically with the WKB approximations to the functions and to their logarithmic derivatives. The ordinary WKB method is found to give only a crude approximation to the exact solutions in the needed range of energies and radii. The WKB formulas modified by changing $L(L+1)$ into $(L+\frac{1}{2})^2$ are very much better for small energies and small radii but they are not reliable as the region of positive kinetic energies is approached. In some cases the $(L+\frac{1}{2})^2$ method is worse than that using $L(L+1)$. The superiority of the $(L+\frac{1}{2})^2$ method for low energies and constant radii is traced to the fact that in such cases it is identical with the Carlini and Laplace approximations to Bessel functions. From this relationship and graphical comparisons given below errors in the WKB$(L+\frac{1}{2})^2$ approximations can be estimated.

1.

THE radial equation for two particles moving under an inverse square repulsive field of force is:

$$\frac{d^2\bar{F}}{d\rho^2}+\left\{1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right\}\bar{F}=0, \qquad (1)$$

where

$$\rho=kr\,;\ k=\mu v/\hbar\,;\ \eta=(ZZ'/\text{“137”})(c/v)=1/ka$$
$$a=\hbar^2/\mu ZZ'e^2;\quad \text{“137”}=\hbar c/e^2=1/\alpha \qquad (2)$$

and r=distance between particles, v=relative velocity, μ=reduced mass, $L\hbar$=angular momentum of state considered; Ze and $Z'e$ are the charges on the particles; $\bar{F}$ is an arbitrary constant times r times the radial factor of the wave function.

If the inverse square law applies for all r then one needs only the regular solutions $\bar{F}=F$ of the above equation. These are well-known hypergeometric series used by Gordon and Mott[1] in the discussion of Rutherford's formula. These exact forms are convenient for some types of analytical calculation but are cumbersome to use in many problems requiring numerical estimates. In the usual discussion of nuclear collisions one has to suppose that for small values of $r(r<r_0)$ the repulsive inverse square field is changed into another attractive field which represents the "nuclear well." For these smaller values of r the radial function multiplied by r does not satisfy Eq. (1). Instead it satisfies

$$\left[\frac{d^2}{d\rho^2}+1-\frac{2}{\mu v^2}V(r)-\frac{L(L+1)}{\rho^2}\right]\bar{F}=0, \qquad (3)$$

where V is the potential energy at distance r. In

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[1] W. Gordon, Zeits. f. Physik **48**, 180 (1928); N. F. Mott, Proc. Roy. Soc. **A118**, 542 (1928).

applications one needs regular solutions of (3) joined to general solutions of (1) in such a way as to have $\bar{F}$ and its derivative continuous at $r=r_0$. This requires the introduction of solutions for $\bar{F}$ which are linearly independent of F and which are thus irregular solutions of (1) at $\rho=0$. These solutions will be called G. In order to define F and G we have to specify the normalization of the functions and for G we must further define in some way the arbitrary additive constant with which F may enter into G. Now Eq. (1) is a special case of the much studied confluent hypergeometric equation.[2] It is known[1] that the regular solutions of (1) are asymptotic at $\rho=\infty$ to const. $\sin(\rho-L\pi/2-\eta\ln 2\rho+\sigma_L)$ where $\sigma_L=\arg\Gamma(L+1+i\eta)$ and that the irregular solutions are asymptotic at $\rho=\infty$ to forms differing from the above only by the insertion of an arbitrary constant phase into the argument of the sine. It is convenient to standardize both F and G by the requirement that the asymptotic values for large ρ shall be

$$F\sim\sin(\rho-L\pi/2-\eta\ln 2\rho+\sigma_L);\qquad G\sim\cos(\rho-L\pi/2-\eta\ln 2\rho+\sigma_L).\tag{4}$$

Thus F and G are defined by making both have unit amplitude for large ρ and by making the phase of G lead the phase of F by 90°. For the special case of $L=0$ practically explicit forms for G were worked out by Sexl[3] but they are not in a convenient form for calculation on account of the occurrence of imaginaries in real expressions.

The asymptotic forms of the solutions are brought out clearly by using Whittaker's function[2] $W_{\kappa,m}(z)$. Eq. (1) is reduced to the equation of the confluent hypergeometric function by the substitutions

$$z=2i\rho;\quad \kappa=i\eta;\quad m=L+\tfrac{1}{2}.\tag{5}$$

A regular solution of (1) is the function $M_{\kappa,m}(z)$. This may be expressed as a linear combination of $W_{\kappa,m}(z)$ and of $W_{-\kappa,m}(-z)$.

The formula given by Whittaker and Watson at the end of their 16.41 gives an expression for $W_{\kappa,m}(z)$ in terms of $M_{\kappa,m}(z)$ and $M_{\kappa,-m}(z)$ valid for $-\pi/2<\arg z<3\pi/2$; similarly Example 1 immediately following gives $W_{-\kappa,m}(-z)$ in terms of $M_{-\kappa,m}(-z)$, $M_{-\kappa,-m}(-z)$ valid for $-3\pi/2<\arg(-z)<\pi/2$.

We let $\arg z=\pi/2$ and $\arg z=-\pi/2$ so as to correspond to (5) and we have by Kummer's first formula

$$M_{-\kappa,m}(-z)=M_{\kappa,m}(z)\exp\{-(m+\tfrac{1}{2})\pi i\}$$

and

$$M_{-\kappa,-m}(-z)=M_{\kappa,-m}(z)\exp\{(m-\tfrac{1}{2})\pi i\}.$$

We have thus two relations between $W_{\kappa,m}(z)$, $W_{-\kappa,m}(-z)$ and $M_{\kappa,m}(z)$, $M_{\kappa,-m}(z)$.

These can be solved for the latter two quantities and one has

$$M_{\kappa,m}(z)=e^{-\pi i\kappa}\frac{\Gamma(2m+1)}{\Gamma(\frac{1}{2}+m-\kappa)}W_{-\kappa,m}(-z)-e^{\pi i(m-\kappa-\frac{1}{2})}\frac{\Gamma(2m+1)}{\Gamma(\frac{1}{2}+m+\kappa)}W_{\kappa,m}(z).$$

This formula is almost the same as that given in Example 2 of Whittaker and Watson's 16.41, but differs from it by the signs of κ in the exponents. Now $M_{\kappa,m}(z)$ gives a regular solution of (1) if m is real and positive. Similarly $M_{\kappa,-m}(z)$ gives a power series but its first term is $z^{-m+\frac{1}{2}}$. When m is a positive integer $+\frac{1}{2}$ this latter solution is not independent of the first. The asymptotic expansion of $W_{\kappa,m}$ determines the asymptotic form of $M_{\kappa,m}(z)$ and determines the factor by which $M_{\kappa,m}(z)$ should be multiplied to give F as defined by Eq. (4). One thus finds

$$F=\tfrac{1}{2}(Y+Y^*)=\tfrac{1}{2}\left|\frac{e^{\pi i\kappa/2}\Gamma(\frac{1}{2}+m-\kappa)}{\Gamma(2m+1)}\right|e^{-(\pi i/2)(m+\frac{1}{2})}M_{\kappa,m}(z),\tag{6}$$

$$G=\frac{1}{2i}(Y-Y^*)=\tfrac{1}{2}\left|\frac{e^{\pi i\kappa/2}\Gamma(\frac{1}{2}+m-\kappa)}{\Gamma(2m+1)}\right|e^{-(\pi i/2)(m+3/2)}\overline{M}_{\kappa,m}(z),$$

[2] E. T. Whittaker and G. N. Watson, *Modern Analysis*, third edition, Chap. XVI.

[3] Th. Sexl, Zeits. f. Physik **56**, 72 (1929). See especially p. 83. Cf. also Th. Sexl, Zeits. f. Physik **81**, 163 (1933).

where

$$Y=[\Gamma(\tfrac{1}{2}+m-\kappa)/\Gamma(\tfrac{1}{2}+m+\kappa)]^{\frac{1}{2}}\exp\{(\pi i/2)(m+\tfrac{1}{2}-\kappa)\}W_{\kappa,m}(z),$$
$$Y^{*}=[\Gamma(\tfrac{1}{2}+m+\kappa)/\Gamma(\tfrac{1}{2}+m-\kappa)]^{\frac{1}{2}}\exp\{-(\pi i/2)(m+\tfrac{1}{2}+\kappa)\}W_{-\kappa,m}(-z). \quad (7)$$

The function $\overline{M}_{\kappa,m}$ is defined by Eq. (6) and is analogous to $M_{\kappa,m}$. If m is not a positive half-integer it may be expressed as

$$\overline{M}_{\kappa,m}(z)=\frac{i[\cos 2\pi m+\exp(-2\pi i\kappa)]}{\sin 2\pi m}M_{\kappa,m}(z)-\frac{2i\Gamma(2m+1)\Gamma(2m)}{\Gamma(\tfrac{1}{2}+m+\kappa)\Gamma(\tfrac{1}{2}+m-\kappa)}e^{\pi i(m-\kappa)}M_{\kappa,-m}(z). \quad (8)$$

If m is a positive half-integer the value of $\overline{M}$ can be obtained by passing to the limit in the above Eq. (8).

Substituting into Eq. (6) and expressing the results in terms of ρ, L for integral values of L we have

$$F_L=C_Le^{-i\rho}\rho^{L+1}\left\{1+\frac{L+1-i\eta}{1!(2L+2)}2i\rho+\frac{(L+1-i\eta)(L+2-i\eta)}{(2L+2)(2L+3)}\frac{(2i\rho)^2}{2!}+\cdots\right\}, \quad (8')$$

where

$$(2L+1)!C_L=2^L[L^2+\eta^2]^{\frac{1}{2}}[(L-1)^2+\eta^2]^{\frac{1}{2}}\cdots[1+\eta^2]^{\frac{1}{2}}(2\pi\eta)^{\frac{1}{2}}(e^{2\pi\eta}-1)^{-\frac{1}{2}}. \quad (9)$$

Similarly one finds from the second Eq. (6)

$$G_L=\text{R.P. }D_Le^{-i\rho}\rho^{-L}\left\{1+\frac{L+i\eta}{2L}\frac{2i\rho}{1!}+\cdots+\frac{(L+i\eta)\cdots(-L+1+i\eta)}{(2L)!}\frac{(2i\rho)^{2L}}{(2L)!}\right\}$$
$$+pD_L\left[\ln 2\rho+2\gamma-\sum_{L}^{2L+1}s^{-1}+\sum_{1}^{L}\frac{s}{s^2+\eta^2}+\text{R.P. }\Gamma'(-i\eta)/\Gamma(-i\eta)\right]F_L/C_L$$
$$-pD_L\,\text{R.P. }e^{-i\rho}\rho^{L+1}\sum_{s=1}^{\infty}\frac{(2L+1)!\Gamma(L+1+s-i\eta)(2i\rho)^s}{(2L+1+s)!s!\Gamma(L+1-i\eta)}\sum_{t=1}^{s}\left[\frac{1}{t}+\frac{1}{2L+1+t}-\frac{1}{L+t-i\eta}\right], \quad (10)$$

where

$$(2L+1)D_LC_L=1;$$
$$\gamma=\text{Euler's constant}=0.57721 \quad (11)$$

and

$$p=(2L+1)(e^{2\pi\eta}-1)C_L^2/\pi, \quad (12)$$

while R.P. means real part.

