Polarization Effects in the Positron Theory

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Some of the consequences of the positron theory for the special case of impressed electrostatic fields are investigated. By imposing a restriction only on the maximum value of the field intensity, which must always be assumed much smaller than a certain critical value, but with no restrictions on the variation of this intensity, a formula for the charge induced by a charge distribution is obtained. The existence of an induced charge corresponds to a polarization of the vacuum, and as a consequence, to deviations from Coulomb's law for the mutual potential energy of point charges. Consequences of these deviations which are investigated are the departures from the Coulombian scattering law for heavy particles and the displacement in the energy levels for atomic electrons moving in the field of the nucleus.

INTRODUCTION

ACCORDING to Dirac's theory of the positron an electromagnetic field will, in general, induce a charge and current distribution due to the creation and annihilation of electron-positron pairs. The induced fields produced by the electron-positron distribution may be regarded phenomenologically as corresponding to supplementary terms in Maxwell equations. Since one must demand the validity of these equations in sufficiently weak and slowly varying fields, i.e., the vanishing of the susceptibility of vacuum, these terms must depend upon higher powers of the field intensities and their derivatives. Both linear and nonlinear terms will be present. The latter correspond to a failure in the superposition principle for electromagnetic waves, and correspond, therefore, to the scattering of light by light, and of light by an electrostatic field, i.e., to the interaction in general of electromagnetic fields. The linear terms, on the other hand, correspond to a nonvanishing susceptibility of the vacuum. These terms are of importance whenever the fields vary appreciably in a distance of the order of $\hbar/mc$, under which circumstances an appreciable polarization of the electron-positron distribution can exist. It is this effect which we wish to investigate. In order to do so our considerations will be confined to the simplest possible circumstances, namely, those attendant upon an impressed rapidly varying external electrostatic field.[1]

Since the critical field strength of the theory is $F_c=m^2c^3/e\hbar$, our considerations will be valid only insofar as $|\mathcal{E}|$ may be supposed $\ll F_c$. The critical length of the theory is $\lambda_0=\hbar/mc$; no restrictions on the variations of the field in distances of this order will be imposed. In the impact of two point charges (protons for example) rapidly varying fields occur, and one may expect therefore, as has been pointed out by Furry and Oppenheimer,[2] deviations from the Coulomb law of force between them. Similar deviations will occur, also, for atomic electrons moving in the field of a nucleus, and these deviations will give rise to displacements in the position of the atomic energy levels.

The non-Maxwellian terms given by the theory of the positron do not in any way account for the stability of the elementary electrical particle. On the contrary, these terms correspond to an increase in its electromagnetic proper energy. The supplementary terms given by the positron theory are not, therefore, completely correct. However, terms of the same general character are capable of giving a stable electron, and it is of interest to determine in how far such terms lead to experimentally detectable results.

The formulism of the theory developed by Heisenberg will be used in our treatment of the problem.[3] For the case of electrostatic fields, these methods are fully equivalent to those of Furry and Oppenheimer,[2] since apart from the ambiguities introduced by gauge and Lorentz transformations, the infinite constant polarization of vacuum which appeared in their work can be simply ignored, and the finite deviations can be given an unambiguous meaning. It is more convenient, however, to work with expressions which are unconditionally convergent. Heisenberg, following a suggestion of Dirac,[4] was able to obtain

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[1] The generalization of these results to rapidly varying electromagnetic fields is to be found in a paper of Serber appearing in this issue.

[2] Furry and Oppenheimer, Phys. Rev. **45**, 245 (1934).

[3] Heisenberg, Zeits. f. Physik **90**, 209 (1934).

[4] Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934).

such convergent expressions for the expectation values of physical quantities in the theory of the positron. This convergence is achieved by consideration of the density matrix in nondiagonal form, and subtraction from it of all singular contributions to the expectation values of electromagnetic and dynamical quantities. We shall not be concerned here with the arguments according to which the density matrix is defined, nor with the question of the uniqueness of the result. The definition is, of course, consistent with the vanishing of the susceptibility of vacuum in weak and slowly varying fields.

I. The Induced Charge Density

A general formula for the induced charge, arising from the existence of static external charges, and valid in those regions of space in which the external field intensity is small compared with the critical field, will be obtained. One considers the field to be adiabatically increased from zero to some constant value corresponding to the presence of the existing external charges, and determines the modified wave functions for the electrons by a first-order perturbation method. In accordance with the method of Dirac and Heisenberg a density matrix R_S is then defined such that $R_S=R-\frac{1}{2}R_F$, where R is the density matrix corresponding to the Dirac hole theory and R_F the density matrix for the distribution in which every energy level is occupied. All of these matrices are defined in nondiagonal form in the coordinates, the time, and the spin variables, and the result for the matrix R_S has been obtained by Heisenberg.[5] The first term of this expression is identified immediately with the highest singularity on the light cone. Heisenberg then finds the remaining singularities by developing the integrand in powers of $g(\mathbf{g}=\mathbf{p}'-\mathbf{p}''$ where $\mathbf{p}'$ and $\mathbf{p}''$ are the momenta corresponding to the electron and positron) valid for slowly varying fields. The result of the integration then shows that the remaining singularities correspond to the terms in the lowest power of g, and these terms are in consequence to be subtracted from the matrix R_S to form the matrix r which is to be identified with the physically significant portion of the density matrix representing only the deviations from that distribution in which all negative energy states are occupied and all positive states unoccupied. In the process of this identification, according to which just those terms corresponding to the lowest power g are taken to represent the remaining singularities, and arbitrary constant appearing in the formulation of these singular terms is fixed so that the induced charge over all space vanishes and the polarization in slowly varying fields is zero.

