addition, a number of explorations have been made with negative results. Naturally the greatest amount of effort was spent in trying to find the transition in the sodium series. NaCl gave nothing at room temperature with two different set-ups, the first to 32,000 and the second to 50,000. NaI is the member of the series that would be expected to show the transition most readily; this gave negative results at room temperature with four different set-ups, to 46,000, 49,000, 48,000, and 57,000, respectively. However, it has already been mentioned in the previous paper that the shearing curve of NaI begins to bend over at 50,000, and I believe the transition is not far away.

Another interesting salt of the alkali halides series is CsF; this crystallizes with the NaCl structure, although all the other caesium salts are body-centered cubic. One might expect to be able to force the transition to the body-centered structure by high pressure. Two separate set-ups with CsF, for which I am again indebted to Dr. R. W. G. Wyckoff, gave negative results, the first to 42,000 at room temperature, and the second to 56,000 at 137°. It would thus appear that the fluorides are qualitatively different from the other halides. In view of the failure to obtain the transition with CsF, it seemed hardly worth while to explore the fluorides of any of the other alkali metals.

In addition to these, sugar gave negative results to 40,000 at room temperature, and BaS to 39,000 at room temperature. PbTe, on the other hand, probably has a small transition at a mean temperature of 18,000, but this is not certain; there is no further transition up to 39,000. The explorations mentioned in this paragraph were made with a preliminary form of apparatus, only one-third as sensitive as the final form.

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Intranuclear Forces

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Quantitative information as to the nature of intranuclear forces has been obtained by considering the hydrogen and helium isotopes, using a Hamiltonian of the exchange type based on the neutron-proton model with interactions between all the particles. The interactions between like particles are taken to be the same for protons and for neutrons except for the small Coulomb force between protons which accounts for at least half of the difference in the binding energies of He^3 and H^3. It is not inconsistent with existing experimental evidence to assume the range of such forces to be about the same as the range of neutron-proton forces. By making these assumptions and by using simple attractive potentials, it is found that the forces between like particles are given by a potential with depth not greater than 30 mc^2 and with an effective radius of action which must be less than 2.8×10^{-13} cm to be consistent with experimental data for the binding energies. A more accurate determination of the constants based on assuming a close correlation of the three- and four-body problems with "equivalent" two-body problems gives the depth of the proton-neutron potential as 74 mc^2 and of the like-particle potentials as 26 mc^2 and a range of 2.2×10^{-13} cm for the forces. Reasons for believing in the reliability of the "equivalent" two-body method for determining approximate binding energies for three- and four-body problems are discussed. It is shown furthermore that if the exchange operators multiplying the neutron-proton interaction potentials are assumed to be linear combinations of the Majorana and Heisenberg types, it is possible to explain the large scattering of neutrons on protons at low velocities. The proportion of the Heisenberg to the Majorana operator in the linear combination necessary to obtain the correct scattering is about one-fifth. It is then shown that in the three- and four-body problems such an operator can to a good approximation be represented by an equivalent Majorana operator and that because of this fact the results obtained for nuclear energies and for the magnitude and range of the forces (in which Majorana operators were used) remain the same except that the depth of the potential for like particles is increased from 26 to 41 mc^2. The latter value yields a scattering intensity for protons in hydrogen which agrees qualitatively with the experimental results.

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Section I. Introduction

WE propose to use the binding energies of the hydrogen and helium isotopes to fix the parameters in a simple exchange type nuclear Hamiltonian based on the neutron-proton model with interactions between all pairs of particles. Several authors have found evidence for large interactions between like particles in the odd-even properties of nuclei.[1] In a previous investigation[2] the same problem was studied, but subject to the assumption, which we do not retain, that the interaction between like particles could be treated as a small perturbation in comparison with the neutron-proton interaction.

The Hamiltonian has the form

$$H=-\tfrac{1}{2}\Sigma\Delta_i-\Sigma J(r_{\nu_i\pi_j})P_{\nu_i\pi_j}-\Sigma' K(r_{\nu_i\nu_j})P'_{\nu_i\nu_j}$$
$$-\Sigma' L(r_{\pi_i\pi_j})P'_{\pi_i\pi_j}+(2\pi)^{1/2}/8\Sigma' 1/r_{\pi_i\pi_j} \quad (1)$$

with the units $m_e c^2$ (510,000 ev) for energy and $h/2\pi c(m_e m_\pi)^{1/2}$ (8.97×10^{-13} cm) for length. There exist several possibilities for the exchange operators P and P' depending on the general character of the forces between the nuclear particles. If these forces are first attractive as two particles approach and then strongly repulsive upon closer approach, the exchange operators may be replaced by unity. The inner repulsive regions prevent the collapse of heavy nuclei which would otherwise occur when the forces are of the ordinary (Wigner) type. Because of the mathematical difficulties connected with the correct treatment of the repulsive regions, this model has not yet been adequately investigated.