The above formulas could be used for the computation of F_L and G_L but it is inconvenient to do so on account of the occurrence of imaginaries. It is obvious from Eq. (8′) that

$$F_L=C_L\rho^{L+1}\Phi_L, \quad (13)$$

where Φ_L is a power series in ρ the first term of which is 1. Substitution of this form into the differential equation satisfied by F_L leads to a recurrence formula between the coefficients of the series and one finds

$$\Phi_L=\sum_{L+1}^{\infty}A_j\rho^{j-L-1};\quad A_{L+1}=1 \quad (14)$$

$$A_j=(2\eta A_{j-1}-A_{j-2})(j+L)^{-1}(j-L-1)^{-1}. \quad (15)$$

For G_L Eq. (10) shows that this function consists of three parts: (a) a power series in ascending powers of ρ, the first term of which is $D_L\rho^{-L}$; (b) $pD_L(F_L/C_L)\ln 2\rho$; (c) an additive term in F_L. The differential equation does not determine G_L from a knowledge of parts (a) and (b) because an arbitrary term in F_L can be added to any solution and thus the power series of part (a) is not uniquely defined. But by requiring that the coefficient of ρ^{L+1} in this power series should vanish it is uniquely determined by substituting it together with $p(F_L/C_L)\ln 2\rho$ into the differential equation. The term in F_L is then also uniquely determined by the coefficient of ρ^{L+1} in Eq. (10). Thus if this coefficient is qD_L/C_L then the term in F_L is $qD_L\rho^{L+1}\Phi_L$. Collecting terms in ρ^{L+1} in Eq. (10) one finds

$$q=p\left[\sum_{1}^{L}\frac{s}{s^2+\eta^2}-\sum_{1}^{2L+1}\frac{1}{s}+\text{R.P.}\frac{\Gamma'(-i\eta)}{\Gamma(-i\eta)}\right]+(-)^{L+1}\frac{2^L}{(2L)!}\sum_{-L}^{+L}\text{I.P.}\frac{2^n(i\eta+n-1)\cdots(i\eta-L)}{(L+n)!(L-n+1)}. \quad (16)$$

We thus have

$$G_L=D_L\rho^{-L}\Theta_L, \quad (17)$$

$$\Theta_L=\Psi_L+\rho^{2L+1}(p\ln 2\rho+q)\Phi_L, \quad (18)$$

where Ψ_L is a power series in ρ beginning with the term 1, having no term in ρ^{2L+1} and having its coefficients determined by the differential equation. It is found that

$$\Psi_L=\sum_{-L}^{\infty}a_j\rho^{j+L};\quad a_{-L}=1;\quad a_{L+1}=0 \quad (19)$$

$$a_j=[2\eta a_{j-1}-a_{j-2}-p(2j-1)A_j]\times(j+L)^{-1}(j-L-1)^{-1}. \quad (20)$$

The computation of the logarithmic derivative of the gamma-function can be made either by

$$\text{R.P.}[\Gamma'(1+i\eta)/\Gamma(1+i\eta)]=-1/(1+\eta^2)+(1-\gamma)+(S_3-1)\eta^2-(S_5-1)\eta^4+\cdots;$$

$$S_k=\sum_{1}^{\infty}n^{-k} \quad (21)$$

for small η of the order of 1 or less, or else by Stirling's series for $\Gamma'(z)/\Gamma(z)$ as given in Whittaker and Watson's 12.33 for larger η. Differentiating expressions (14), (18) with respect to ρ we also have formulas for the derivatives

$$F_L'=C_L\rho^L\Phi_L^*,\quad G_L'=D_L\rho^{-L-1}\Theta_L^*, \quad (22)$$

$$\Phi_L^*=\sum_{L+1}^{\infty}jA_j\rho^{j-L-1}, \quad (23)$$

$$\Theta_L^*=\Psi_L^*+\rho^{2L+1}(p\ln 2\rho+q)\Phi_L^*+\rho^{2L+1}p\Phi_L, \quad (24)$$

$$\Psi_L^*=\sum_{-L}^{\infty}ja_j\rho^{j+L}. \quad (25)$$

By means of Eqs. (14), (15), (16), (17), (18), (19), (20), (22), (23), (24), (25), the computation of the functions can be carried out by ordinary substitutions.

The functions Φ, Φ^*, Θ, Θ^*, Ψ, Ψ^* vary more slowly than F, G, F', G' both with ρ and with η. It proved practical to calculate Φ, Φ^*, Ψ, Ψ^* at large intervals by means of their power series expansions and to interpolate for the remaining values of ρ, η.

2. Approximations for Small Energies

It was found that the tabulated values of Φ, Φ^*, Ψ, Ψ^* vary slowly in the region of small ρ and $1/\eta$ if the product $\rho\eta=\rho/ka=r/a$ is kept constant. This way of varying ρ, η corresponds to keeping the radius constant and varying the energy. Now it is well known that if the kinetic energy of a particle at ∞ is negligible in comparison with the Coulombian energy one can approximate the wave functions by means of Bessel functions of order $2L+1$ having for argument in our case $i(8\rho\eta)^{\frac{1}{2}}$. It turns out that this approximation is a very good one in a large region, and it proved practical to expand the exact solutions in series of Bessel functions as well as to extend the connection to the irregular function.

The differential Eq. (1) can be transformed by the substitutions

$$F=\zeta f,\quad \zeta=i(8\rho\eta)^{\frac{1}{2}} \quad (26)$$

into

$$\left[\frac{d^2}{d\zeta^2}+\frac{1}{\zeta}\frac{d}{d\zeta}+1-\frac{(2L+1)^2}{\zeta^2}+\frac{\zeta^2}{16\eta^2}\right]f=0. \quad (27)$$

Here the last term may be regarded as small in the region of low energies. If it is neglected one is left with Bessel's equation. The procedure will now be to expand the solution of (27) in a power series in $1/\eta^2$, i.e., in a power of the energy. Thus

$$f=f^{(0)}+(1/16\eta^2)f^{(1)}+(1/16\eta^2)^2f^{(2)}+\cdots \quad (28)$$

and

$$L(f^{(s)})=-\zeta^2f^{(s-1)}, \quad (29)$$

where

$$L=d^2/d\zeta^2+d/\zeta d\zeta+1-\nu^2/\zeta^2,\quad \nu=2L+1. \quad (29')$$

If we set $f^{(0)}=J_\nu$ we can obtain a regular solution of Eq. (1) by solving Eq. (29) always discarding the irregular solutions of these equations. However, it should be remembered that to any $f^{(s)}$ one can add J_ν multiplied by an arbitrary constant involving $1/16\eta^2$ to the 0th and higher powers. Thus an additional restrictive condition must be imposed on each $f^{(s)}$ in order that ζf should differ from F only by a factor independent of ζ and η. This condition is obtained from the exact expression for F given by Eq. (8). We substitute for ρ in terms of ζ and we expand in $1/\eta$. We see then that the lowest power of ζ which occurs with η^{-2p} is $\zeta^{\nu+1+2p}$. Thus each $f^{(s)}$ should be a power series having no terms of order lower than $\zeta^{\nu+s}$. This determines the regular solution uniquely through the successive approximations obtainable from Eq. (29) because the addition of a term in J_ν to any $f^{(s)}$ brings in a term in ζ^ν. The solution of any Eq. (29) can be carried out by using

$$\begin{aligned} L(\zeta^{\mu+1}J_\nu') &= \mu^2\zeta^{\mu-1}J_\nu' + [-2(\mu+1)\zeta^\mu + 2\nu^2\mu\zeta^{\mu-2}]J_\nu, \\ L(\zeta^\mu J_\nu) &= 2\mu\zeta^{\mu-1}J_\nu' + \mu^2\zeta^{\mu-2}J_\nu. \end{aligned} \tag{30}$$

Each approximation is then obtained as a sum of terms of the type $\zeta^\mu J_\nu$, $\zeta^{\mu+1}J_\nu'$. Or else one may use

$$\begin{aligned} L(\zeta^\mu J_\nu) &= (\mu^2+2\mu\nu)\zeta^{\mu-2}J_\nu - 2\mu\zeta^{\mu-1}J_{\nu+1}, \\ L(\zeta^\mu J_{\nu+1}) &= 2\mu\zeta^{\mu-1}J_\nu + (\mu-1)(\mu-2\nu-1)\zeta^{\mu-2}J_{\nu+1} \end{aligned} \tag{30'}$$

and obtain it in terms of $\zeta^\mu J_\nu$, $\zeta^\mu J_{\nu+1}$. On account of the recurrence relation between $J_{\nu+2}$, $J_{\nu+1}$, J_ν one can transform the result into a linear combination of terms in $\zeta^\mu J_{\nu+2}$, $\zeta^\mu J_\nu$. This is convenient for our calculations because $J_{\nu+2}$ is related to Φ_{L+1} in the same way as J_ν is to Φ_L. We thus obtain

$$\begin{aligned} f = J_\nu + \eta^{-2}&\left\{\left[-\frac{(\zeta/2)^4}{12(\nu+1)} - \frac{\nu-1}{24}(\zeta/2)^2\right]J_{\nu+2} - \frac{(\zeta/2)^4}{12(\nu+1)}J_\nu\right\} \\ &+\frac{1}{24\eta^2}\left\{\left[\frac{(\zeta/2)^6}{5(\nu+1)} + \frac{(\nu+2)(\nu+3)(5\nu-7)}{120(\nu+1)}(\zeta/2)^4 + \frac{(\nu+2)(\nu+3)(5\nu-7)(\nu-1)}{240}(\zeta/2)^2\right]J_{\nu+2}\right. \\ &\left.+\left[\frac{7-5\nu}{60(\nu+1)}(\zeta/2)^6 - \frac{(\nu-1)(\nu+3)(5\nu-7)}{240(\nu+1)}(\zeta/2)^4\right]J_\nu\right\} + \cdots. \end{aligned} \tag{31}$$

It is also possible to work out the above expansion directly as an expansion of Eq. (8) and expressing the sth term in terms of $s(s-1)\cdots(s-p+1)$. The series arranges itself then as a sum of terms in powers $\zeta^p J_{\nu+p}$ and then by use of recurrence relations can be brought to the form (31). This calculation is lengthy and need not be mentioned in more detail.