We now use Heisenberg's result to form the matrix r valid for rapidly varying fields. Subtracting the terms corresponding to the singularities and defining $(x_\rho=x_\rho'-x_\rho''$, $x_i'=x^{i'}$, $x_0'=-x^{0'}=ct'$, $x_\rho'+x_\rho''=2\mathfrak{y}_\rho$, $\mathbf{p}'=\mathbf{k}+\mathbf{g}/2$, $\mathbf{p}''=\mathbf{k}-\mathbf{g}/2)$ one obtains the result

$$(x'k'|r|x''k'')=-\frac{1}{256\lambda_0^2\pi^6\hbar}\int d\mathbf{r}'\int d\mathbf{g}\int d\mathbf{K}\frac{e}{c}A^0(\mathbf{r}')\{\exp i(\mathbf{r}'\cdot\mathbf{g}-\mathfrak{y}\cdot\mathbf{g}-\mathbf{K}\cdot\mathbf{x})\}$$
$$\times\{(\mathbf{K}\cdot\mathbf{g})^{-1}[(1+K^2+\tfrac{1}{2}\mathbf{K}\cdot\mathbf{g})(1+K^2+\tfrac{1}{4}g^2+\mathbf{K}\cdot\mathbf{g})^{-1/2}-(1+K^2-\tfrac{1}{2}\mathbf{K}\cdot\mathbf{g})(1+K^2+\tfrac{1}{4}g^2-\mathbf{K}\cdot\mathbf{g})^{-1/2}]$$
$$-\tfrac{1}{4}g^2(1+K^2)^{-3/2}+\tfrac{1}{4}(\mathbf{K}\cdot\mathbf{g})^2(1+K^2)^{-5/2}\}+\text{conj.}, \qquad (1)$$

where $\lambda_0=\hbar/mc$, k' and k'' are the spin variables, all lengths are expressed in units of λ_0, and all momenta in units of mc. Since all infinities are now removed, one can go to the diagonal of the matrix r before performing the integrations. Integrating over the azimuthal angle of K, going to the diagonal ($x_\rho=0$, $k''=k'$), adding the conjugate, and simplifying, one obtains

$$(\mathfrak{y}k'|r|\mathfrak{y}k')=-\frac{1}{32\pi^5\lambda_0^2\hbar}\int d\mathbf{r}'\int\frac{d\mathbf{g}}{g}\int KdK\frac{e}{c}A^0(\mathbf{r}')\{\exp i\mathbf{g}\cdot(\mathbf{r}'-\mathfrak{y})\}$$
$$\times\int_{-1}^{1}\frac{du}{u}\{(1+K^2+\tfrac{1}{2}Kgu)(1+K^2+\tfrac{1}{4}g^2+Kgu)^{-1/2}-\tfrac{1}{8}Kgu(g^2(1+K^2)^{-3/2}-K^2g^2u^2(1+K^2)^{-5/2})\}, \qquad (2)$$

[5] Heisenberg, reference 3, Eq. (33).

where $u=\cos\vartheta$ is a new integration variable (ϑ is the polar angle between the vectors $\mathbf{K}$ and $\mathbf{g}$). When the integral over K is performed it is found that the subtractive terms are just sufficient to remove the divergences of the principal term. The result of this integration gives

$$(\mathfrak{y}k'|r|\mathfrak{y}k')=\frac{1}{256\pi^5\lambda_0\hbar}\int d\mathbf{r}'\int g^2dg\frac{e}{c}A^0(\mathbf{r}')\{\exp i\mathbf{g}\cdot(\mathbf{r}'-\mathfrak{y})\}\int_{-1}^{1}(1-u^2)\log\{(1+\tfrac{1}{4}g^2)^{1/2}+\tfrac{1}{2}gu\}du. \quad (3)$$