If the inner repulsive regions are lacking, the burden of keeping the nuclear volume and binding energy proportional to the total number of particles is put on the exchange operators. These may have the form[3]

$$P_{ij}=(1-g)P_{ij}{}^M+gP_{ij}{}^H, \quad (2)$$

a linear combination of the Majorana and Heisenberg exchange operators, and[4]

$$P_{ij}{}'=\tfrac{1}{3}(1+2P_{ij}{}^M)=-\tfrac{1}{3}\sigma_i\cdot\sigma_j. \quad (3)$$

Because the wave functions for the hydrogen and helium isotopes are symmetrical in the space coordinates of like particles, we may replace P' by unity for our purposes without loss of generality. It is known from the calculations in FI that the effective width and depth of the neutron-proton potential well required to account for the binding energies of the deuteron and the alpha-particle depend only slightly on whether the forces are of the Wigner or of the Majorana type. Thus, if g is small in comparison with unity, it is possible to compare directly the exchange type nuclear model for the hydrogen and helium isotopes with a model based on a Hamiltonian containing only ordinary forces. For this reason a calculation with the exchange model has a certain generality in that it gives at once quantitative information about the alternative model with attractive forces of the Wigner type and also yields a good qualitative picture of the magnitude of the attractive potential wells when these are associated with inner repulsive regions.

Section II. Potentials

To obtain a manageable problem we set $K(r)$ equal to $L(r)$ and suppose that $K(r)$ and $J(r)$ are proportional. The first assumption finds some support in the fact that at least half of the small difference between the binding energies of H^3 and He^3 can be accounted for as resulting from the Coulomb interaction between the protons in He^3. The second assumption is arbitrary, but does not conflict with the experimental evidence as yet available. The definition of the Hamiltonian is completed by the assumption

$$J(r)=A_{\nu\pi}e^{-\alpha r^2}, \quad K(r)=L(r)=A_\nu e^{-\alpha r^2}. \quad (4)$$

The experimental energies have the values[5] $E(H^2)\leqq-4.0$, $E(H^3)\leqq-15.8$, $E(He^4)\leqq-54.0$. From the deuteron equation, $H\psi=E\psi$ with

[1] K. Guggenheim, J. de physique **5**, 253 (1934); L. A. Young, Phys. Rev. **47**, 972 (1935); H. Bethe, unpublished lectures.

[2] E. Feenberg, Phys. Rev. **47**, 850 (1935), denoted by FI in text; Phys. Rev. **47**, 857 (1935), denoted by FII in text. For discussion of related problems see L. H. Thomas, Phys. Rev. **47**, 903 (1935) and W. V. Houston, Phys. Rev. **47**, 942 (1935).

[3] $P_{ij}{}^M$ interchanges the space coordinates i and j in a function to which it is applied; $P_{ij}{}^H$ interchanges both space and spin coordinates.

[4] In Eq. (3) σ_i represents the Pauli spin operator for the ith particle. The equivalence of the two expressions for $P_{ij}{}'$ can be deduced from a relation derived by Dirac in his *Quantum Mechanics*, first edition, p. 215, Eq. (35). See also Van Vleck, Phys. Rev. **48**, 367 (1935) for a discussion of the operator $P_{ij}{}'$.

[5] Chadwick and Goldhaber, Nature **134**, 237 (1934); Oliphant, Kempton and Rutherford, Proc. Roy. Soc. **A149**, 406 (1935).

TABLE I. $A_{\nu\pi}(\alpha)$ *for* $E(II^2)=-4.0$.

