On the Fermi Theory of β-Radioactivity

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A comparison of Fermi's formula for the distribution in energy of the electrons and positrons emitted by radioactive bodies with the observed spectra seems to show that a basic factor in it, the statistical factor, is not asymmetric enough. Since about the same degree of asymmetry is common to the spectra of light and heavy nuclei and of positron and electron emitters, it cannot be ascribed to another factor in the Fermi formula, depending on the nuclear field. A weight factor is introduced to provide the required asymmetry by changing the form of the Fermi interaction energy. It is shown that two almost equivalent points of view can be employed in attacking this problem and that a certain uniqueness in the form of the interaction law can be obtained within the requirements laid down by Fermi. The modified distribution formula, which holds strictly only for light nuclei, is then shown to give a much more satisfactory agreement with the data than the original formula.

§1. INTRODUCTION

FOLLOWING a suggestion by Pauli that the continuous character of the β-disintegration spectra may be due to the emission of a second, unobserved particle together with each electron, Fermi[1] has developed a theory of β-decay, which is in general agreement with the experimental facts. This second particle, the neutrino, makes possible the fulfillment of the conservation laws, and owes its ability to escape detection to its zero charge, and to the smallness of its mass and its magnetic moment.[2]

The mathematical formulation of the Fermi theory is quite analogous to the quantum theory of radiation. Corresponding to the transition of an atom from an excited state to the normal state with emission of a light quantum, there is the transition of a heavy particle in the nucleus from a "neutron state" to a "proton state" with emission of an electron and a neutrino. This can be represented by the reaction:

$$N \rightarrow P + e^- + n. \tag{1}$$

In the case of positron emitters, on the other hand, one has:

$$P \rightarrow N + e^+ + n' \tag{2}$$

representing the transition from the proton state of the heavy particle to the neutron state. Here the second particle should properly be called, according to Fermi's formulation of the theory, an "antineutrino."[3]

One can write the result Fermi obtains for the probability of emission of an electron or positron with an energy (including the rest energy) between W and $W+dW$ and a maximum energy W_0:[4]

$$P(W)dW = G^2|M|^2F(Z, W)(W_0-W)^2 \times (W^2-1)^{1/2}WdW. \tag{3}$$

From this follows the lifetime τ according to:

$$1/\tau = \int_1^{W_0} P(W)dW. \tag{4}$$

In (3), G is a dimensionless constant which measures the strength of the coupling between the heavy particle and the "electron-neutrino field." To explain the slowness of β-decay, Fermi has to assume this constant to be very small, of the order of magnitude 10^{-13}.[5] M is a matrix element (containing the wave functions of the heavy particle), which enters into the theory in the same way that the matrix element of the dipole moment enters radiation theory. Fermi connects the two curves of Sargent's[6] well-known graph relating the lifetime and the maximum energy with "allowed" and "forbidden" transitions, corresponding to different values of $|M|^2$ and in analogy to the dipole and quadripole transitions of radiation theory. $F(Z, W)$ is given by:

$$F(Z, W) = \{4/[\Gamma(3+2\gamma)]^2\}(2p\rho)^{2\gamma} \times e^{\pi\alpha ZW/p}|\Gamma(1+\gamma+i\alpha ZW/p)|^2, \tag{5}$$

[1] E. Fermi, Zeits. f. Physik **88**, 161 (1934).

[2] For attempts to observe the neutrino through its magnetic moment, compare J. Chadwick and D. E. Lea, Proc. Camb. Phil. Soc. **30**, 59 (1934); M. E. Nahmias, Proc. Camb. Phil. Soc. **31**, 99 (1935).

[3] G. C. Wick, Atti Lincei **19**, 319 (1934). See also §3.

[4] Here and in the following are used the rational relativistic units. The unit of energy is mc^2; of time, $mc^2/\hbar$; of length, the Compton wavelength, $\hbar/mc$. Ordinary formulae are put into these units by setting m, $\hbar$ and c equal to unity and e^2 to the fine-structure constant $\alpha = e^2/\hbar c$.