By introducing now the potential in terms of a continuous charge distribution

$$A^0(\mathbf{r}')=-A_0(\mathbf{r}')=\lambda_0^2\int\frac{\rho(\mathbf{r}'')d\mathbf{r}''}{|\mathbf{r}'-\mathbf{r}''|},\quad\int A_0(\mathbf{r}')\{\exp i\mathbf{r}'\cdot\mathbf{g}\}d\mathbf{r}'=\frac{4\pi\lambda_0^2}{g^2}\int\rho(\mathbf{r}')\{\exp i\mathbf{r}'\cdot\mathbf{g}\}d\mathbf{r}',$$

where $\rho(\mathbf{r}')$ is the charge density per unit volume measured in ordinary units, introducing the Laplacian, and summing over the spin variables k'

$$(\mathfrak{y}|r|\mathfrak{y})=\frac{\alpha}{16\pi^4e}\int\frac{d\mathbf{g}}{g^2}\int\rho(\mathbf{r}')d\mathbf{r}'\Delta\{\exp i\mathbf{g}\cdot(\mathbf{r}'-\mathfrak{y})\}\int_0^1(1-u^2)\log\{1+\tfrac{1}{4}g^2(1-u^2)\}du, \quad (4)$$

where $\alpha=e^2/\hbar c$. The Laplacian considered as an operator on $\{\exp i\mathbf{g}\cdot\mathbf{r}'\}$ may be introduced as an operator on $\rho(\mathbf{r}')$ by partial integration. Integrating over the angle variables of g, one then obtains

$$(\mathfrak{y}|r|\mathfrak{y})=-\frac{\alpha}{4\pi^3e}\int\Delta\rho(\mathbf{r}')d\mathbf{r}'\int_0^\infty\frac{db}{b}\frac{\sin\{2b|\mathbf{r}'-\mathfrak{y}|(1-u^2)^{-1/2}\}}{|\mathbf{r}'-\mathfrak{y}|}\int_0^1(1-u^2)\log(1+b^2)du. \quad (5)$$

Integrating now over b, and expressing all lengths again in ordinary units, one obtains an expression for the matter density given by

$$(\mathfrak{y}|r|\mathfrak{y})=(\alpha/4\pi^2e)\int U(|\mathbf{r}'-\mathfrak{y}|)\Delta\rho(\mathbf{r}')d\mathbf{r}', \quad (6)$$

where

$$U(R)=\frac{1}{R}\int_0^1(1-u^2)li\{\exp-(2R/\lambda_0)(1-u^2)^{-1/2}\}du,$$
$$li(e^{-y})=\int_0^{e^{-y}}\frac{dt}{\log t}=\int_\infty^1 e^{-yz}\frac{dz}{z}=\mathcal{E}_i(y). \quad (7)$$

The induced charge is then

$$\delta\rho(\mathfrak{y})=(\alpha/4\pi^2)\int U(|\mathbf{r}'-\mathfrak{y}|)\Delta\rho(\mathbf{r}')d\mathbf{r}'. \quad (8)$$

The function $U(R)$ has an essential singularity at $R=0$, and falls rapidly to zero for values of R different from zero. Its behavior for small values of the argument can be found from the development of the logarithmic function

$$li(e^{-y})=\gamma+\log y+\sum_{n=1}^{\infty}\frac{(-1)^n y^n}{n(n!)}. \quad (8a)$$

Thus for small values of R

$$U(R)\sim(2/3R)[\gamma+\log R/\lambda_0+(5/6)+0(R)], \quad (9)$$

where $\gamma=0.5772$ is Euler's constant. One can show also that for large values of R

$$U(R)\sim-\frac{\pi^{1/2}}{4}\frac{\lambda_0^{3/2}}{R^{5/2}}e^{-2R/\lambda_0}[1+0(R^{-1})]. \quad (10)$$

The contribution to the induced charge exists therefore in those regions of space where the Laplacian of the inducing charge has a finite value and in a narrow fringe of space of the order of the Compton wavelength bordering on those regions. Since $U(R)$ is negative for all values of R, the sign of the induced charge is the same as that of the inducing charge in those regions of space where the Laplacian of the latter is negative. One may now obtain an approximate expression for the induced charge by developing

$\Delta\rho(\mathbf{r}')$ about the point $\mathbf{r}'=\mathfrak{y}$. If the field is slowly varying, only the first term in this development need be retained, and since the integral over all space of $U(R)$ is $-4\pi\lambda_0^2/15$, one obtains the approximate result

$$\delta\rho(\mathfrak{y})=-(\alpha/15\pi)\lambda_0^2\Delta\rho(\mathfrak{y}). \tag{11}$$

This result agrees with Heisenberg's determination (reference 3, Eq. 40) obtained from the matrix R_S as a result of retaining only one term in the development in powers of g beyond those which represent the singularities.

As a consequence of the existence of an induced charge in the immediate neighborhood of space varying external charges, and the fact that this induced charge is polarized in the manner shown by Eq. (8), the mutual potential energy of external charges must deviate from Coulomb's law when the charges approach one another within a distance of the order of the Compton wavelength. Between like charges, for example, the deviation from the Coulomb potential will be given by the interaction energy of the external charges with that portion of the induced charge due to the neighboring charge which lies within a sphere of which the line connecting the external charges is a radius. Since the induced charge outside of this sphere is of opposite sign to that of the inducing charges this deviation from Coulomb's law must be such as to increase the mutual forces acting between the charges. For charge distributions this deviation could be calculated directly from Eq. (8). For point charges this deviation is most easily determined from the matrix representing the interaction energy. This determination will be made in the next section.