The contour integral for $W_{i\eta,\,m}(2i\rho)$ used as the definition of this function by Whittaker and Watson can be related to the Bessel function contour integral used by them in their 17.2 by observing that

$$(1+i\zeta^2/4\eta\tau)^{i\eta+m-\frac{1}{2}} = e^{-\zeta^2/4\tau}\exp\left\{\frac{1}{\eta}\left[\frac{i}{2}\left(\frac{\zeta^2}{4\tau}\right)^2 + (m-\tfrac{1}{2})\frac{i\zeta^2}{4\tau}\right] + \cdots\right\}.$$

The exponential can be expanded in powers of $1/\eta$ and the result is a sum of powers of ζ and Bessel functions of order higher than ν. However, care must be taken to have the path of integration stay inside the circle of convergence for the power expansion of $\ln(1+i\zeta^2/4\eta\tau)$. This point of view is apparently more troublesome than the use of Eqs. (30) or (30′), but it may have its advantages.

If one omits all but the first term in (31) one obtains an approximation to Φ_L which applies for very large η, i.e., for very small energies. The limiting value of Φ_L for large η and fixed $\rho\eta$ will be written $\mathfrak{F}_L$. We have

$$\mathfrak{F}_L=(2L+1)\,!(\zeta/2)^{-\nu}J_\nu(\zeta)=(2L+1)\,!(x/2)^{-\nu}I_\nu(x),\quad x=(8\rho\eta)^{\frac{1}{2}},\tag{32}$$

where I_ν is as defined in Whittaker and Watson's 17.7. Similarly

$$\Phi_L=(2L+1)\,!(\zeta/2)^{-\nu}f,\tag{32$'$}$$

where f is the expansion (31); the connection of f and Φ_L is defined uniquely by the coefficients of lowest powers of ρ in both expressions. Expressing all J_ν in terms of the $\mathfrak{F}_L$ in Eq. (31) one obtains an expansion of Φ_L in terms $\mathfrak{F}_L$ and $\mathfrak{F}_{L+1}$. We thus have

$$\begin{aligned}\Phi_L=\mathfrak{F}_L&+\frac{1}{\eta^2}\left\{\left[\frac{(x/2)^6}{12(\nu+1)}-\frac{\nu-1}{24}(x/2)^4\right]\frac{\mathfrak{F}_{L+1}}{(\nu+1)(\nu+2)}-\frac{(x/2)^4}{12(\nu+1)}\mathfrak{F}_L\right\}\\&+\frac{1}{24\eta^4}\left\{\left[\frac{(x/2)^8}{5(\nu+1)}+\frac{(\nu+2)(\nu+3)(7-5\nu)}{120(\nu+1)}(x/2)^6-\frac{(\nu+2)(\nu+3)(7-5\nu)(\nu-1)}{240}(x/2)^4\right]\frac{\mathfrak{F}_{L+1}}{(\nu+1)(\nu+2)}\right.\\&\left.\qquad+\left[-\frac{7-5\nu}{60(\nu+1)}(x/2)^6+\frac{(\nu-1)(\nu+3)(7-5\nu)}{240(\nu+1)}(x/2)^4\right]\mathfrak{F}_L\right\}+\cdots.\end{aligned}\tag{33}$$

For $L=0$ the term in η^{-6} to be added in the above expansion is

$$\frac{\rho^2\eta^2}{23{,}040\eta^6}\left\{\left[-\frac{320}{9}\rho^3\eta^3-\frac{640}{7}\rho^2\eta^2-\frac{1280}{21}\rho\eta\right]\mathfrak{F}_0+\left[\frac{320}{27}\rho^4\eta^4+\frac{5056}{63}\rho^3\eta^3+\frac{2560}{21}\rho^2\eta^2+\frac{1280}{21}\rho\eta\right]\mathfrak{F}_1\right\}.\tag{33$'$}$$

Numerical calculation of successive terms in η^{-2}, η^{-4} shows that the above series converges rapidly even in regions where the power series in ρ requires a large number of terms for its computation. Thus for example for $\rho\eta=0.631$ and $\rho=1.259$ the successive terms of (33) are 1.778, -0.394, $+0.028$ giving $\Phi_0=1.412$ which agrees with $\Phi_0=1.412$ as computed by the power series. By means of Eq. (32) the power series calculations for small ρ were checked satisfactorily.

The recurrence formulae for Bessel functions give the limiting values $\mathfrak{F}_{L+1}$ in terms of $\mathfrak{F}_L$, $\mathfrak{F}_{L-1}$. The relation is

$$\left[1+2\frac{(x/2)^2}{\nu^2-1}\right]\mathfrak{F}_L=\frac{(x/2)^4}{\nu(\nu+1)^2(\nu+2)}\mathfrak{F}_{L+1}+\mathfrak{F}_{L-1}.\tag{34}$$

Similarly recurrence formulae give

$$\mathfrak{F}_L{}^*=\frac{1}{2}\left[1+\nu+\frac{2(x/2)^2}{\nu+1}\right]\mathfrak{F}_L-\frac{(x/2)^4}{(\nu+1)^2(\nu+2)}\mathfrak{F}_{L+1}=\frac{1}{2}\left[1-\nu+\frac{2(x/2)^2}{1-\nu}\right]\mathfrak{F}_L+\nu\mathfrak{F}_{L-1}.\tag{34$'$}$$

These relations also determine $\mathfrak{F}_L$, $\mathfrak{F}_{L+1}$ in terms of $\mathfrak{F}_L{}^*$ and $\mathfrak{F}_{L+1}{}^*$. The relations

$$\mathfrak{F}_L{}^*=\frac{1}{2}\left(\nu+1+x\frac{d}{dx}\right)\mathfrak{F}_L,\tag{35}$$

$$\frac{1}{2}\left(\nu+1+x\frac{d}{dx}\right)(x/2)^p\mathfrak{F}_L=\frac{1}{2}\left(\nu+1+p+\frac{(x/2)^2}{\nu+1}\right)(x/2)^p\mathfrak{F}_L-\frac{(x/2)^{4+p}}{(\nu+1)^2(\nu+2)}\mathfrak{F}_{L+1},\tag{35$'$}$$

$$\frac{1}{2}\left(\nu+1+x\frac{d}{dx}\right)(x/2)^p\mathfrak{F}_{L+1}=(\nu+2)(x/2)^p\mathfrak{F}_L+\left[-\frac{(x/2)^2}{\nu+1}-\frac{\nu+3-p}{2}\right](x/2)^p\mathfrak{F}_{L+1},\tag{35$''$}$$

determine through (33) the series $\Phi_L{}^*$. Using the relations (34$'$) one can also express it in terms of $\mathfrak{F}_L{}^*$, $\mathfrak{F}_{L+1}{}^*$.

Using Eq. (10) and collecting terms of lowest power in $1/\eta$ we obtain an approximation to

$$\mathfrak{G}_L = -\frac{2}{(2L)!}\left(\frac{x}{2}\right)^{2L+1} K_{2L+1}(x), \tag{36}$$

where K is defined in Whittaker and Watson's 17.71. In our case $K_\nu = (-)^L(\pi/2)H_\nu^{(1)}(\zeta)$. This formula checks satisfactorily the values of $\mathfrak{G}_L$ for small ρ. An expansion of Θ_L similar to (33) is possible but is more complicated. Substitution of K_ν for J_ν in Eq. (31) gives an irregular solution of the differential equation, but we have no proof that this is the irregular solution which is wanted. In special cases numerical trial gave satisfactory results with this expansion.

Both F_L and G_L can be approximated by means of formulae of the type of Wentzel, Kramers and Brillouin (WKB). One can try to use the differential Eq. (1) and to compute the phase integral directly. Such results we will call WKB $L(L+1)$ so as to indicate that the last term is used without any change. In the region of negative kinetic energy the appropriate approximations are

$$F = \tfrac{1}{2}Q^{-\frac{1}{2}} \exp\left\{-\int_\rho^{\rho_0} Q d\rho\right\}; \quad G = Q^{-\frac{1}{2}} \exp\left\{\int_\rho^{\rho_0} Q d\rho\right\}, \tag{37}$$

where $-Q^2$ stands for the coefficient of F in Eq. (1) and ρ_0 is the classical turning point, i.e., the positive root of Q^2. Kramers[4] showed that in the case of attractive coulomb fields the correct phase in the region of positive kinetic energies is obtained by using $(L+\frac{1}{2})^2$ instead of $L(L+1)$ in the phase integral. It was subsequently shown by Uhlenbeck and Young[5] that a change from $L(L+1)$ to $(L+\frac{1}{2})^2$ improves the approximation also so far as absolute values of the function are concerned for attractive Coulomb fields. The results of Uhlenbeck and Young are essentially due to a combination of the effect noted by Kramers and of the fact that the $(L+\frac{1}{2})^2$ modification makes the wave function have the correct power dependence on r at small r. *For repulsive fields* the power dependence on r is also given correctly for F by the $(L+\frac{1}{2})^2$ modification but *Kramers' argument about the phase is not applicable* because the regions of positive and negative kinetic energies changed places. We find, *nevertheless*, by trial that *for low energies the* WKB $(L+\frac{1}{2})^2$ *method gives very good results* and on considering the connection with the preceding expansions in Bessel functions a reason for this became apparent.