[5] This G is related to the g of Fermi by: $G = gm^2c/(2\pi^2)^{\frac{1}{2}}\hbar^3$.

[6] B. W. Sargent, Proc. Roy. Soc. **A139**, 659 (1933).

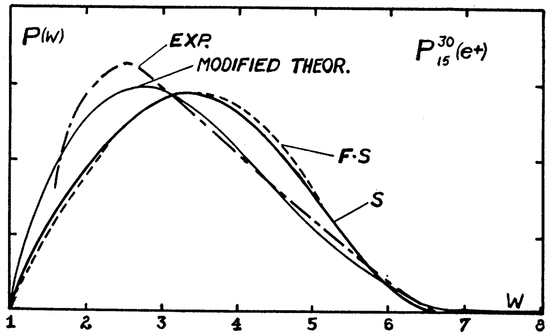

FIG. 1.

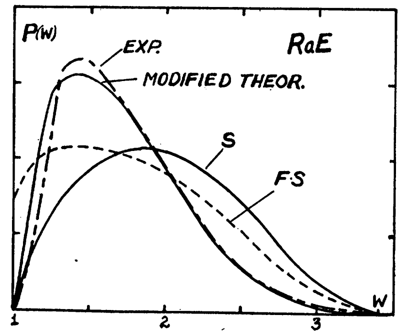

FIG. 2.

FIG. 1. Comparison of the positron distributions given by the statistical factor S, the Fermi formula FS and the modified Fermi formula (see § 4) with the experimental curve of Ellis and Henderson.[19] For S, FS and the experimental curve, $W_0=6.54$. For the modified distribution, $W_0=8$. The areas under the curves are made equal to each other.

FIG. 2. Comparison of the electron distributions given by the statistical factor S, the Fermi formula FS and the modified Fermi formula (see § 4) with the experimental curve for Ra E given in the article of Sargent.[12] For all curves, $W_0=3.4$. The areas under each are made equal.

where ρ is the radius of the nucleus, $p=(W^2-1)^{\frac{1}{2}}$ and $\gamma=(1-\alpha^2Z^2)^{\frac{1}{2}}-1$. This represents the influence of the Coulomb field of the nucleus on the energy distribution; for $Z=0$ it reduces to unity.[7] The remainder of (3) will be referred to as the "statistical factor" and will be symbolized by S.

Such a statistical factor will always occur in the expression for the energy distribution between two particles.[8] It is proportional to the volume in phase space or the number of states into which each particle can go, thus: $S\sim p_e^2p_n^2dp_edp_n$. This is put in terms of energies for the purpose of comparison with observed spectra and is made independent of quantities connected with the unobserved neutrino by taking into account energy conservation.

§2. Comparison with Experiment

Whatever success the Fermi theory may have in explaining the form of the electron and positron spectra is mainly due to the statistical factor. S plotted against W gives a distribution curve having the general shape of the observed distributions (see Figs. 1 and 2). The asymmetry of the curve comes from the difference in the masses of the two emitted particles.[9] The factor $F(Z, W)$ has little effect on the form of the distribution curves for light nuclei (compare S and SF in Fig. 1).[10] For the heavy nuclei, $F(Z, W)$ has quite an appreciable effect (see Fig. 2). Although it shifts the maximum toward lower electron energies as the experimental data require, it brings in the serious difficulty that it predicts too many slow electrons.[11] Because fast electrons are less influenced by the nuclear attraction, these effects of F will be less marked for spectra with higher maximum energy.

The comparison in Figs. 1 and 2 of the theoretical and experimental distributions for typical, noncomplex positron and electron spectra[12] seems to indicate that the basic

[7] Z must be taken positive for electron emitters and negative for positron emitters. Following Fermi we have taken the mass of the neutrino zero. For a finite mass μ, $F(Z, W, \mu)$ is a more complicated function, which for $Z=0$ reduces to $1+\mu/W(W_0-W+\mu)$.

[8] Compare G. E. Uhlenbeck and S. Goudsmit, Physica, in press.