$A_{\nu\pi}/\alpha$	3.45	3.66	3.88	4.11	4.35	4.60	4.85
$1/\alpha^{1/2}$	0.12	0.15	0.18	0.21	0.24	0.27	0.30
$A_{\nu\pi}/\alpha$	5.10	5.36	5.71	6.16	6.65	7.17	7.70
$1/\alpha^{1/2}$	0.33	0.36	0.40	0.45	0.50	0.55	0.60

$E=-4.0$, $A_{\nu\pi}$ is obtained as a function of α by means of a series of numerical integrations: Table I makes possible the ready determination of the lowest eigenvalue of any two-particle Schrödinger equation with an arbitrary error function type of potential[6] and will be used for that purpose in Section IV. The function $A_{\nu\pi}(\alpha)$ does not depend on the value assigned to g. This is true because the wave function ψ for the deuteron is symmetrical in both the space and spin coordinates of the two particles and hence in this case

$$\{(1-g)P^M+gP^H\}\psi=\psi. \tag{5}$$

When, however, the spins of unlike particles are not completely parallel, the properties of the model do depend on g. In Sections III and IV the discussion is based on the simplifying assumption that the forces between unlike particles are of the pure Majorana type ($g=0$). The general case in which g differs from zero is considered in Section V.

SECTION III. THE SIMPLE VARIATIONAL METHOD

With the Gaussian type of wave function

$$\varphi_0(1;2,3)=Ne^{-(\nu/2)(r_{12}^2+r_{13}^2)-(\mu/2)r_{23}^2},\ (\mathrm{H}^3), \tag{6}$$

$$\varphi_0(1,2;3,4)$$

$$=Ne^{-(\nu/2)(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2)-(\mu/2)(r_{12}^2+r_{34}^2)},\ (\mathrm{He}^4), \tag{7}$$

the average value of the Hamiltonian operator (omitting the Coulomb terms and setting $g=0$) can be written

$$E^0(\mathrm{H}^3)=(2+p)\alpha\sigma-2A_{\nu\pi}\{p(4-p)/(1+2p)\}^{3/2}\times(\sigma/(\sigma+1))^{3/2}-A_\nu(p\sigma/(p\sigma+1))^{3/2} \tag{8}$$

for the three-body problem and

$$E^0(\mathrm{He}^4)=(3/2)(2+p)\alpha\sigma-4A_{\nu\pi}(2p-p^2)^{3/2}\times(\sigma/(\sigma+1))^{3/2}-2A_\nu(p\sigma/(p\sigma+1))^{3/2} \tag{9}$$

for the alpha-particle. These differ from the corresponding expressions in FI (Eqs. (18) and (19)) in the presence of the term proportional to A_ν and in a more suitable choice of the variation parameters.[7] The contribution from the Coulomb interaction to the energy has the form

$$\Delta E=\tfrac{1}{4}(2\alpha p\sigma)^{1/2} \tag{10}$$

for both He^3 and He^4. The values of p and σ which minimize (8) and (9) must be found by trial. However the dependence on p is slight, the best value generally being between 0.9 and 1.0.

Let $A_\nu(\alpha, \mathrm{H}^3)$ and $A_\nu(\alpha, \mathrm{He}^4)$ represent the functions defined by the conditions that the accurate lowest eigenvalues of the three- and four-body problems have the experimental values $E(\mathrm{H}^3)=-15.8$ and $E(\mathrm{He}^4)=-54.0$, respectively. Eqs. (8) and (9) can be used to obtain upper bounds on $A_\nu(\alpha, \mathrm{H}^3)$ and $A_\nu(\alpha, \mathrm{He}^4)$. Consider first Eq. (9). There exists a function $A_\nu{}^0(\alpha, \mathrm{He}^4)$ such that if A_ν is replaced by $A_\nu{}^0(\alpha, \mathrm{He}^4)$ in (9) the minimum value of $E^0(\mathrm{He}^4)+\Delta E$ with respect to p and σ coincides with the experimental energy. By the minimum property of the variation method, we have

$$A_\nu(\alpha, \mathrm{He}^4)\leqq A_\nu{}^0(\alpha, \mathrm{He}^4). \tag{11}$$

We find that $A_\nu{}^0(\alpha, \mathrm{He}^4)$ is practically constant with the value 30 mc^2 on the range $10\leqq\alpha\leqq50$. In the same way we find that $A_\nu{}^0(\alpha, \mathrm{H}^3)$ varies from the value 30 mc^2 at $\alpha=10$ to 100 mc^2 at $\alpha=50$. Furthermore the difference $A_\nu{}^0(\alpha, \mathrm{H}^3)-A_\nu(\alpha, \mathrm{H}^3)$ is much greater than the corresponding difference $A_\nu{}^0(\alpha, \mathrm{He}^4)-A_\nu(\alpha, \mathrm{He}^4)$. This can be seen from the fact that the binding energy in the three-body problem is a small difference between large potential and kinetic energy terms. Hence the energy is sensitive to errors in the wave function. The same statement is true for the alpha-particle, but in a much less extreme degree. Thus the correct value of A_ν is less than 30 mc^2 and the correct value of α is greater than 10 ($1/\alpha^{1/2}<2.8\times10^{-13}$ cm).