The values of F_L, G_L obtained by WKB $(L+\frac{1}{2})^2$ are given by

$$F_L = (2\cos u)^{-\frac{1}{2}} \sin^{L+1}\frac{u+u_0}{2} \cos^{-L}\frac{u-u_0}{2} \exp\left\{-\eta\left(\frac{\pi}{2}-u\right)+(L+\tfrac{1}{2})\frac{\cos u}{\cos u_0}\right\}, \quad F_L G_L = 1/2Q \tag{38}$$

where

$$\sin u = (\rho-\eta)[\eta^2+(L+\tfrac{1}{2})^2]^{-\frac{1}{2}}; \quad \sin u_0 = \eta[\eta^2+(L+\tfrac{1}{2})^2]^{-\frac{1}{2}} \tag{38'}$$

and u is supposed to be in the first or fourth quadrant while $0<u_0<\pi/2$. This formula is convenient for the computation of F_L. Passing to the limit of small ρ, keeping $\rho\eta$ at a fixed value, one obtains the value of $C_L\mathfrak{F}_L$. Since on the other hand Φ_L is related to J_ν by Eq. (32) this also means that we obtain an approximation for J_ν. We obtain in this way

$$J_\nu(ix) \cong \frac{(ix)^\nu \exp(\nu^2+x^2)^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}(\nu^2+x^2)^{\frac{1}{4}}[\nu+(\nu^2+x^2)^{\frac{1}{2}}]^\nu}, \tag{38''}$$

which will be recognized as Carlini's formula.[6] This is supposed to be valid only for large ν being obtained as a first term of an expansion in descending powers of ν. But even for $\nu=1$ this formula is found, by trial, to be remarkably accurate, deviating from the correct value by not more than 8.5 percent. For $\nu=3$ which corresponds to $L=1$ the agreement is much better, the maximum discrepancy being only 2.8 percent. This agreement applies for all values of x and

[4] H. A. Kramers, Zeits. f. Physik **39**, 828 (1926).

[5] L. A. Young and G. E. Uhlenbeck, Phys. Rev. **36**, 1154 (1930).

[6] G. N. Watson, *Bessel Functions*, pp. 7, 226.

implies a corresponding agreement of the WKB $(L+\frac{1}{2})^2$ method for any radius if the energy is made low enough.

Corresponding to (38″) the approximation to Θ by means of K leads to an approximation for K_ν given by

$$-2(\nu^2+x^2)^{\frac{1}{2}}K_\nu I_\nu=1, \qquad (38''')$$

where I_ν is as determined by (38″). This approximation is also good. It is apparently a new one mathematically although physical considerations make this approximate connection of K_ν with I_ν very obvious. The exact values of K_ν and I_ν satisfy this formula to within a few percent from $x=0$ to $x=10$.

Keeping η constant and making ρ very small one obtains the region of small radii at constant energy. Here the F_L function can be approximated by $C_L\rho^{L+1}$ and as has been already mentioned the correct power of ρ is obtained by the WKB $(L+\frac{1}{2})^2$ method. We find also that the coefficient of ρ^{L+1} obtained by this approximation agrees well with C_L. Thus for $\eta=1$ this coefficient is 0.120 while $C_L=0.108$; for $\eta=3.162$ the coefficient is 2.35×10^{-4} while the exact value is 2.17×10^{-4}. This agreement causes as well an agreement of G_L with its WKB $(L+\frac{1}{2})^2$ approximation because in this region the approximations satisfy $F_LG_L=\rho/(2L+1)$ and the same formula is obeyed by the exact solutions for small ρ. It is thus seen that both the F and G functions are reproduced quite well by the WKB $(L+\frac{1}{2})^2$ approximation for energies and radii which correspond to the particles being well inside their mutual potential barrier.

It should be remembered, on the other hand, that the $(L+\frac{1}{2})^2$ modification raises the barrier in comparison with its actual value. It is thus not a good approximation in the proximity of the classical turning point, not only because the WKB formulae are always poor close to the turning point but also because the position of the turning point is given incorrectly by the $(L+\frac{1}{2})^2$ modification.

The Carlini formula is obtainable from Meissel's first expansion[6] through the use of Stirling's series and the rapid convergence of this series even for small ν is in part responsible for the practical nature of Carlini's formula for small ν. It is of interest to note that the procedure used in deriving successive terms of Meissel's formula is very closely related to that used in discussing the WKB method. The essential difference lies in the fact that the exponent is expanded in descending powers of ν by Meissel and no such convenient parameter is used in deriving the WKB approximations. The first two terms in the exponent of Meissel's formula given by the first line of V_ν on p. 227 of Watson's Bessel functions give, as is found by trial, a very good approximation which for our purposes is superior to Carlini's. It is not obvious why an expansion in $1/\nu$ should give results so superior to those obtained by treating Eq. (1) by the procedure of WKB.

3. Discussion of Numerical Results[7]

In tables B, C, D, E the coefficients for Φ_L and Ψ_L, are given for $L=0$, 1 and for values of η so chosen that logarithms of η^{-1} extend from $\bar{2}.0$ to 0.5 in steps of 0.1. The values of η^{-1} that were used did not correspond quite exactly to the tabulated values of the logarithms. In some cases it is necessary to know precisely the value of η used in the computation of coefficients and for this reason a separate column giving these values is given in the tables. For a_j, $L=0$, this was not necessary because the same η's were used for them as for A_j, $L=0$. Coefficients for Φ_L^* and Ψ_L^*, namely jA_j and ja_j can be also obtained from these tables.

Table A gives Φ_L, Φ_L^*, Ψ_L, Ψ_L^* for $L=0$, 1 obtained using these coefficients. They are calculated for values of ρ chosen in steps of 0.1 on a logarithmic scale to the base 10 from $\ln\rho=\bar{2}.0$ up to $\ln\rho=\bar{1}.9$ and in some cases up to $\ln\rho=0.9$. In this work the values of ρ^n that were used in the computation of the series were each good only to four significant figures.

The probable percentages of error in the values of the series given in Table A are indicated for certain columns of the table as explained in the note at its heading. This was estimated for a large number of them considering: (1) the accu-

[7] The bulk of these tables will be published in the December issue of the Journal of Terrestrial Magnetism and Atmospheric Electricity together with a brief explanation and a collection of formulas. These tables are referred to as A, B, C, D, E, F, G, H, I. In distinction to these the tables of the present paper are numbered: I, II, III, IV, V, VI, VII, VIII.

racy in the coefficients and (2) the slowness of convergence of the series. For Φ and Ψ it was assumed that the first digit discarded in each A_j or a_j was 9. The sum of the squares of the contributions of the errors in individual terms was considered to be the square of the "total error" if the last few terms were small in comparison to previous terms. In case this was not so, additional coefficients were roughly determined to the stage where this condition did exist. The additional contribution of these approximate coefficients was squared and added to the quantity obtained from considering errors in each term. The square root of this sum was used as the "total error." "Total errors" were determined for Φ^* and Ψ^* in the same manner. The error in the powers of ρ was not properly taken into account in these estimates but the fact will probably be of no great importance. The percentages of "total error" increase to the value given for the column at the right more rapidly as the column is approached.

A large number of checks were made on Φ, Φ^*, Ψ, Ψ^* and p_L both for $L=0$ and $L=1$. Use was made of the relation

$$F'G-G'F=1$$

or its equivalent

$$\Phi^*\Theta-\Theta^*\Phi=2L+1.$$

One hundred thirty-five such random checks were made for $L=0$, $\rho\eta\leq 1.259$, $\rho\leq 0.7944$; 145, for $L=1$, $\rho\eta\leq 3.162$, $\rho\leq 1.585$. The majority of these checks were made with values of p_L *slightly* erroneous in a manner to be discussed more fully later. This fact probably does not vitiate the values of $F'G-G'F$ so obtained to any great extent since the checks in Tables VII, VIII (which are a fair sample of those obtained in general) were gotten by using accurate p_L's and do not vary appreciably from original checks with less accurate p_L's. The maximum inaccuracy in $F'G-G'F$ due to this cause was 0.5 percent which was found for $\eta=1$, $L=0$. In other cases the correction due to using quite accurate p_L's had less effect. Thus for $L=0$, $\rho\eta=0.6310$ the effect was less than 0.04 percent and less than 0.3 percent for $L=1$, $\ln\eta^{-1}=0.0$.

The values of $F'G-G'F$ for $L=0$ ranged from 0.9836 to 1.0291. Ninety satisfied the check to 0.1 percent and 135 to 1 percent. The remainder were for $\rho\eta=1.000$ to $\rho\eta=1.259$ where deviations are to be expected. For $L=1$ values lay between 2.82 and 3.24. One hundred eighteen of these satisfied the check to 0.1 percent and 140 to 1 percent. The remainder were for values of $\rho\eta$ from 1.259 to 3.162.

p_L and q_L are given in Table F together with the values of η to which they correspond. The p_0's are accurate as given. The p_1's are accurate to 1 in the last figure. The bothersome part of determining the q_L's is the calculation of R.P. $\Gamma'(1+i\eta)/\Gamma(1+i\eta)$. If one uses the difference equation for the gamma-function this may be written

$$\text{R.P.}\,\frac{\Gamma'(1+i\eta)}{\Gamma(1+i\eta)}=-\sum_{j=1}^{s-1}\frac{j}{j^2+\eta^2}+\text{R.P.}\,\frac{\Gamma'(s+i\eta)}{\Gamma(s+i\eta)}.$$

The R.P. $\Gamma'(1+i\eta)/\Gamma(1+i\eta)$ was determined from the above equation by using Stirling's series for $\Gamma'(s+i\eta)/\Gamma(s+i\eta)$ with $s=2$ and $s=6$. This was done for $\eta\geq 1$. Agreement between the two methods was exceedingly good. For $\eta<1$ the calculation was made as above for $S=6$ and this checked satisfactorily with results by Eq. (21). For $\eta\geq 1$ the values for $s=6$ were used inclusive of the first figure in which they disagreed from $s=2$. The difference between $s=2$ and $s=6$ was taken as the probable error. The error due to using a finite number of digits in the other terms in q_L was also taken into account. The square root of the sum of the squares of all these errors was calculated and used as a measure of the error in q. The values of q are so tabulated that the last figure given is not in doubt by more than 5 and generally by less, using the above way of estimating accuracy.

Values of C_L, D_L for $L=0$ and $L=1$ together with the η's to which they correspond are given in Table H.

Tables discussed thus far are auxiliary to the determination of the regular and irregular wave functions, F_L, G_L and their derivatives F_L', G_L'. The latter four quantities are given by Eqs. (13), (17), (18), (22), (24).