[9] For a neutrino mass equal to the electron mass the corresponding statistical factor would give a perfectly symmetrical curve. The maximum is shifted toward lower electron energies for smaller neutrino masses. The data, as we shall see, require a strongly asymmetric curve, so that in the present form of the theory, we certainly must take the neutrino mass zero, which Fermi had already done on other grounds.

[10] The theory proposed by Beck and Sitte (Zeits. f. Physik **86**, 105 (1933)) takes as the second particle a positron, which is then again captured by the nucleus. Because now the masses of the two particles between which the energy is divided are *equal*, one must depend entirely on the nuclear field to provide the asymmetry in the distribution. It seems then difficult to account for the similarity of the experimental distributions for heavy and light nuclei and for positron and electron emitters.

[11] G. Beck and K. Sitte, Zeits. f. Physik **89**, 259 (1934); R. L. Dolecek, Phys. Rev. this issue, p. 13. We want to thank Dr. Dolecek for allowing us to see his paper before its publication.

[12] The data for P used here are taken from C. D. Ellis and W. J. Henderson, Proc. Roy. Soc. **A146**, 206 (1934):

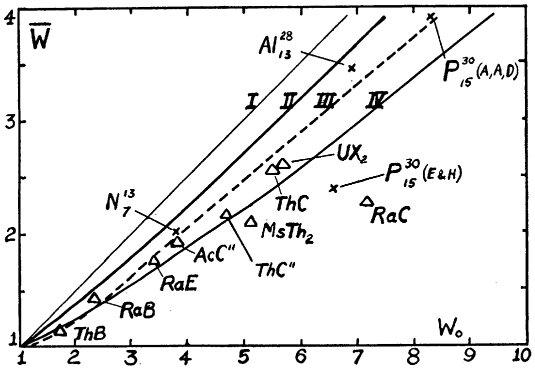

FIG. 3. Comparison of the experimental average energies with theory: I, perfectly symmetrical distribution; II, distribution given by S ($\mu=0$, $Z=0$); III, distribution given by FS with $\mu=0$, $Z=82.2$ (applicable only to the experimental points given by the triangles); IV, distributions given by the modified Fermi theory (see §4) with $\mu=0$, $Z=0$. ${}_{15}P^{30}$ (E & H) is the value taken from Ellis and Henderson's[19] results, and ${}_{15}P^{30}$ (A, A & D) from Alichanian, Alichanow and Dzelepow's data. The experimental values for W_0 are used; if the theoretical end points (see Figs. 5, 6 and 7) were used for ${}_{13}Al^{28}$, ${}_{7}N^{13}$ and ${}_{15}P^{30}$ (A, A & D), these points would lie almost on curve IV

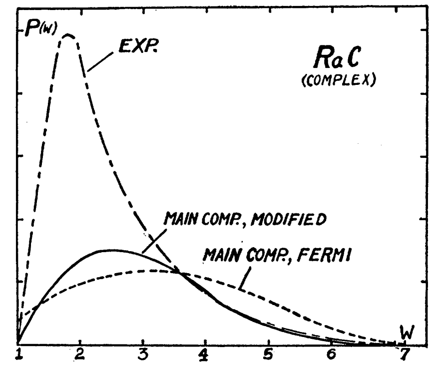

FIG. 4. The main component (weight 0.5) of the complex Ra C spectrum as given by the Fermi and the modified (see §4) theories compared with the observed Ra C spectrum. $W_0=7.16$ for all curves. The area of each component is made equal to one-half the area under the experimental curve.