II. Deviation from the Coulomb Potential

Consider two fixed point charges $Z'e$ and $Z''e$ separated by a distance R. In the mutual field of these charges electrons and positrons are created and annihilated, and the equilibrium value of the electron density is obtained from the formulae of the preceding section in which $A_0(\mathbf{r}')$ represents the electrostatic potential of an electron at the point $\mathbf{r}'$ in this electrostatic field. The energy represented by the electron-positron distribution is given by

$$E=\text{Spur}\,(x'k'|Hr|x''k''), \tag{12}$$

where H represents the matrix of the Hamiltonian function, and r is Heisenberg's density matrix after subtraction of the singularities. The evaluation of the Spur in Eq. (12) yields two terms, one of which represents the electromagnetic energy of the electron-positron distribution, namely,

$$E_e=4e\lambda_0^3\int d\mathfrak{y}A^0(\mathfrak{y})(\mathfrak{y}|r|\mathfrak{y}), \tag{12a}$$

where $\mathfrak{y}$ as in the preceding section is in units of λ_0, and the other of which represents the kinetic energy of the distribution. These terms are of opposite sign, and the latter has a magnitude equal to 1/2 the former. Consequently, we may obtain the total energy of the distribution by the introduction of the density matrix as given by Eq. (3) into Eq. (12a) and dividing the result by a factor 2. One would, of course, obtain the same result for the total energy by using the perturbation matrix $V_{r\rho}$ for transitions between negative and positive energy states, and calculation of the total energy from the expression

$$E=-\sum_{r\rho}|V_{r\rho}|^2/(E_r-E_\rho),$$

where r and ρ index respectively the positive and negative energy states. Using the former procedure one obtains

$$E=\frac{\alpha\lambda_0}{128\pi^5}\int d\mathfrak{y}A^0(\mathfrak{y})\int d\mathbf{r}'\int g^2dgA^0(\mathbf{r}')\{\exp i\mathbf{g}\cdot(\mathbf{r}'-\mathfrak{y})\}\int_{-1}^{1}(1-u^2)\log\{(1+\tfrac{1}{4}g^2)^{1/2}+\tfrac{1}{2}gu\}du. \tag{13}$$

The potentials may now be introduced. With all quantities in natural units one has

$$\int d\mathfrak{y}A^0(\mathfrak{y})\{\exp -i\mathbf{g}\cdot\mathfrak{y}\}=\frac{4\pi e}{g^2\lambda_0}\{Z'\{\exp -\tfrac{1}{2}\mathbf{R}\cdot\mathbf{g}\}+Z''\{\exp\tfrac{1}{2}\mathbf{R}\cdot\mathbf{g}\}\},$$

$$\int d\mathbf{r}'A^0(\mathbf{r}')\{\exp i\mathbf{g}\cdot\mathbf{r}'\}=\frac{4\pi e}{g^2\lambda_0}\{Z'\{\exp\tfrac{1}{2}\mathbf{R}\cdot\mathbf{g}\}+Z''\{\exp -\tfrac{1}{2}\mathbf{R}\cdot\mathbf{g}\}\}.$$

Thus, the energy as represented by Eq. (13) has terms dependent on Z'^2 and on Z''^2 alone and on the

product $Z'Z''$. The former represent the contribution to the proper energy of the charges, and like the proper energy due to its own field the proper energy of the electron-positron distribution due to the point charges is infinite.[6] The interaction energy represented by the terms containing the product $Z'Z''$ is, however, finite for finite distances of separation. One obtains

$$E_{\text{Int.}}=\frac{Z'Z''\alpha e^2}{8\pi^3\lambda_0}\int\frac{dg}{g^2}\{\{\exp i\mathbf{R}\cdot\mathbf{g}\}+\{\exp -i\mathbf{R}\cdot\mathbf{g}\}\}\cdot\int_{-1}^{1}(1-u^2)\log\{(1+\tfrac{1}{4}g^2)^{1/2}+\tfrac{1}{2}gu\}du$$
$$=\frac{Z'Z''\alpha e^2}{\pi^2\lambda_0 R}\int\frac{\sin Rgdg}{g}\int_0^1(1-u^2)\log\{1+\tfrac{1}{4}g^2(1-u^2)\}du. \tag{14}$$

By integrating over g and introducing ordinary units for R

$$E_{\text{Int.}}=-(\alpha/\pi)Z'Z''e^2U(R), \tag{15}$$

where $U(R)$ is defined as in Eq. (7). Since $U(R)$ is always negative the interaction energy is always positive for like charges, as it should be according to the discussion of the previous section. Asymptotic values for $E_{\text{Int.}}$ are easily obtained from Eqs. (9) and (10) for both large and small values of R. The values are:

$$E_{\text{Int.}}\cong-(2\alpha/3\pi)(Z'Z''e^2/R)[\gamma+\log R/\lambda_0+5/6+0(R)] \quad R \text{ small} \tag{16}$$
$$\cong(\alpha\lambda_0^{3/2}/4\sqrt{\pi})(Z'Z''e^2/R^{5/2})e^{-2R/\lambda_0}[1+0(R^{-1})] \quad R \text{ large.} \tag{17}$$