Figs. 1, 2 give the values of Θ_0 and Θ_1, respectively. The two families of curves in each graph correspond to constant energies ($\eta=\text{const.}$) and constant radii ($\rho\eta=\text{const.}$). Some of the

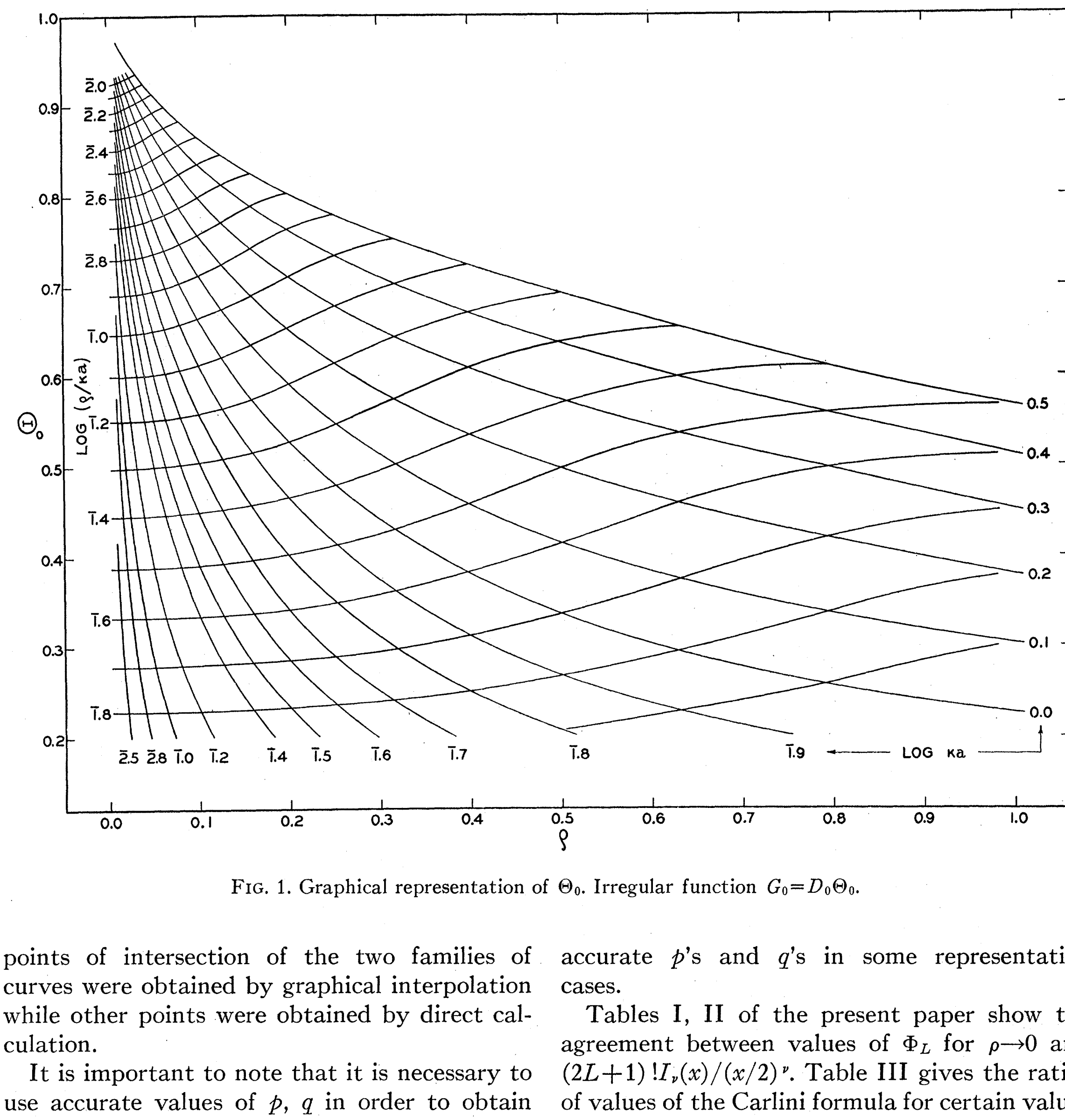

FIG. 1. Graphical representation of Θ_0. Irregular function $G_0 = D_0\Theta_0$.

points of intersection of the two families of curves were obtained by graphical interpolation while other points were obtained by direct calculation.

It is important to note that it is necessary to use accurate values of p, q in order to obtain correct results for Θ because this quantity is often obtained as a difference of two nearly equal numbers. A portion of these values were calculated by using values of p, q, $\ln 2\rho$ which for one reason or another differed slightly from the final accurate values given in the tables.

It is believed that the values of Θ used in plotting the curves of Figs. 1, 2 are good to better than 2.5 percent for $L=0$ and better than 0.5 percent for $L=1$ and in most cases the accuracy is probably higher than just stated. These estimates of accuracy were arrived at by using accurate p's and q's in some representative cases.

Tables I, II of the present paper show the agreement between values of Φ_L for $\rho \rightarrow 0$ and $(2L+1)!I_\nu(x)/(x/2)^\nu$. Table III gives the ratios of values of the Carlini formula for certain values

TABLE I. *Values of* $\mathfrak{F}_0/\Phi_0$.

$\rho\eta$	ρ	Φ_0	$\mathfrak{F}_0$	$\mathfrak{F}_0/\Phi_0$
0.01000	0.01000	1.01002	1.01003	1.00001
0.01585	0.01585	1.01589	1.01593	1.00004
0.02512	0.01000	1.02531	1.02532	1.00001
0.03981	0.01585	1.04029	1.04033	1.00004
0.06310	0.01000	1.06442	1.06444	1.00002
0.1000	0.01000	1.1034	1.10339	0.99999
0.1995	0.01000	1.1671	1.16709	0.99999
0.2512	0.01000	1.2730	1.27313	1.00011
0.3981	0.01000	1.4545	1.45458	1.00005
0.6310	0.01000	1.7783	1.77860	1.00017
1.000	0.01000	2.3947	2.3948	1.00004
1.259	0.01259	2.9132	2.91336	1.00006
1.585	0.01585	3.6823	3.68246	1.00004
1.995	0.01995	4.8641	4.86371	0.99992
2.512	0.02512	6.7583	6.75734	0.99986

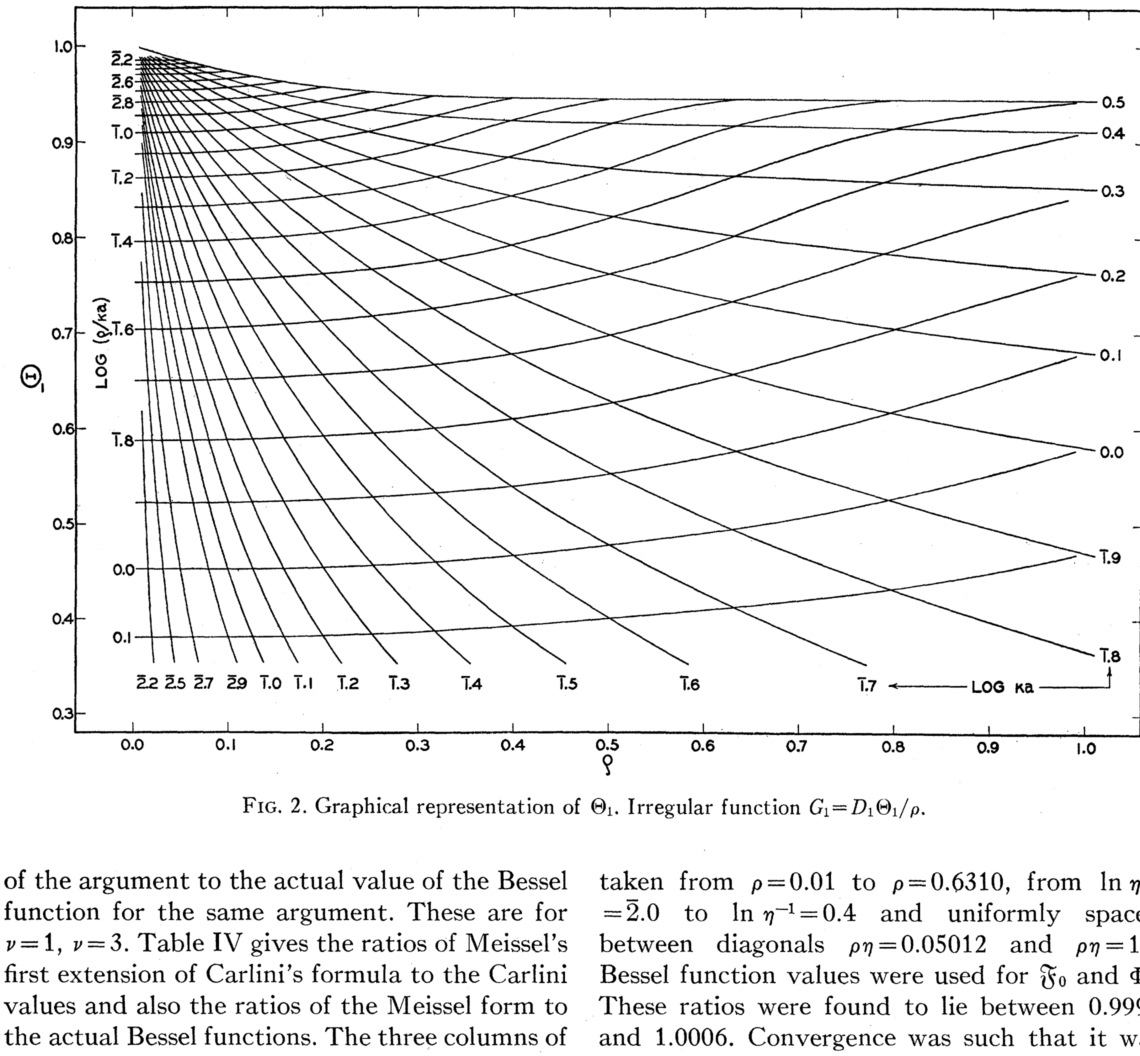

FIG. 2. Graphical representation of Θ_1. Irregular function $G_1 = D_1\Theta_1/\rho$.

of the argument to the actual value of the Bessel function for the same argument. These are for $\nu=1$, $\nu=3$. Table IV gives the ratios of Meissel's first extension of Carlini's formula to the Carlini values and also the ratios of the Meissel form to the actual Bessel functions. The three columns of $\nu=1$, $\nu=3$ give the results of using 1, 2 and 3 terms in Meissel's exponent V.