SECTION IV. THE "EQUIVALENT" TWO-BODY METHOD

To improve upon these inequalities we have recourse to the method of the "equivalent" two-

[6] FI, Eq. (4).

[7] In Eq. (8) $4\alpha\sigma=5\nu+\mu$, $2\alpha\sigma p=\nu+2\mu$; in Eq. (9) $2\alpha\sigma=3\nu+\mu$, $\alpha\sigma p=\nu+\mu$.

body problem described and used in FI. The method is not rigorous, but its general correctness is supported by the considerations advanced in FI and by the results of attempts to improve on Eqs. (8) and (9) with more complicated variational calculations (see appendix). The procedure involves first replacing p by unity in both (8) and (9) in order to reduce the two different potential energy terms to the same form. In FI because there was only one kind of potential energy term, this simplification was unnecessary. The resulting simplified expressions have exactly the form taken by the corresponding expression for the variational problem associated with the equation

$$\{d^2/dr^2+E'+Be^{-\beta r^2}\}\varphi=0 \qquad (12)$$

when the approximate wave function $\varphi=e^{-\nu r^2/2}$ is used to evaluate the energy integral. The formal analogy is made exact by the identifications

$$\left.\begin{aligned}\beta&=2\alpha\\ B&=2A_{\nu\pi}+A_\nu\end{aligned}\right\}\mathrm{H}^3,\qquad \left.\begin{aligned}\beta&=3\alpha\\ B&=2(2A_{\nu\pi}+A_\nu)\end{aligned}\right\}\mathrm{He}^4. \qquad (13)$$

These identifications establish a correspondence between the three and four-body problems and "equivalent" two-body problems which can be solved accurately. To proceed further we insert into (12) the experimental value for E' and, using Table I, compute B and hence A_ν as a function of α for both H^3 and the alpha-particle. We designate these functions by the symbols $A_\nu'(\alpha, H^3)$ and $A'(\alpha, He^4)$. Small corrections to make up for the inaccuracy resulting from setting $p=1$ are easily determined by means of Eqs. (8) and (9). Results are exhibited in Tables II and III. Table II includes a column giving the binding energy of H^3 as a function of α when $A_\nu(\alpha)=A_\nu'(\alpha, He^4)$. At the point of intersection $(A_\nu'(\alpha, H^3)=A_\nu'(\alpha, He^4))$ α has the value 17 $(1/\alpha^{1/2}=2.17\times10^{-13}$ cm), $A_{\nu\pi}=74.3$ mc^2 and $A_\nu=26$ mc^2. The figures given by Young[1] can be compared with the $\alpha=10$ column in Table II by noting that $\frac{1}{3}(2A_{\nu\pi}+A_\nu)$ corresponds to the depth of the rectangular potential function used in Young's calculation. There is satisfactory qualitative agreement.

All the results of this and the preceding sections can be extended without difficulty to other forms of the attractive potentials, in particular, to the rectangular potential well and the straight exponential function. The interesting generalization of taking different radii of action for the forces between like and between unlike particles can also be treated by the same methods, but would involve a large amount of numerical computation.

TABLE II. *The "equivalent" two-body method for the determination of $A_\nu'(\alpha, He^4)$.*

α	$A_{\nu\pi}$	A_ν'	$-E'(He^4)$	C.E.(He^4)*	$-E'(H^3)$
10	49.7	29	55.4	1.4	
20	84.4	24	55.6	1.6	15.2
30	117.1	19	55.8	1.8	13.7
40	148.8	12	56.0	2.0	12.8

TABLE III. *The "equivalent" two-body method for the determination of $A'(\alpha, H^3)$.*

α	$A_{\nu\pi}$	A_ν'	$-E'(H^3)$	C.E.(He^3)*
10	49.7	22	15.8	1.2
20	84.4	27	15.8	1.4
30	117.1	31	15.8	1.5
40	148.8	33	15.8	1.7

* C. E. in Tables II and III—Coulomb energy.