statistical factor does not give a sufficiently asymmetrical distribution. This is further confirmed when one considers the average energies defined by:

$$\bar{W}=\tau\int_1^{W_0} WP(W)dW$$

as a function of W_0 (see Fig. 3). The experimental average energies deviate systematically from the theoretical values computed from the purely statistical distributions in a direction corresponding to greater asymmetry (the straight line represents the average energies of perfectly symmetrical distributions). Taking $F(Z, W)$ into account helps the agreement with the average energies of the β-ray spectra of the heavy elements, although the actual form as we have seen is not represented very well. For the positron emitters the slight (in their case) influence of Z tends to make the distributions more symmetric instead of less as required. It is true that perhaps many of the experimental points are not directly comparable with the theory because of the complex nature of the corresponding spectra. The spectrum of Ra C, for instance, is quite surely the result of a superposition of several spectra having various upper limits. An attempt was made, following Ellis and Mott,[13] to construct the Ra C curve out of properly weighed components computed from the theory. The result depends very much on what one takes for the populations, which are quite uncertain. One is only sure that the component with the greatest upper limit has at least a population 0.5. We have therefore drawn in Fig. 4 only this main component of the theoretical curve, giving it one-half the area of the experimental curve. Because of the lack of asymmetry of the theoretical curve it is clear that no agreement can be obtained.

§3. THE FERMI INTERACTION "ANSATZ"

To see how the Fermi theory can be modified so as to agree better with experiment it is necessary to examine more closely the fundamental "Ansatz" made for the interaction of the heavy particle with the electron-neutrino field. In analogy to radiation theory the interaction term of the total Hamiltonian is made up of the scalar product of two polar four-vectors, and is written:

compare with this Fig. 5, §4. The Ra E curve was obtained from B. W. Sargent, Proc. Camb. Phil. Soc. **24**, 538 (1932).

[13] C. D. Ellis and N. F. Mott, Proc. Roy. Soc. **A141**, 502 (1933), composed the Ra C spectrum of curves made geometrically similar to the Ra E distribution curve. The upper limit and the weight of each component were calculated from the known energy levels of the product nucleus (Ra C′) and their populations as obtained from γ-ray data. These populations must be revised according to the later experimental results of Rutherford, Lewis and Bowden, Proc. Roy. Soc. **A142**, 347 (1933).

$$H=Q[A_0+(\vec{\alpha}_{\text{heavy}}\cdot\vec{A})]+Q^*[A_0^*+(\vec{\alpha}_{\text{heavy}}\cdot\vec{A})^*]. \quad (6)$$

Here Q and Q^* are the Heisenberg matrices:

$$Q=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad Q^*=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

which operate on the inner coordinate ρ of the heavy particle, which in turn determines whether the particle is a neutron $(\rho=+1)$ or a proton $(\rho=-1)$. Q corresponds to the proton to neutron transition and Q^* to the reverse. The components of $\vec{\alpha}_{\text{heavy}}$ are the Dirac matrices which operate on the spin coordinate of the heavy particle.[14] The four vector A is built up out of the quantized wave functions ψ and φ of the electron and neutrino according to:

$$A_0=\psi\delta\varphi,\quad \vec{A}=\psi\delta\vec{\alpha}\varphi, \quad (7)$$

where $$\delta=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}.$$

The above choice of A is made to insure that a neutron to proton transition is accompanied by the creation of an electron and a neutrino. This becomes clear when ψ and φ are developed as follows:

$$\begin{aligned}\psi&=\sum_s a_s\psi_s, & \varphi&=\sum_\sigma b_\sigma\varphi_\sigma,\\ \psi_s^*&=\sum_s a_s^*\psi_s^*, & \varphi^*&=\sum_\sigma b_\sigma^*\varphi_\sigma^*.\end{aligned} \quad (8)$$

According to the method of second quantization,[15] a_s^* and b_σ^* are operators indicating the creation of an electron in state s and a neutrino in state σ; a_s and b_σ are correspondingly the destruction operators. Introducing (8) into (7), one sees that H will now contain the combinations $Q^*a_s^*b_\sigma^*$ and Qa_sb_σ. The first combination is responsible for the electron emission and the second for the positron emission. To see this one must remember that in order to obtain a definite upper limit of the spectrum, Fermi must assume that the negative energy states of *both* the electron and the neutrino are filled in the Dirac fashion. In the electron emission term, $Q^*a_s^*b_\sigma^*$, s and σ refer to the positive states of the light particles. In the positron emission term, the s and σ must be negative states because the destruction of an electron and a neutrino in negative states is equivalent to the creation of a positron and an antineutrino.