No tables are given here for the verification of Eqs. (33), (34) and (34′). In testing (33) 51 ratios of $\Phi_0 = f(\mathfrak{F}_0, \mathfrak{F}_1)$ to the actual Φ_0 were taken from $\rho=0.01$ to $\rho=0.6310$, from $\ln \eta^{-1} = \bar{2}.0$ to $\ln \eta^{-1} = 0.4$ and uniformly spaced between diagonals $\rho\eta = 0.05012$ and $\rho\eta = 10$. Bessel function values were used for $\mathfrak{F}_0$ and Φ_1. These ratios were found to lie between 0.9990 and 1.0006. Convergence was such that it was not necessary to consider the term in η^{-6}. The term in η^{-6} was computed for a point beyond the barrier given by $\rho\eta = 0.6310$, $\rho = 1.259$. The first three terms were 1.778, −0.397, 0.028. The term

TABLE II. *Values of* $\mathfrak{F}_1/\Phi_1$.

$\rho\eta$	ρ	Φ_1	$\mathfrak{F}_1$	$\mathfrak{F}_1/\Phi_1$
0.01000	0.01000	1.00500	1.00515	1.00015
0.01585	0.01259	1.00793	1.00817	1.00024
0.02512	0.01259	1.01261	1.01278	1.00017
0.03981	0.01259	1.02004	1.01983	0.99979
0.06310	0.01259	1.03192	1.03198	1.00006
0.1000	0.01259	1.0509	1.0510	1.00010
0.1995	0.01000	1.1039	1.1038	0.99991
0.2512	0.01259	1.1321	1.1321	1.00000
0.3981	0.01000	1.2155	1.2156	1.00008
0.6310	0.01000	1.3582	1.3582	1.00000
1.000	0.01000	1.6118	1.6120	1.00012
1.259	0.01259	1.8122	1.8123	1.00006
1.585	0.01585	2.0933	2.0934	1.00005
1.995	0.01995	2.4976	2.4976	1.00000
2.512	0.02512	3.0986	3.0991	1.00016

TABLE III. *Values of* $\phi_L/\mathfrak{F}_L \cdot \phi_L =$ *Carlini's approximation to* $\mathfrak{F}_L$.

$x=(8\rho\eta)^{\frac{1}{2}}$	$\mathfrak{F}_0$	ϕ_0	$\phi_0/\mathfrak{F}_0$	$\mathfrak{F}_1$	ϕ_1	$\phi_1/\mathfrak{F}_1$
0.0	1.00000	1.0844	1.0844	1.00000	1.0281	1.0281
0.4				1.01004	1.0373	1.0269
0.8	1.08216	1.1126	1.0281	1.04065	1.0654	1.0238
1.2				1.09331	1.1144	1.0193
1.6	1.35601	1.3348	0.9844	1.17061	1.1871	1.0141
2.0				1.27644	1.2877	1.0088
2.4	1.91510	1.8506	0.9663	1.41621	1.4219	1.0040
3.2	2.95891	2.8610	0.9669	1.82942	1.8230	0.9965
3.6				2.12562	2.1124	0.9938
4.0	4.87973	4.7363	0.9706			
4.4				2.98394	2.9554	0.9904
4.8				3.59824	3.5595	0.9892
5.2				4.38500	4.3347	0.9885
5.6				5.39605	5.3318	0.9881
6.0				6.70012	6.6186	0.9878

TABLE IV. *Ratios of Meissel expansion to Carlini formula.*

$x=(8\rho\eta)^{\frac{1}{2}}$	$\nu=1$	$\nu=1$	$\nu=1$	$\nu=3$	$\nu=3$	$\nu=3$
0.0	0.92214	0.92214	0.92214	0.97271	0.97271	0.97271
0.8	1.00069	0.97039	0.98068	0.97796	0.97642	0.97604
1.6	1.03886	1.03357	1.04465	0.98922	0.98557	0.98560
2.4	1.03925	1.04138	1.04564	0.99957	0.99621	0.99659
3.2	1.03068	1.03358	1.03638	1.00532	1.00300	1.00353
4.0	1.03006	1.03258	1.03483	1.01016	1.00880	1.00941
4.8	1.02634	1.02837	1.03094	1.01205	1.01146	1.01188
Ratios of Meissel expansion to actual Bessel functions for same x and ν						
0.0	0.99997	0.99997	0.99997	1.0000	1.0000	1.0000
0.8	1.0288	0.99766	1.0082	1.0012	0.99966	0.99927
1.6	1.0227	1.0174	1.0284	1.0031	0.99942	0.99944
2.4	1.0042	1.0063	1.0104	1.0036	1.0002	1.0006
3.2	0.9966	0.9994	1.0021	1.0018	0.99946	0.99999
4.0	0.9998	1.0022	1.0044	0.97906	0.97780	0.97833
4.8				1.0012	1.0006	1.00098

due to η^{-6} was 0.003. This indicates that for some values of $\rho\eta$ the first three terms give a good approximation to Φ_0 even beyond the barrier.

In testing (33) for $L=1$ fifty-six values of the ratio of $\Phi_1=f(\mathfrak{F}_1, \mathfrak{F}_2)$ to Φ_1 were taken between $\rho=0.01$ and $\rho=3.981$, $\ln \eta^{-1}=\bar{2}.0$ and $\ln \eta^{-1}=0.4$, evenly spaced between the diagonals $\rho\eta=0.05012$ and $\rho\eta=3.981$. The Bessel function values of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ were used. Out to $\rho=0.7944$ the values of the ratio lay between 0.9993 and 1.0078. This expansion is good within three percent up to $\rho=2.512$, for $\ln \eta^{-1}=0.0$, 0.4 but breaks down for higher ρ. Eqs. (34), (34′) were also verified by numerical trial in a large number of cases. The terms beyond η^{-4} were not considered in these calculations.

The verification of Eq. (36) is shown in Table V. The error in Θ due to the errors in the p's and q's should be small for these values. In fact for $\rho=0.01$ the values for quite accurate p's and q's should not vary from those used by more than 0.17 percent for $L=0$ or by more than 0.24 percent for $L=1$. Similarly the extension of Carlini's result to K_ν given by Eq. (38‴) is verified in Table VI.

A relation giving Θ_0 as a function of $\mathfrak{G}_0$, $\mathfrak{G}_1$ was obtained by analogy with Eq. (33). Thus substituting

$$(x/2)^{-4s}[(\nu+1)\cdots(\nu+2s-1)]^2\nu(\nu+2s)\mathfrak{G}_{L+s}$$

for $\mathfrak{F}_{L+s}$ one obtains an expansion for Θ_L which gives an irregular solution of the differential equation for G. We have no proof that this expansion is equal to Θ because it may differ from it by terms involving the regular solution.

TABLE V. *Values of $\mathfrak{G}_L/\Theta_L$ for $L=0,1$.*

$\ln \eta^{-1}$	ρ	$\mathfrak{G}_0$	Θ_0	$\mathfrak{G}_0/\Theta_0$	$\mathfrak{G}_1$	Θ_1	$\mathfrak{G}_1/\Theta_1$
$\bar{2}.0$	0.01995				0.2386	0.240	0.994
$\bar{2}.0$	0.01000	0.1399	0.136	1.029	0.4496	0.4486	1.0022
$\bar{2}.2$	0.01000	0.2287	0.229	0.9987	0.5866	0.5856	1.0017
$\bar{2}.3$	0.01000				0.6486	0.6479	1.0011
$\bar{2}.4$	0.01000	0.3331	0.333	1.0003			
$\bar{2}.5$	0.01000	0.3880	0.388	1.0000	0.7518	0.7512	1.0008
$\bar{2}.6$	0.01000	0.4438	0.445	0.9973			
$\bar{2}.7$	0.01000	0.5520	0.551	1.0019	0.8333	0.8302	1.0037
$\bar{2}.9$	0.01000				0.8917	0.8870	1.0053
$\bar{1}.0$	0.01000	0.6489	0.6473	1.0025	0.9093	0.9084	1.0010
$\bar{1}.1$	0.01000	0.6917	0.6908	1.0013	0.9266	0.9260	1.0006
$\bar{1}.2$	0.01000	0.7322	0.7305	1.0023			
$\bar{1}.3$	0.01000				0.9592	0.9522	1.0074
$\bar{1}.4$	0.01000	0.8064	0.799	1.0093			
$\bar{1}.5$	0.01000	0.8281	0.8273	1.0010	0.9704	0.9694	1.0010
$\bar{1}.7$	0.01000				0.9809	0.9805	1.0004
$\bar{1}.8$	0.01000	0.9141	0.894	1.0225			
$\bar{1}.9$	0.01000				0.9995	0.9876	1.0120
0.0	0.01000	0.9477	0.9256	1.0239	1.0148	0.9901	1.0249

It is remarkable that this expansion gives good numerical results. Thus for $\rho\eta=0.1000$ the ratio of the values of Θ_0 computed by this expansion to the accurate values increases from 1 to 1.0046 as $\ln(1/\eta)$ varies from $\bar{1}.0$ to 0.0. There is a similar good agreement for $\rho\eta=0.3162$. For $\rho\eta=0.10$ and $\ln(1/\eta)=0.5$ the ratio is 1.3 without using the term in η^{-6} and 2.4 if the term in η^{-6} is used. For such high energies the expansion is not satisfactory but it is apparently safe and accurate in the low energy range.