SECTION V. THE MIXED EXCHANGE OPERATOR

The experimental cross section for the elastic scattering of slow neutrons in hydrogen has about the value[8] 30×10^{-24} cm^2, which exceeds by a factor of eight the theoretical cross section[9] computed from the neutron-proton potential determined in the preceding section. One of the writers has attempted to explain this descrepency by supposing that the potential depends on the state of the system and on the velocity of the colliding particles in such a way that the interaction potential for scattering is smaller than for binding and decreases with increasing relative velocity.[10] This hypothesis, if correct, should apply equally well to the interaction of two protons. Thus we should expect the anomalous proton-proton interaction potential required to explain the anomalous scattering of protons in hydrogen[11] to be smaller than $K(r)$. In fact $K(r)$ is very much too small to explain White's results.[12] Thus the

[8] Dunning, Pegram, Fink and Mitchell, Phys. Rev. **47**, 970 (1935); Bjerge and Westcott, Proc. Roy. Soc. **A150**, 790 (1935).
[9] FII, Eq. (5).
[10] FII.
[11] M. G. White, Phys. Rev. **47**, 573 (1935).
[12] R. D. Present, to be submitted to the Physical Review.

hypothesis advanced in FII must be abandoned.

Wigner[13] has suggested that the very large experimental neutron-proton scattering cross section indicates a dependence of the interaction potential on spin orientation. Eq. (2) represents the mathematical formulation of Wigner's suggestion.[14] In the scattering problem based on the exchange operator of Eq. (2), there occur four distinct wave functions:

$$\psi_1(12)=\varphi(r_{12})\delta(m_1,\tfrac{1}{2})\delta(m_2,\tfrac{1}{2}), \tag{14}$$

$$\psi_2(12)=\varphi_a(r_{12})\delta(m_1,\tfrac{1}{2})\delta(m_2,-\tfrac{1}{2}) +\varphi_b(r_{12})\delta(m_1,-\tfrac{1}{2})\delta(m_2,\tfrac{1}{2}), \tag{15}$$

and two additional functions obtained by reversing the directions of the spin coordinates in ψ_1 and ψ_2. It is readily found that φ and $\varphi_a+\varphi_b$ are solutions of the equation

$$\{\nabla_{12}^2+W+J(r_{12})P_{12}^M\}\psi(r_{12})=0, \tag{16}$$

while $\varphi_a-\varphi_b$ satisfies the equation

$$\{\nabla_{12}^2+W+(1-2g)J(r_{12})P_{12}^M\}\psi(r_{12})=0. \tag{17}$$

An appropriate set of boundary conditions is provided by the following statement: the functions φ and φ_a have the form of an incident plane wave plus a scattered wave: the incident wave does not occur in φ_b which contains only a scattered wave. The scattering intensity obtained from ψ_2 is simply the sum of the intensities associated with φ_a and φ_b because there is no interference between scattered waves associated with orthogonal spin wave functions. From this fact can be deduced that the statistical weights of the scattering intensities obtained from the potential fields $J(r)$ and $(1-2g)(Jr)$ are in the ratio three to one. The scattering cross section for slow neutrons can be made as large as desired by choosing g to bring the lowest eigenvalue of Eq. (17) sufficiently near the top of the potential hole, or, equally well, by taking g so large that Eq. (17) just fails to have a discrete state. For the effective radius determined in the preceding section these conditions are met with $g\sim 0.2$.

The effect of the mixed operator on the three- and four-body problems may be studied most simply when g is small by means of a first order perturbation calculation. When g vanishes, the alpha-particle wave function can be written in the form

$$\psi(1,2;3,4)=\varphi(x_1,x_2;x_3,x_4)\times S(m_1,m_2)S(m_3,m_4) \tag{18}$$

with

$$S(m_1,m_2)=1/2^{1/2}\{\delta(m_1,\tfrac{1}{2})\delta(m_2,-\tfrac{1}{2}) -\delta(m_2,\tfrac{1}{2})\delta(m_1,-\tfrac{1}{2})\}. \tag{19}$$

The average value of the mixed neutron-proton potential is simply

$$4\sum_{\text{Spin}}\int\cdots\int\psi(1,2;3,4) \times\{(1-g)P_{13}^M+gP_{13}^H\}\psi(1,2;3,4)d\tau, \tag{20}$$

which reduces to

$$4(1-g/2)\int\cdots\int\varphi(x_1,x_2;x_3,x_4) \times J(r_{13})P_{13}^M\varphi(x_1,x_2;x_3,x_4)d\tau \tag{21}$$

when the summation over the spin coordinates is carried out. Thus in this problem for small values of g the mixed operator can be replaced by an "equivalent" pure Majorana operator:

$$(1-g)P^M+gP^H\rightarrow(1-g/2)P^M. \tag{22}$$

The same result is obtained from the analogous treatment of the three-body problem.