Of course it is arbitrary to connect the neutrino with the emission of an electron and the antineutrino with the positron emission. Physically one could just as well interchange the roles of the neutrino and antineutrino. Formally this can be done by constructing a four vector A' in which the wave function of one particle is coupled with the complex conjugate of the other wave function; this will produce in the Hamiltonian combinations $Q^*a_s^*b_\sigma$ and $Qa_sb_\sigma^*$. The only possibility for such a four vector is:

$$A_0'=\psi\varphi^*,\quad \vec{A'}=\psi\vec{\alpha}^*\varphi^*, \quad (9)$$

which is analogous to the familiar current four vector. The choices (7) and (9) are the only two possible polar four-vectors involving products of wave functions only. This can perhaps most easily be seen with the spinor notation.[16] The ψ becomes then the pair of spinors $\psi_{\dot{m}}$, χ_l and φ correspondingly $\Psi_{\dot{m}}$, X_l. Combining these into four vectors one finds immediately that there are essentially only four possible, of which two are polar and two axial. The two polar ones are:

$$a_{\dot{m}l}=\psi_{\dot{m}}X_l-\Psi_{\dot{m}}\chi_l,\quad a_{\dot{m}l}'=\psi_{\dot{m}}\Psi_l+X_{\dot{m}}\chi_l$$

which correspond to A and A', respectively.

One can easily verify that the use of (9) instead of (7) in the Fermi theory gives *exactly* the same results if the neutrino mass μ is zero, as should also be expected from the physical standpoint. If μ is not zero the statistical factor in the distribution formula is the same whether A or A' is used. In the factor $F(Z, W, \mu)$ there are some sign changes so that for $Z=0$ the factor referred to in footnote 7 becomes $1-\mu/W(W_0-W+\mu)$. The influence of this factor is very

[14] Fermi neglects the terms involving $\vec{\alpha}_{\text{heavy}}$ because they are roughly of the order v/c times the magnitude of the first term. Here v is the velocity of the *heavy* particle in the nucleus, so that v/c is about 0·1. One uses here the Dirac equation for the heavy particle only formally to show the analogy with radiation theory. See also Dolecek, reference 11.

[15] See Pauli, *Handbuch der Physik*, Vol. 24, No. 1, p. 199. In Fermi's paper the formulae defining a_s and a_s^* have to be interchanged.

[16] See, e.g., Laporte and Uhlenbeck, Phys. Rev. **37**, 1380 (1931).

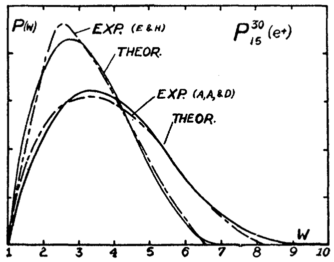

FIG. 5.

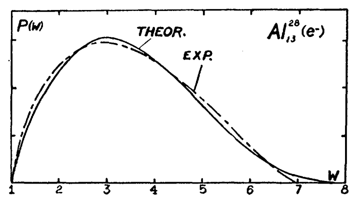

Fig. 6.

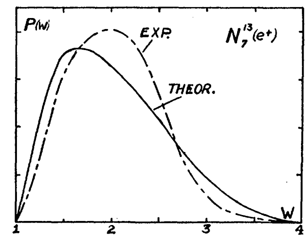

FIG. 7.

FIG. 5. The experimental distributions for the positrons from $_{15}P^{30}$ as given by Ellis and Henderson[19] with $W_0=6.54$ and by Alichanian, Alichanow and Dzelepow[19] with $W_0=8.34$. According to the modified Fermi theory, $W_0=8$ and 10.6.

FIG. 6. Comparison of the distributions of the electrons from radio-aluminum as given by the experiments of Alichanian, Alichanow and Dzelepow[19] and by the modified theory. Experimental $W_0=6.97$. Theoretical $W_0=8.8$.