In applications of the functions one needs the values of F, G as well as the values of the logarithmic derivatives F'/F and G'/G. In view of the fact that the WKB method of approximation is popular and expedient we compare here in some special cases the WKB approximations with the exact values. Figs. 3 and 4 give ratios of approximations to the accurate values of F and G for $L=0$ and for two fixed values of

TABLE VI. *Values of $-2(\nu^2+x^2)^{\frac{1}{2}}I_\nu K_\nu$ for $\nu=1$, $\nu=3$.*

$x=(8\rho\eta)^{\frac{1}{2}}$	$\nu=1$	x	$\nu=3$
0.02	0.99935	0.1	1.00192
0.1	0.99155	0.2	0.99975
0.3	0.97659	0.4	0.99895
0.6	0.95325	0.6	0.99792
0.9	0.95846	0.8	0.99677
1.0	0.96215	1.0	0.99564
2.0	0.99495	1.5	0.99364
3.0	1.00405	2.0	0.99315
4.0	1.00465	2.5	0.99377
5.0	1.00377	3.0	0.99493
6.0	1.00292	3.5	0.99620
7.0	1.00226	4.0	0.99734
8.0	1.00179	4.5	0.99917
9.0	1.00144	5.0	0.99900
10.0	1.00118		
13.0	1.00072		
16.0	1.00048		

the parameter η. It will be noted that the $(L+\frac{1}{2})^2$ modification of the approximation method is in these cases vastly superior to the direct use of $L(L+1)$ in the formulas. The potential barrier in Fig. 3 is at $\rho=2$ and thus corresponds to a point lying $1\frac{1}{2}$ of the marked divisions outside the right end of the figure. For Fig. 4 the potential barrier is reached at log $\rho=\bar{1}.801$. It will be noted that the $(L+\frac{1}{2})^2$ approximation is satisfactory at the barrier and even at values of ρ exceeding 2η. On the other hand the ordinary WKB method breaks down in the well-known manner at the barrier. In Fig. 5 the radius is kept constant and the energy is varied for $\rho\eta=0.631$, $L=0$. It will be noted that here also the $(L+\frac{1}{2})^2$ approximation is superior to that using $L(L+1)$ but at the same time it is clear that even this WKB approximation is consistently incorrect at small energies. This is to be expected from the limiting values obtained by means of the Carlini formula. The barrier is reached in this figure at log $\rho=0.05$. As the barrier is approached there is not much choice between the $(L+\frac{1}{2})^2$ and the $L(L+1)$ approximations for G. In Tables VII and VIII similar comparisons are made for $L=1$. The end column refers to the value of $3(F'G-G'F)$ which corresponds to the values of F', F, G', G used in these calculations. If these values were exact all numbers in the last column would be exactly 3. The amount by which they differ from 3 may be used as a criterion of accuracy. Table VII corresponds to a fixed radius and Table VIII to a fixed energy. The approximations are usually better for $L=1$ than for $L=0$.

Comparison of approximate and exact values for logarithmic derivatives is made graphically in Figs. 6, 7, 8, 9. In these figures the exact values of $\rho F'/F=\Phi^*/\Phi$ and $\rho G'/G=\Theta^*/\Theta$ as well as WKB approximations to these quantities are plotted against ρ. Fig. 6 shows the condition for fixed energy ($\eta=1$) and $L=0$. In the range covered the use of $(L+\frac{1}{2})^2$ is better than that of $L(L+1)$. It will be noted that there is an appreciable region of small ρ where the $L(L+1)$ method gives wrong signs for G'/G. This figure should be compared with Fig. 3 for the direct values of the quantities. In Fig. 7 the radius is kept constant by keeping $\rho\eta=0.631$ for $L=0$. In this case G'/G is approximated somewhat better

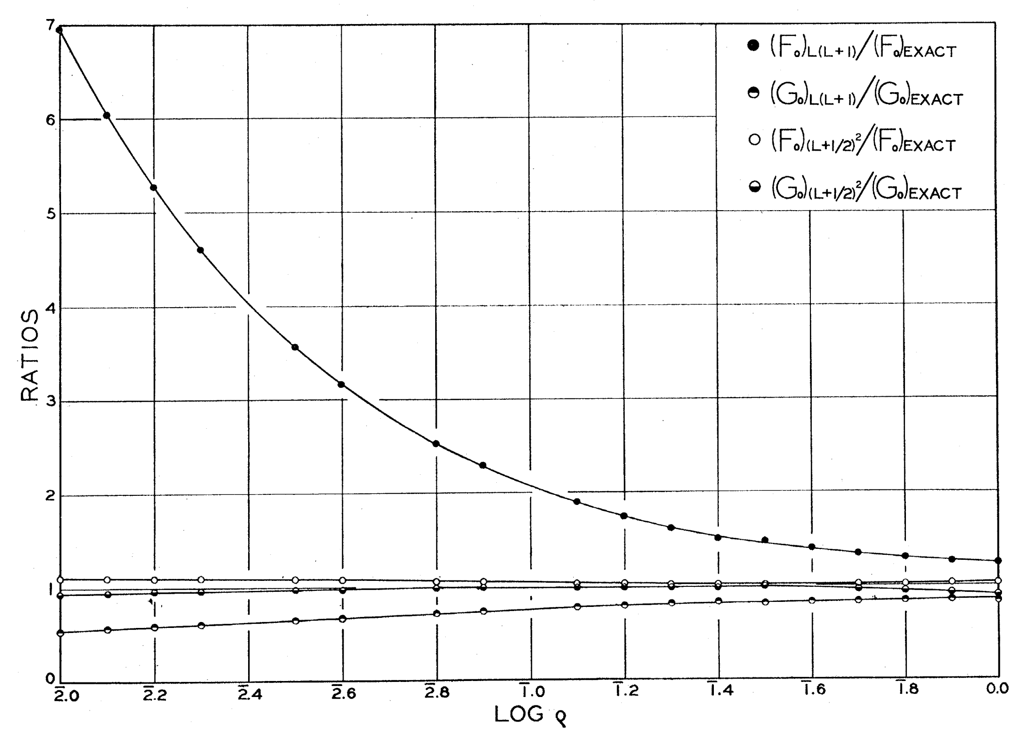

FIG. 3. Ratios of WKB approximations to exact values of F and G for $L=0$, log $\eta=0.0$.

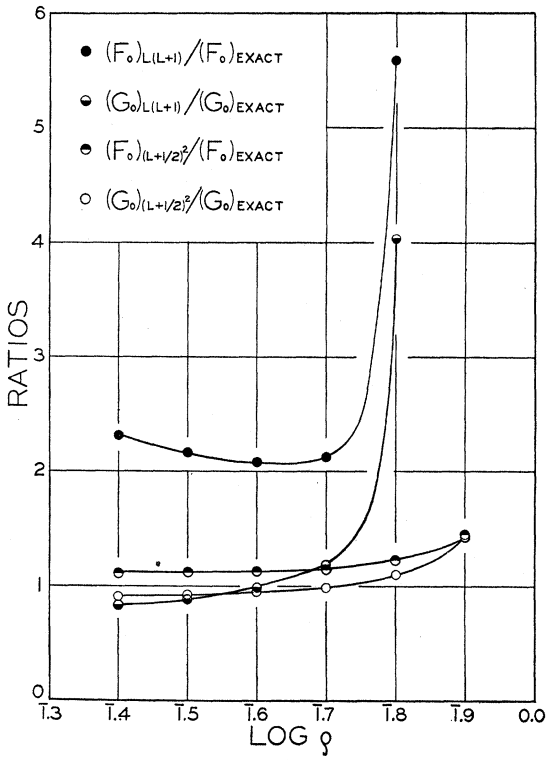

FIG. 4. Ratios of WKB approximations to exact values of F and G for $L=0$, $\log\eta=\bar{1}.5$.

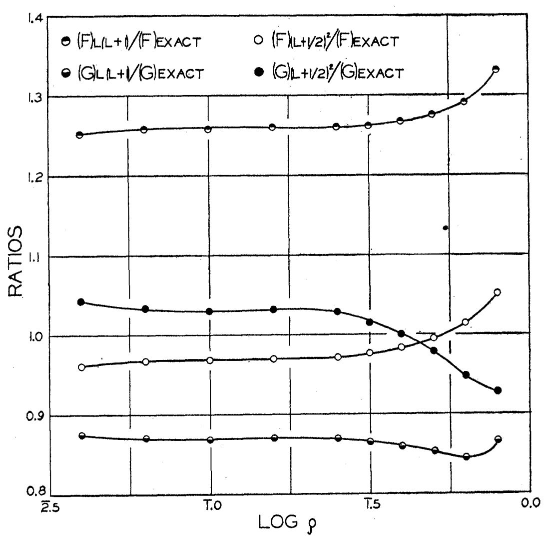

FIG. 5. Ratios of WKB approximations to exact values of F and G for $L=0$, $\rho\eta=0.631$.

by using $L(L+1)$ in the WKB formulas than by using $(L+\frac{1}{2})^2$ at large values of ρ. However, at small values of ρ the $(L+\frac{1}{2})^2$ method is beyond doubt the better. This figure corresponds to Fig. 5 in the values of the parameters used. In Fig. 8 an analogous condition is illustrated for $L=1$, $\rho\eta=1.259$. The values of the parameters used for this figure correspond to Table VII. In Fig. 9 the energy is kept constant for $L=1$, $\eta=1$. Here also the $L(L+1)$ approximation is better than the $(L+\frac{1}{2})^2$ type for G'/G and large ρ. Otherwise the $(L+\frac{1}{2})^2$ method is the better. The analogous comparison of F, G with their approximations was made in Table VIII. It will be noted that the comparisons of approximate and exact values of F and G cannot be used reliably for a discussion of the logarithmic derivatives. Comparisons for many other values of parameters were made in a manner similar to the cases presented here. The same general features are borne out by these calculations. Thus the approximations are usually better for $L=1$ than for $L=0$

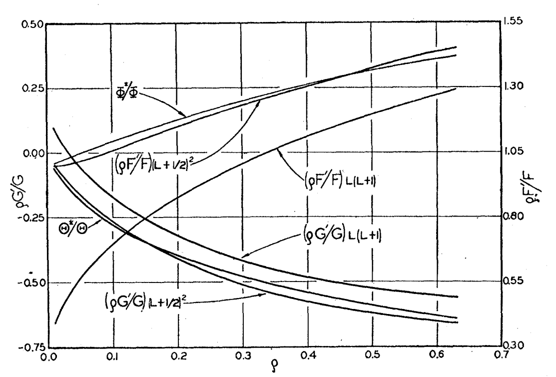

FIG. 6. Comparison of WKB approximations with exact values of F'/F, G'/G for $L=0$, $\log\eta=0.0$.

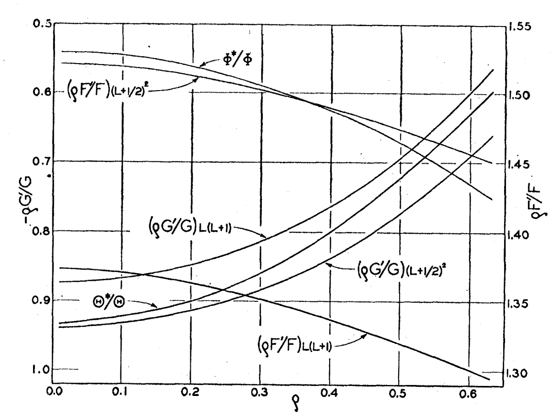

FIG. 7. Comparison of WKB approximations with exact values of F'/F, G'/G for $L=0$, $\rho\eta=0.631$.