A power series development for $E_{\text{Int.}}$ valid for all values of R, which is useful in particular for small and intermediate values, may be obtained from Eq. (14). By adding and subtracting a term in $\log(1+\frac{1}{4}g^2)$ to the integrand of this equation, it may be written

$$E_{\text{Int.}}=\frac{Z'Z''\alpha e^2}{\pi\lambda_0 R}\int\frac{\sin Rgdg}{g}\left\{(4/3)\log a-\int du[\tfrac{2}{3}\log a-(1-u^2)\log(a+bu)]\right\}, \tag{18}$$

where $a=(1+\frac{1}{4}g^2)^{1/2}$, $b=\frac{1}{2}g$. Since

$$\int_0^\infty\frac{\sin Rgdg}{g}\log(1+\tfrac{1}{4}g^2)=-\pi li(e^{-2R})$$

the first term of Eq. (18) is

$$-\frac{2\alpha}{3\pi}\frac{Z'Z''e^2}{\lambda_0 R}li(e^{-2R})=\frac{2\alpha}{3\pi}\frac{Z'Z''e^2}{\lambda_0 R}\frac{e^{-R}}{(2R)^{1/2}}W_{-1/2,\,0}(2R),$$

where $W_{-1/2,\,0}(2R)$ is the confluent hypergeometric function.[7] The integrand of the second term of Eq. (18) may be developed in powers of b/a valid for all values of g. Integrating first over u and then making the development one obtains

$$\int\frac{\sin Rgdg}{g}\int_{-1}^{1}du\{\tfrac{2}{3}\log a-(1-u^2)\log(a+bu)\}$$
$$=\int\frac{\sin Rgdg}{g}\left\{\frac{16}{9}-\frac{2}{3}\frac{a^2}{b^2}+\frac{4}{3}\log a+\left(\frac{a^3}{3b^3}-\frac{a}{b}-\frac{2}{3}\right)\log(a+b)-\left(\frac{a^3}{3b^3}-\frac{a}{b}+\frac{2}{3}\right)\log(a-b)\right\}$$
$$=\frac{2}{3}\sum_{n=1}^{\infty}\int\frac{\sin Rgdg}{g}\left(\frac{1}{n}-\frac{3}{2n+1}+\frac{1}{2n+3}\right)\left(\frac{b}{a}\right)^{2n}.$$

This integral may be evaluated by complex methods to give a power series in R. Combining the results for the first and second terms of Eq. (18) and returning to ordinary units one obtains finally

[6] The contribution to the electrostatic proper energy of a point charge may be obtained by letting $R\to0$ in Eq. (15); it is positive and has a slightly worse singularity than the classical term, since $U\to(\log R)/R$. These polarization terms also lead to divergent positive terms of order α in the proper energy of an electron, giving, in fact, integrals of the form $\alpha\int^\infty(dg/g)\log(1/g)$.

[7] Whittaker and Watson, *Modern Analysis*, 4th ed., p. 339.

for the interaction energy

$$E_{\text{Int.}}=-\frac{2\alpha}{3\pi}\frac{Z'Z''e^2}{R}\Big[li(e^{-2R/\lambda_0})+\frac{3}{2}e^{-2R/\lambda_0}\sum_{n=1}^{\infty}2^n \cdot\sum_{\sigma=0}^{n-1}\sum_{\mu=0}^{n-\sigma-1}\frac{(-2)^{\mu-\sigma}\Gamma(n+\tfrac{1}{2})}{n(2n+1)(2n+3)\Gamma(\tfrac{1}{2})}\frac{(n+\sigma-1)!}{(n+\mu+\sigma)!(n-\mu-\sigma-1)!\sigma!\mu!}\left(\frac{R}{\lambda_0}\right)^{\mu}\Big]. \quad (19)$$

The power series development of the logarithmic function valid for all finite values of the argument is given by Eq. (8a); therefore, one has in Eq. (19) a power series development of $E_{\text{Int.}}$ valid for all finite values of R. One can show that

$$\sum_{\sigma=0}^{n-1}\frac{(-2)^{-\sigma}}{(n+\sigma)(n-\sigma-1)!\sigma!}=\frac{\Gamma(\tfrac{1}{2})}{2^n\Gamma(n+\tfrac{1}{2})}$$

and

$$\sum_{n=1}^{\infty}\frac{1}{n(2n+1)(2n+3)}=\tfrac{2}{3}(5/6-\log 2)$$

as a consequence of which the second term of the bracket in Eq. (19) approaches the value $(5/6-\log 2)$ as R approaches zero. Combining with the first few terms of the series development for li (e^{-2R/λ_0}) one observes that Eq. (19) gives again Eq. (16) for $E_{\text{Int.}}$ valid for small values of R. It is of interest to note that this result agrees in form with that of Furry and Oppenheimer[8] in which the divergences of the previous method are contained in the constant K of their equations.