FIG. 7. Distributions of positrons from radio-nitrogen as given by the experiments of Alichanian, Alichanow and Dzelepow[19] and by the modified theory. Experimental $W_0=3.84$, theoretical $W_0=4.4$.

small and does *not* improve the agreement with experiment. Hereafter we will always take $\mu=0$.

§4. A MODIFICATION OF THE INTERACTION ANSATZ

As we saw in §2, the experimental data both for light and heavy nuclei, whether they are electron or positron emitters, seem to give the impression that the basic distribution is more asymmetrical than the one provided by the statistical factor S. If we may believe this, one must require even for $Z=0$ in addition to the always present statistical factor a weight factor which gives preference to low electron energies or high neutrino energies. This requirement could be fulfilled in a simple, empirical way by multiplying (3) by a power of the neutrino energy W_0-W. In terms of the formalism of the Fermi theory, this is accomplished by introducing derivatives of the neutrino wave functions into the Ansatz for the interaction energy.

We will show that one can construct essentially only one polar four vector involving the wave functions of the electron and the derivatives of the neutrino wave functions. To do this, we shall employ the second point of view, mentioned in §3 in which the emission of an antineutrino is connected with the electron emission. Using the γ_i matrices and the notation of Pauli,[17] we can form only the following three polar four vectors:

$$\begin{aligned} &B_i=\psi\dagger\partial\varphi/\partial x_i, \\ &C_i=\psi\dagger M_{i\alpha}\partial\varphi/\partial x_\alpha, \quad M_{ij}=\gamma_i\gamma_j\delta_{ij}, \\ &D_i=\psi\dagger\gamma_i(\gamma_\alpha\partial\varphi/\partial x_\alpha). \end{aligned} \tag{10}$$

Remembering that the wave equation of the neutrino, with $\mu=0$, is: $\gamma_\alpha\partial\varphi/\partial x_\alpha=0$, one sees that D_i is identically zero. Furthermore, the first component of C_i, for example, is:

$$\psi\dagger\gamma_1\left(\gamma_2\frac{\partial\varphi}{\partial x_2}+\gamma_3\frac{\partial\varphi}{\partial x_3}+\gamma_4\frac{\partial\varphi}{\partial x_4}\right) =\psi\dagger\gamma_1\left(-\gamma_1\frac{\partial\varphi}{\partial x_1}\right)=-\psi\dagger\frac{\partial\varphi}{\partial x_1},$$

so that $C_i=-B_i$. The only vector which remains, B_i, can be written with the Dirac matrices, leaving off a factor i:

$$B_0=\psi^*\beta\partial\varphi/\partial t, \quad \vec{B}=\psi^*\beta\,\mathrm{grad}\,\varphi. \tag{11}$$

Besides the vectors (10), there are, of course, possible three more if we adopt the original viewpoint of Fermi, connecting the electron with the neutrino emission, which again reduce to one when the neutrino wave equation is applied. Since, for $\mu=0$, the two points of view always give the same result there remains really only the vector B_i.[18]

Substituting the vector B_i for A_i in the derivation of the Eq. (3) for the distribution, following Fermi's calculations closely and making

[17] Pauli, *Handbuch der Physik*, Volume 24, No. 1, p. 220.

[18] With the spinor notation both sets of vectors are obtained at the same time just as $a_{\dot{m}l}$ and $a_{\dot{m}l}'$ in §3. One can then make the analogous reduction to one vector and in this way the proof of the uniqueness of B_i is perhaps made more convincing.

the same approximations, one obtains with $Z=0$:

$$P(W)dW = G^2|M|^2(W_0-W)^4(W^2-1)^{\frac{1}{2}}WdW. \quad (12)$$

If Z were not put equal to zero one has again to multiply (12) with the factor $F(Z, W)$ given by (5).