It is clear from Eq. (13) that the calculations in Section IV determine essentially the quantity $2A_{\nu\pi}+A_\nu$ and not A_ν directly. Consequently

$$2A_{\nu\pi}+A_\nu\sim(2-g)A_{\nu\pi}+\bar{A}_\nu, \tag{23}$$

and

$$\bar{A}_\nu\sim A_\nu+gA_{\nu\pi} \tag{24}$$

if $\bar{A}_\nu$ is written for the amplitude of the interaction between like particles when g differs from zero. With $1/\alpha^{1/2}\sim 2.2\times 10^{-13}$ cm, $A_\nu\sim 26$, $A_{\nu\pi}\sim 74$ and $g\sim 0.2$, we get

$$\bar{A}_\nu\sim 41mc^2. \tag{25}$$

Present[12] has computed the anomalous scattering of protons in hydrogen to be expected from

[13] Unpublished; mentioned by Professor Bethe at the Washington Conference on Nuclear Physics.

[14] Van Vleck, Phys. Rev. **48**, 367 (1935), note 20.

the potential defined by Eq. (25) and finds rough qualitative agreement with White's measurements.[11] To summarize: Wigner's suggestion that the neutron-proton interaction potential depend on spin orientation does not appreciably change the binding energy problem, but does enable us to understand the remarkable experimental results on the scattering of slow neutrons and fast protons in hydrogen. It should be mentioned that the theoretical cross section for elastic scattering of fast neutrons in hydrogen is somewhat larger than the rather uncertain experimental value.

Appendix

Comparison of Improved Variational Calculations for Two- and Three-Body Problems

Further justification for the use of the method of the "equivalent" two-body problem for calculating energies for three-body problems is obtained when the results of attempts to improve the variational calculations in the two problems are compared. The "equivalent" two-body problem is determined in both the Wigner and Majorana cases by identifying certain constants in the expressions for the energy obtained when simple Gaussian-type wave functions are used. This identification is possible because the energy expressions in this "zeroth" approximation have the same form.

We have used three different methods for improving the variational calculations beyond the zeroth stage. In the first method the original Gaussian functions are multiplied by polynomials in the various distances. For the two-body problem we then have for the wave function (taking only quadratic terms)

$$\varphi(r)=e^{-(\nu r^2/2)}\{a_0+a_1\nu r^2\}. \tag{26}$$

A similar wave function for the three-body problem is given by the expression

$$\varphi(1,2,3)=e^{-(\nu/2)(r_{12}^2+r_{13}^2)-(\mu/2)r_{23}^2}\{C_0+C_1\nu(r_{12}^2+r_{13}^2)+C_2\nu r_{23}^2\}. \tag{27}$$

The calculations were made by using the Majorana operator for the three-body problem (H^3) including the neutron-neutron attraction term. The parameter p was set equal to unity in order to simplify the calculation of the matrix elements. With a given β, B was determined to give $E'=-16$ for the exact eigenvalue of the "equivalent" problem. The corresponding values of $A_{\nu\pi}$, A_ν, α for the three-body problem are taken from Eq. (13). In the three-body problem, the inclusion of the quadratic terms gives an improvement in the energy of only a small fraction of an mc^2 unit over the value obtained with the simple Gaussian function. The fact that the more flexible three-parameter function is no better than the two-parameter Gaussian function simply means that the quadratic terms serve only to correct ν and μ. When these parameters are both varied, the quadratic terms lose their importance. Correspondingly, for the "equivalent" two-body problem, the quadratic term gives no appreciable improvement. This result is of value because it brings out the closeness of the correlation between the three-particle problem and the associated "equivalent" problem.

In general, computations with Majorana interaction terms in the Hamiltonian are more involved than those with the ordinary Wigner type of potential. Moreover, as we show in the next paragraph, there is a Wigner problem closely related to every Majorana problem. For the purpose of showing the close correlation of three-body problems to their "equivalent" two-body problems it is just as instructive and very much simpler to use Wigner interaction potentials.