As has already been mentioned, the results of the theory are valid only insofar as the field intensities can be assumed to be much smaller than the critical field F_c. Thus the results obtained for the deviation from Coulomb potential given above by Eq. (15) can be expected to hold for protons only for distances of separation such that $R\gg\lambda_0(\alpha)^{\frac{1}{2}}$.

III. Scattering of Heavy Particles

The deviations from Coulomb potential considered in the last section become important when the distance of separation of the particles is of the order of the Compton wavelength. These deviations are in principle observable, and may be detected in the scattering of high energy particles. For the case of protons the classical distance of closest approach in units of the Compton wavelength is of the order α for incident particles having energies of the order of one million electron volts. For energies of this order of magnitude one is already beyond the range of validity of the theory, yet, for incident particles of this and somewhat lower energies, deviations from the law of scattering valid for the strict Coulomb potential should be found.

The scattering law for the new potential is easily obtained. Since the deviations from Rutherford scattering will be important only for incident particles of high energy the Born approximation may be used. The perturbation parameter $rV(r)/\hbar v\sim Z'Z''e^2/\hbar v$, the smallness of which determines the validity of the Born method, is for the case of incident protons having kinetic energies of the order of one million electron volts equal approximately to 20α. For this and higher energies the Born approximation will be fairly accurate.

The scattered amplitude according to the Born formula for unit incident flux scattered by a fixed scattering center is

$$f(\theta)=-\frac{2M}{\hbar^2}\int_0^{\infty}\frac{\sin Kr}{Kr}V(r)r^2dr, \quad (20)$$

where $K=(2/\lambda)\sin(\theta/2)$, $\lambda=\hbar/mv$, M is the mass of the incident particle, v its initial velocity, and θ the angle with respect to the direction of the incident beam at which the scattering is observed. Let $Z'e$ and $Z''e$ denote the charge of the incident and scattering particles, respectively. Then from Eq. (15) the mutual potential energy (Coulomb potential plus the deviation from Coulomb potential) is

$$V(r)=(Z'Z''e^2/r)[1-(\alpha/\pi)rU(r)]. \quad (21)$$

All of the integrations can be performed strictly

[8] Furry and Oppenheimer, Phys. Rev. **45**, 245 (1934), Eq. (4.8). A typographical error has put the factor π in the numerator instead of the denominator in two terms of this expression. Also λ_0 here is 2π times the λ_0 used in the expressions given above.

with the result that one obtains for $f(\theta)$

$$f(\theta)=\frac{Z'Z''e^2}{2Mv^2}\csc^2\frac{\theta}{2}\left[1+\frac{\alpha}{3\pi}F(k)\right], \qquad (22)$$

where

$$F(k)=\frac{4}{k^2}-\frac{5}{3}+\left(1-\frac{2}{k^2}\right)\left(1+\frac{4}{k^2}\right)^{1/2}\times\log\frac{(1+\frac{1}{4}k^2)^{1/2}+\frac{1}{2}k}{(1+\frac{1}{4}k^2)^{1/2}-\frac{1}{2}k},$$

$$k=2(Mv/mc)\sin(\theta/2).$$

m is the electron mass, and the negative sign of $f(\theta)$ is dropped since only its absolute value is important. The first term of Eq. (22) gives the ordinary Rutherford scattering; the second term gives the deviation from this scattering law. For small values of k, $F(k)\sim\frac{1}{5}k^2$, and, therefore, the deviation from the Rutherford scattering is finite for all values of θ. For large values of k

$$F(k)\sim 2\log k-(5/3).$$

The form of the function $F(k)$ shows that the deviations from the usual scattering law for particles of a given incident velocity rises rapidly with increasing θ in the neighborhood of $\theta=0$, and then remains practically constant throughout the greater portion of the angular range increasing with θ only as the logarithm of $\sin(\theta/2)$. The practical constancy of the deviation together with its small magnitude precludes the possibility of any easy experimental verification of the new potential.[9]

The intensity of scattering into a given solid angle is given by the square of the scattered amplitude $f(\theta)$. For the case of proton scattering by protons (the scattering of protons by hydrogen nuclei) the scattered amplitude in relative coordinates is given by Eq. (22) with M equal to one-half the proton mass. Considering the hydrogen nuclei as initially stationary compared with the high energy incident protons, the intensity of scattering into a solid angle $d\Omega$ measured with respect to the direction of the incident beam is

$$I(\theta)d\Omega=4|f(2\theta)|^2\cos\theta d\Omega. \qquad (23)$$

Exchange may be taken into account in the usual manner by introducing $|f(2\theta)+f(\pi-2\theta)|^2$ for $|f(2\theta)|^2$.