In Figs. 1, 5, 6 and 7 the distribution given by (12) is compared with the experimental data[19] for ${}_{15}P^{30}$, ${}_{7}N^{13}$ (positron emitters) and ${}_{13}Al^{28}$ (electron emitter). In these cases the factor F is negligible, and it seems that (12) gives the required asymmetry. Because it appears that the experimental values for the upper limits are not very certain, and because the curve (12) tails off very slowly to zero near W_0, so that the end would be hardly observable, we have adjusted the upper limits somewhat.[20] In Fig. 3 this would shift the points for P and Al to the right and bring them nearly on the theoretical curve IV. For Ra E (12) also represents the observed distribution rather well as is seen in Fig. 2. However, in this case F is *not* negligible and should be included. As in the Fermi theory the theoretical distribution will then *not* go to zero for $W=1$, so that the change in the interaction energy does *not* remove this difficulty.

§5. Concluding Remarks

1. The choice of B in the interaction energy fulfills just as A all the *a priori* requirements which Fermi set down at the beginning of his paper. Besides the fact that B seems to give better agreement with the data, it may have significance that in B enter the energy and impulse operators of the neutrino. The neutrino has been introduced to fulfill the conservation laws, so that we may perhaps say that the energy and the impulse of the neutrino are its only observable properties.

2. We have also considered other forms of the interaction energy involving derivatives both of the electron and the neutrino wave functions, with the hope of diminishing the influence of $F(Z, W)$ for the heavy elements near $W=1$. By using the neutrino wave equation the number of independent polar four vectors containing the derivatives of both wave functions narrows down to two:

$$K_i = \frac{\partial\psi^\dagger}{\partial x_\alpha}\gamma_i\frac{\partial\varphi}{\partial x_\alpha}, \quad L_i = \frac{\partial\psi^\dagger}{\partial x_\alpha}\gamma_\alpha\frac{\partial\varphi}{\partial x_i}.$$

For $Z=0$ L_i reduces to B_i. We did not succeed however, either with these vectors or with a linear combination of them, in bringing down the distribution curve near $W=1$ and keeping the required asymmetry at the same time.

3. Our modification of the form of the interaction energy does not of course affect Fermi's explanation of Sargent's law, nor does it change the order of magnitude of G. The essential difficulty of the Fermi theory as pointed out by Nordsieck and Tamm[21] therefore remains. Because of the smallness of G the interaction between the neutron and the proton through the electron-neutrino field is much too weak to account for the experimental results on neutron-proton scattering. Nordsieck computes the cross section for inelastic scattering (the colliding neutron is transformed into a proton, emitting an electron and a neutrino) and finds a convergent result of the order G^2d^2 where $d=e^2/mc^2$. The elastic collision cross section comes out even smaller ($\sim G^4d^2(M/m)^2\alpha^{-2}$) if the integral occurring in it, which diverges as $\int^\infty dp/p$, may be cut off at $p\sim 1/\alpha$ and considered of the order of magnitude unity. With the modified interaction these calculations are changed essentially only in the fact that the integral in the elastic cross section diverges as $\int^\infty pdp$. If we may again take the upper limit as $1/\alpha$ this would make the elastic cross section at least $(137)^4$ times as large as Nordsieck's value. This is, however, by far not enough to remove the discrepancy.

[19] Ellis and Henderson, Proc. Roy. Soc. **A146**, 206 (1934); Alichanow, Alichanian and Dzelepow, Zeits. f. Physik **93**, 350 (1935). The experimental curves for P disagree with each other. We have shown them both in Figs. 1 and 5. The fact that each can be fitted by a theoretical curve indicates that they have the same kind of asymmetry.

[20] Increasing the upper limits will also tend to remove the discrepancy between Chadwick's value for the mass of the neutron and the value given by Curie and Joliot (Nature **133**, 721 (1934)). Raising the upper limit and the kinetic energy of the neutron together by about 1.5 MEV would bring down the Curie-Joliot mass to the Chadwick value. If (12) really represents the form of the β-ray spectrum, an error of this order of magnitude can easily be understood, especially for the spectra with high upper limits.

[21] A. Nordsieck, Phys. Rev. **46**, 234 (1934); Ig. Tamm, Nature **133**, 981 (1934).