Let us consider the following closely related three-body problems. Call problem (a) a problem in which we have only neutron-proton potential terms of the Wigner type in the Hamiltonian:

$$-Ae^{-\alpha r_{12}^2}-Ae^{-\alpha r_{13}^2}. \tag{28}$$

In problem (b) take only neutron-proton potential terms of the Majorana type:

$$-Ae^{-\alpha r_{12}^2}P_{12}^M-Ae^{-\alpha r_{13}^2}P_{13}^M. \tag{29}$$

Let problem (c) be the same as (b) except for the addition of a neutron-neutron potential term to the Hamiltonian:

$$-(A-C)e^{-\alpha r_{12}^2}P_{12}^M-(A-C)e^{-\alpha r_{13}^2}P_{13}^M-2Ce^{-\alpha r_{23}^2}. \tag{30}$$

The close relation between problems (a) and (b) was shown in FI; (a) gives slightly lower energies than (b). On the other hand, problems (b) and (c) are both associated with the same "equivalent" two-body problem provided the parameter p is set equal to unity since then for both $B=2A$. Problem (c) gives slightly lower energies than problem (b) since in passing from the "equivalent" problem for (c) to that for (b) the parameter p is fixed at unity. We thus see that problem (a) and problem (c) are very closely related. For the rest of the discussion we shall use the simple Wigner-type Hamiltonian of problem (a) since the results so obtained are quite general and because to do so greatly simplifies the computations.

We sought to improve the variational calculations by using wave functions which are the sum of functions of the Gaussian type but with different ν's and μ's. If we take

$$\psi=\psi\sigma+\lambda\psi\sigma', \tag{31}$$

then appropriate wave functions for the two-body problem are obtained by substituting for $\psi\sigma$

$$\psi\sigma=Ne^{-(\nu r^2/2)},\qquad \nu=\beta\sigma, \tag{32}$$

and for $\psi\sigma'$ a similar expression having $\nu'=\beta\sigma'$. The matrix elements then are

$$(\sigma/1/\sigma')_2=\{4\sigma\sigma'/(\sigma+\sigma')^2\}^{3/4}\equiv\gamma,$$
$$(\sigma/\mathrm{H}/\sigma')_2=\gamma\{3\beta\sigma\sigma'/(\sigma+\sigma')-B((\sigma+\sigma')/(\sigma+\sigma'+2))^{3/2}\}. \tag{33}$$

In our wave function for the three-body problem we use for $\psi\sigma$ the function given by Eq. (6), with $\nu=\alpha\sigma(1+n)/(1+2n)$ $n=\mu/\nu$, and for $\psi\sigma'$ a similar expression. The matrix elements are simplified if we specialize the function by taking $n'=\mu'/\nu'=n$, thus reducing the number of parameters by one. This simplification makes it possible to put the matrix elements in the same form as those of the two-body problem. In fact we have

$$(\sigma/1/\sigma')_3=\{4\sigma\sigma'/(\sigma+\sigma')^2\}^{3/2}=\gamma^2,$$
$$(\sigma/\mathrm{H}/\sigma')_3=\gamma^2\left\{3\alpha\frac{2+3n+n^2}{1+2n}\frac{\sigma\sigma'}{\sigma+\sigma'}-2A\left(\frac{\sigma+\sigma'}{\sigma+\sigma'+2}\right)^{3/2}\right\}. \tag{34}$$

The value of n which minimizes the zeroth approximation $(\sigma/\mathrm{H}/\sigma)_3$ to the three-body energy is very nearly $\frac{1}{3}$, independent of A, α and σ. As in the case of the polynomial method, B and β are determined so that E' for the two-body problem has the value -16. A and α are such as to establish an "equivalence" between the two problems in the zeroth order and are given by the relations

$$\beta=(28/15)\alpha, \qquad B=2A. \tag{35}$$

Although A is the amplitude of the neutron-proton interaction, it is not directly connected with the $A_{\nu\pi}$ considered in Section IV since here we have a Wigner Hamiltonian and no interaction between like particles. In this problem A plays the same role as $A_{\nu\pi}+\frac{1}{2}A_\nu$ in the actual problem. Taking the same value of σ and n throughout (σ, best value for the zeroth approximation, $n=\frac{1}{3}$) and varying σ' we obtain about 19 and 17 percent improvement in the two- and three-body problems, respectively. If σ, σ' and n are all varied the improvement is 51 percent in the three-body problem and 86 percent in the two-body problem. A part of this difference must result from the assumption $n'=n$ which restricts considerably the flexibility of the three-body function. Moreover the three-body wave function contains only even powers of the distances r_{12}, r_{13}, r_{23} and hence lacks an essential part of the true solution. For these two reasons it is thought that the difference in the improvements found in the two problems is without significance.