[9] Some recent experimental evidence on the scattering of high energy protons in hydrogen (M. G. White, Phys. Rev. **47**, 573 (1935)) indicates the possible existence of rather large deviations, which in their dependence on energy and angular distribution as well as in the magnitude of the effect bear little resemblance to the predictions given here.

IV. Displacement of Atomic Energy Levels

§1. Secular terms

Because of the failure of the potential energy between electron and nucleus to be strictly Coulombian, the energy levels for the electron are slightly displaced. This displacement may be calculated with the deviation from the Coulombian potential as a small perturbation, or by using the expression for the induced charge to calculate the potential between the nucleus and the induced charge of the electron and the potential between the electron and the induced charge of the nucleus. With the former procedure the displacement in the energy levels is given by

$$\delta E=\int V(r)|\Psi_{nlm}(r\theta\varphi)|^2 d\mathbf{r}, \qquad (24)$$

where $V(r)$ represents the deviation from Coulomb potential, and Ψ_{nlm} the electron wave function. From Eqs. (15) and (7). $V(r)$ for the interaction of an electron with the nucleus is given by

$$V(r)=\frac{Z\alpha e^2}{\pi r}\int_0^1(1-u^2)du\int_\infty^1\left\{\exp-\frac{2rz}{\lambda_0}(1-u^2)^{-1/2}\right\}\frac{dz}{z}. \qquad (25)$$

For the $1s$ level

$$\Psi_{100}(r\theta\varphi)=-[1/2(2\pi)^{1/2}](2/a)^{3/2}e^{-r/a}, \qquad (26)$$

where $a=\hbar^2/\mu e^2Z$, $\mu=mM/(m+M)$. The integration can be performed strictly to the lowest power in $\lambda_0/a=ZM\alpha/(M+m)$ with the result that

$$\delta E_{1s}\cong-8Z^2\alpha^3R/15\pi, \qquad (27)$$

where $R=\mu e^4Z^2/2\hbar^2$ is the Rydberg constant. Similarly for the $2s$ level

$$\Psi_{200}(r\theta\varphi)=\frac{1}{8(2\pi)^{1/2}}e^{-r/2a}\left(\frac{1}{a}\right)^{3/2}\left(\frac{2r}{a}-4\right) \qquad (28)$$

and $$\delta E_{2s} \cong -Z^2\alpha^3 R/15\pi. \tag{29}$$

For p levels the calculation shows that the displacement is of higher order in α, being, in fact, of order α^5. Of the secular terms, therefore, the important ones are the s-terms.[10]

§2. Nonsecular terms

A Fourier resolution of the inducing charge $\rho(rt)$ yields after a Lorentz transformation and substitution into Eq. (8) an expression for a single component of the induced charge given by[11]

$$\delta\rho(K) = X(K)\rho(K), \tag{30}$$

where

$$X(K) = -\frac{\alpha}{2\pi}\int_0^{\pi/2} \cos^3\psi d\psi \cdot \log\left(1+\tfrac{1}{4}K^2\cos^2\psi\right) \tag{31}$$

and the propagation vector K is in units of λ_0^{-1}. For small values of $\mathbf{K}$

$$X(K) \cong -\alpha K^2/15\pi \tag{32}$$

and accordingly, when only small values of K are important in $\rho(rt)$ Eq. (30) yields the result

$$\delta\rho(rt) = (\alpha/15\pi)\lambda_0^2 \Box\rho(rt) \tag{33}$$

an expression which represents the sum of Eq. (11) for slowly varying space components and Heisenberg's result for pure but slowly varying time components.[12] The order of magnitude of the contribution of transitions within the discrete spectrum to the displacement of the $1s$ electron level may be determined then with the help of Eq. (33), yielding a result which is small of order $Z^4\alpha^6$, and, therefore, negligible in comparison with the secular term. We will, accordingly, consider in more detail only the contribution given by transitions to the continuous spectrum.

Confining our considerations as before to the $1s$ level, the displacement is given by

$$\delta E_{1s} = -\int d\mathbf{k}\,|V_{1k}|^2/h\nu_{k1}, \tag{34}$$

where $$V_{1k} = Ze\int [\delta\rho(r)/r]d\mathbf{r} \tag{35}$$

and $\delta\rho(r)$ is the induced charge due to the charge representing the transition $1s$ to k. Using plane waves to represent the states of the continuous spectrum, a procedure justified in what follows since little contribution comes from the small values of k, and Eq. (33) for the induced charge, one obtains

$$\delta E_{1s} \cong -\frac{64\alpha}{225\pi^3 Z}\int \frac{k^2[(1+k^2)^{1/2}-1]dk}{(1+a^2k^2)^2} R. \tag{36}$$