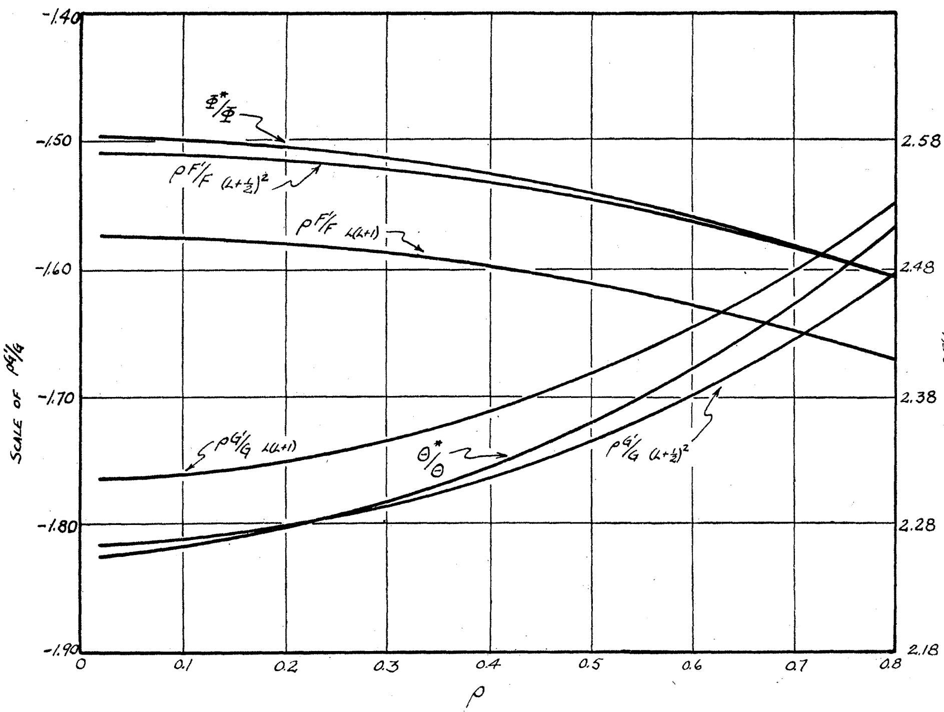

FIG. 8. Comparison of WKB approximations with exact values of F'/F, G'/G for $L=1$, $\rho\eta=1.259$.

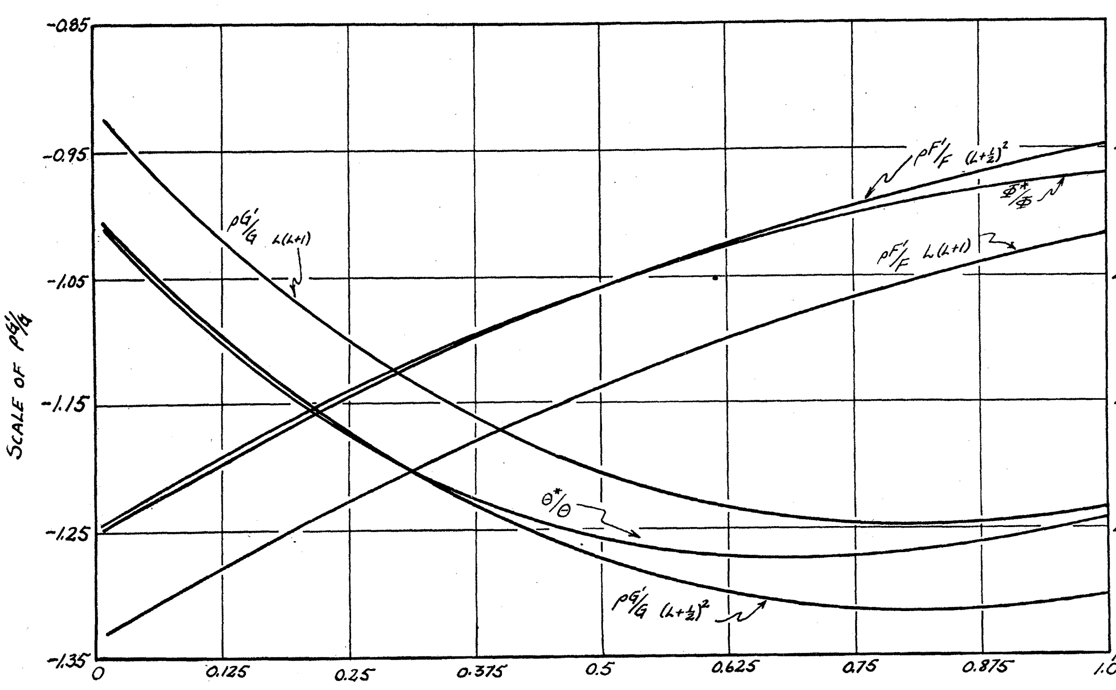

FIG. 9. Comparison of WKB approximations with exact values of F'/F, G'/G for $L=1$, $\log\eta=0.0$.

TABLE VII. *Ratios of WKB approximations to exact values of F, G for $\rho\eta = 1.259$, $L = 1$.*

ρ	η	$\frac{F_{WKB}}{F_{ex}}$ $L(L+1)$	$\frac{F_{WKB}}{F_{ex}}$ $(L+\frac{1}{2})^2$	$\frac{G_{WKB}}{G_{ex}}$ $L(L+1)$	$\frac{G_{WKB}}{G_{ex}}$ $(L+\frac{1}{2})^2$	Check
0.01585	79.44	1.14	0.96	0.91	1.04	2.9986
0.01995	63.10	1.15	0.96	0.90	1.05	3.0017
0.02512	50.12	1.16	0.99	0.89	1.01	3.0010
0.03162	39.81	1.17	1.00	0.88	1.00	3.0030
0.06310	19.95	1.17	1.01	0.88	1.00	3.0007
0.1000	12.59	1.16	1.00	0.88	1.01	3.0001
0.1259	10.00	1.16	1.00	0.89	1.01	2.9996
0.1995	6.310	1.16	1.00	0.89	1.00	2.9985
0.3162	3.981	1.16	1.00	0.89	1.00	3.0018
0.5012	2.512	1.16	1.01	0.89	0.99	2.9948
0.7944	1.585	1.16	1.02	0.88	0.97	2.9945
1.259	1.000	1.16	1.05	0.86	0.92	3.0019
1.585	0.7943	1.20	1.09	0.90	0.93	3.014

for the same ρ, η. Also the $(L+\frac{1}{2})^2$ type of approximation is usually better than the unmodified type of WKB formula. However, there are cases where the opposite is true as has been brought out above. The only absolutely unquestionable region in which the WKB formulas have an easily predictable accuracy is that of small ρ where the $(L+\frac{1}{2})^2$ method holds to within the range of validity of the Carlini formula.

In the calculation of excitation functions of nuclear reactions one is at times concerned only with relative values of F for the same radius and different energies. The above comparison shows that even the $L(L+1)$ method is not bad for such a purpose through considerable ranges of energy. Effects of resonance levels, however, are not represented very well by the approximations

TABLE VIII. *Ratios of WKB approximations to exact values of F, G for $\eta = 1$, $L = 1$.*

ρ	$\frac{F_{WKB}}{F_{ex}}$ $L(L+1)$	$\frac{F_{WKB}}{F_{ex}}$ $(L+\frac{1}{2})^2$	$\frac{G_{WKB}}{G_{ex}}$ $L(L+1)$	$\frac{G_{WKB}}{G_{ex}}$ $(L+\frac{1}{2})^2$	Check
0.01000	1.72	1.05	0.62	0.96	2.9999
0.01585	1.66	1.05	0.64	0.96	3.0000
0.03162	1.56	1.04	0.68	0.97	3.0000
0.05012	1.50	1.04	0.71	0.96	3.0010
0.06310	1.50	1.04	0.70	0.96	3.0000
0.07944	1.44	1.04	0.74	0.96	2.9998
0.1000	1.41	1.04	0.75	0.96	3.0000
0.1259	1.36	1.04	0.78	0.96	3.0000
0.1995	1.33	1.04	0.79	0.96	3.0000
0.3162	1.28	1.04	0.82	0.97	3.0001
0.5012	1.23	1.04	0.85	0.96	3.0001
0.6310	1.21	1.04	0.86	0.96	2.9991
0.7944	1.19	1.04	0.86	0.95	2.9998
1.000	1.17	1.04	0.87	0.94	3.0063
1.259	1.16	1.05	0.86	0.92	3.0019
1.585	1.17	1.07	0.88	0.93	3.050
1.995	1.19	1.11	0.97	0.99	3.64

because these depend on values of F'/F.

In Table VI we give values of $-2(\nu^2+x^2)^{\frac{1}{2}} \times I_\nu(x)K_\nu(x)$ for $L=0$, 1 which give an idea of the accuracy of the approximate Eq. (38‴). By means of the relation

$$\hat{F}\hat{G}/(\hat{F}\hat{G})_{\mathrm{WKB}(L+\frac{1}{2})^2} = -2(\nu^2+x^2)^{\frac{1}{2}} I_\nu K_\nu$$

and this table one obtains the accuracy of the WKB approximation for FG in the limit of small ρ and finite $\rho\eta$. Table III gives directly $(\hat{F})_{\mathrm{WKB}(L+\frac{1}{2})^2}/\hat{F}$ and by multiplying corresponding numbers in the two tables one obtains $\hat{G}/(\hat{G})_{\mathrm{WKB}(L+\frac{1}{2})^2}$. In this limit the approximation is seen to be good to within a few percent and generally better for $L=1$ than for $L=0$.

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Probability of Radiative Processes for Very High Energies

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THE present theory of radiative processes[1] (e.g., radiation of electrons and pair production by photons when passing atomic nuclei), though in quantitative agreement with observations up to several MV, gives far too large effects for higher energies and is therefore not applicable to the discussion of cosmic-ray phenomena. The reasons for this failure can be found in (a) the inapplicability already of classical electrodynamics for too high or too rapidly varying fields; (b) difficulties in the simultaneous application of the principles of quantum mechanics and relativity for field strengths above the critical limit

$$E_c = m^2c^3/e\hbar, \tag{1}$$

that is a potential difference mc^2/e through a distance $\hbar/mc$; (c) breakdown of the superposition principle in Dirac's hole theory at the same limit.

[1] H. Bethe and W. Heitler, Proc. Roy. Soc. **A146**, 183 (1934).