The Hassé method[15] was the third method used to improve the wave functions and the one which gave the greatest improvement in the case of the three-body problem. The Hassé method has the great advantage that it automatically introduces into the wave function essential terms which may be lacking in the zeroth approximation. Suppose that in either problem H is the Hamiltonian and ψ_0 is the Gaussian wave function; we take as our new wave function

$$\psi=(1+\lambda H)\psi_0, \tag{36}$$

where λ is a parameter to be varied. In the two-body problem ψ is a two-parameter function while in the three-body problem it is a three-parameter function.

The matrix elements of H, H^2, H^3 and unity are necessary to the calculation. For the "equivalent" problem they are (with $B=2A$, $\beta=(28/15)\alpha$)

$$\begin{aligned}
&(0/1/0)=1,\\
&(0/\mathrm{H}/0)=2.8\alpha\sigma-2A(\sigma/(\sigma+1))^{3/2},\\
&(0/\mathrm{H}^2/0)=13.07(\alpha\sigma)^2-0.4\alpha A(\sigma/(\sigma+1))^{5/2}(28\sigma+56)\\
&\qquad\qquad+4A^2(\sigma/(\sigma+2))^{3/2},\\
&(0/\mathrm{H}^3/0)=85.37(\alpha\sigma)^3-5.23\alpha^2A(\sigma/(\sigma+1))^{7/2}\\
&\qquad\qquad\times(15\sigma^2+52\sigma+52)\\
&\qquad+11.2\alpha A^2(\sigma/(\sigma+2))^{5/2}(3\sigma+12+4/\sigma)\\
&\qquad-8A^3(\sigma/(\sigma+3))^{3/2}.
\end{aligned} \tag{37}$$

The matrix elements of unity and H are given correctly by Eq. (37) for both problems. Those of H^2 and H^3 in the three-particle problem are (with $n=\frac{1}{3}$)

$$\begin{aligned}
&(0/\mathrm{H}^2/0)=10.67(\alpha\sigma)^2-0.4\alpha A(\sigma/(\sigma+1))^{5/2}(28\sigma+43)\\
&\qquad+2A^2[(\sigma/(\sigma+2))^{3/2}+(\sigma^2/(\sigma^2+2\sigma+15/16))^{3/2}],\\
&(0/\mathrm{H}^3/0)=51.77(\alpha\sigma)^3-\alpha^2A(\sigma/(\sigma+1))^{7/2}\\
&\qquad\qquad\times(64\sigma^2+184.8\sigma+143.3)\\
&+0.2\alpha A^2[(\sigma^2/(\sigma^2+2\sigma+15/16))^{5/2}(84\sigma+258+165/\sigma)\\
&\qquad+(\sigma/(\sigma+2))^{5/2}(84\sigma+258+60/\sigma)]\\
&\qquad-2A^3[(\sigma/(\sigma+3))^{3/2}+3(\sigma^2/(\sigma^2+3\sigma+15/8))^{3/2}].
\end{aligned} \tag{38}$$

We find about 65 percent improvement in the energy for the two-body problem with this method and about 60 percent improvement for the three-body problem. It is interesting to note that the matrix elements have somewhat the same form in the two problems except for different numerical factors and certain correction terms. The numerical values of the matrix elements $(0/\mathrm{H}^2/0)$ and $(0/\mathrm{H}^3/0)$ are quite different in the two problems because of these differences. The fact that the corrections obtained are a few percent smaller for the three-body problem than for the two-body problem is thought to be of no significance because one of the parameters which appear in the three-body problem was fixed at a probable value and not varied further.

These investigations establish the inequality

$$E(\mathrm{H}^3)<E^0(\mathrm{H}^3)+0.6\{E'(\mathrm{H}^3)-E^0(\mathrm{H}^3)\} \tag{39}$$

and lend strong support to the view that $E(\mathrm{H}^3)$ does not differ very much from $E'(\mathrm{H}^3)$.

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[15] H. R. Hassé, Proc. Camb. Phil. Soc. **26**, 542 (1930).