Thus, the integral over k diverges for large k like dk/k, the effect of large components falling off too slowly on account of the k^2 from the d'Alembertian. Aside from the divergence the order of magnitude of the result is $Z^3\alpha^5$. The divergence is, of course, spurious. Convergence may be expected when the true expression for $X(K)$ from Eq. (31) is used. Using again plane waves to represent the states of the continuous spectrum, and Eqs. (30) and (31) for the induced charge, one obtains for the Lth component

$$\delta\rho(Lr) = \frac{e}{8\pi^4}\frac{(2a)^{3/2}\lambda_0^{3/2}}{[1+a^2|[\mathbf{K}+\mathbf{k}|^2]^2} X(L)\{\exp i\mathbf{K}\cdot\mathbf{r}/\lambda_0\} \tag{37}$$

with K and k in units of λ_0^{-1}, $a = (\alpha Z)^{-1}(m+M)/M$, k the space vector of the plane wave, and

$$L^2 = K^2 - (k_0 - E_1/\hbar c)^2,$$

where E_1, is the energy for the $1s$ electron, and k_0 is the time component of the propagation vector of the plane wave. The coefficient of $X(L)$ in Eq. (37) in the limit for large a behaves like $\delta(\mathbf{K}+\mathbf{k})$. Since large values of k are important, one can neglect the binding of the electron which is equivalent then in Eq. (37) to replacing $\mathbf{K}$ by $-\mathbf{k}$ in $X(L)$. Therefore, in this approximation using the definition of X from Eq. (31) and substituting $-\mathbf{k}$ for $\mathbf{K}$ in $\mathbf{L}$

[10] These calculations have been made also by E. Brunner to whom we are indebted for having told us of the results.

[11] The derivation of this result is given in the paper of R. Serber appearing in this issue.

[12] Heisenberg, reference 3, Eq. (44).

$$\delta\rho(Lr)=-\frac{\alpha e}{16\pi^5}\frac{(2a)^{3/2}\lambda_0^{3/2}}{[1+a^2|\mathbf{K}+\mathbf{k}|^2]^2}\{\exp i\mathbf{K}\cdot\mathbf{r}/\lambda_0\}\int_0^{\pi/2}\cos^3\psi d\psi\log[1+\tfrac{1}{4}\{k^2-(k_0-E_1/\hbar c)^2\}\cos^3\psi]. \quad (38)$$

Integrating over all values of $\mathbf{L}$ and introducing $\delta\rho(r)$ into Eq. (35) one obtains for the dominant term of V_{1k}

$$V_{1k}\sim-\frac{4Z\alpha e^2}{15\pi^2}\frac{(2a)^{1/2}\lambda_0^{1/2}}{1+a^2k^2}\log\{1+\tfrac{1}{2}[(1+k^2)^{1/2}-1]\}. \quad (39)$$

For large values of k this expression behaves like $(\log k)/k^2$ instead of like $1/k$ as obtained using the d'Alembertian. The integral for δE_{1s} will, therefore, converge, and the result for the non-secular terms is that δE_{1s} is of the order $Z^3\alpha^5$.

As a consequence of these calculations one finds that the energy levels are depressed, the displacement corresponding to a quantum defect $\Delta=n-n^*$ which is positive. It is not possible, therefore to ascribe any part of the observed discrepancy in the doublet separations of hydrogen to the new potential, since the experimentally observed values of the doublet separations in the Balmer series are always less than those predicted by the relativistic fine-structure theory, whereas, the displacement in the energy levels corresponding to the new potential correspond to an increased separation of the doublet components. Furthermore, the percentage change in the doublet separations would be smaller than the experimentally observed discrepancy of about 3 percent by a factor of at least 10. The nature of the potential necessary to explain the observed results has been considered by Kemble and Present.[13] It needs to be emphasized here, only that the potential to which the electron-positron theory gives rise is not of the requisite character.

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[13] Kemble and Present, Phys. Rev. **44**, 1031 (1933).

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The Raman Spectrum of Heavy Chloroform

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The Raman spectrum of chloroform made from deuterium has been photographed and the line shifts measured. The three types of isotope effects observed in the spectrum of heavy chloroform have been discussed. By treating different groups of atoms in the molecule as entities satisfactory agreement between calculated and observed values for the isotope displacements can be secured. Some idea of the nature of the role played by the chlorines in the C–H "bending" and "stretching" vibrations is obtained.

THE heavy chloroform, made from deuterium by Dr. F. W. Breuer of the Chemistry Department of Pennsylvania State College, was placed at our disposal for the study of its Raman spectrum. It was contained in a sealed glass tube 6 mm in diameter and 30 cm long, wrapped in black insulation tape with the exception of the lower 5 cm adjacent to the flat Pyrex glass window, the portion occupied by the fluid.

The arrangement of apparatus for irradiating the tube was essentially the same as that described by one of us[1] except that the high potential quartz mercury tube was replaced with the 220-volt Hanovia quartz mercury arc in its metal housing turned to the vertical position. As was shown in the previous paper this disposition prevents a temperature gradient across the diameter of the irradiation tube, with its attendant troublesome refraction, obviating as well the necessity of an air blast. The light from the arc

[1] R. W. Wood, *Raman Spectra of Heavy Water*, Phys. Rev. **45**, 392 (1934).