the electrons has been attempted, since no data are available for such a computation. Furthermore it is not known to what extent these metastable atoms are produced locally by recombination and to what extent they are swept down by the vapor from the region of the arc. Webb and Wang[2] found that both factors are important. They further showed that if into a moving plasma similar to that studied here, they introduced sodium vapor, a marked drop in electron temperature immediately resulted; in the case cited from 2300° to 800°K. The most important action of the sodium on the flowing vapor was to eliminate the metastable atoms by excitation of the sodium, and the most probable explanation of the temperature drop is that the elimination of the metastable atoms removed a source of electron energy, resulting in lowered electron temperature.

Webb and Sinclair[2] found that the rate of recombination depended to a large extent on the electron temperature, and came to the conclusion that a somewhat complex interaction between ions and fast and slow electrons was involved in the recombination process. Furthermore, a study of the intensity distribution in the afterglow spectrum leads to the conclusion that considerable kinetic energy may be freed by the process. It therefore seems probable that the recombination process is largely responsible for the energy supply which tends to hold up the electron temperature, either through some process involving many-body collisions, or by the metastable atoms which constitute a large part of the atoms formed by recombination. It may be noted that the recombination has been found by Webb and Sinclair and by Mohler and Boeckner[7] to increase with the pressure, which corresponds to the increase in the final electron temperature level with increase in pressure found here. On the other hand, for a given rate of production, the concentration of metastable atoms also tends to increase with increased pressure, as under these conditions the life of these atoms is practically determined by the diffusion to the wall.

The authors wish to express their appreciation to Mr. T. C. Hardy for his assistance in the taking of the measurements.

[7] Mohler and Boeckner, Bur. Standards J. Research **2**, 489 (1929).

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A Contribution to the Theory of the B. W. K. Method

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In order to put Zwaan's scheme for deriving the Kramers connection formulas on a rigorous basis, differential equations are set up governing the variation in the coefficients used to fit a linear combination of the B. W. K. type approximation functions to an exact solution of Schrödinger's equation in one dimension. Approximate solutions of the differential equations are worked out which lend themselves to the setting up of the connection formulas and give definite upper bounds to the errors involved in their use. The method is also used to set up the Sommerfeld phase integral quantum condition independently of the connection formulas. An upper limit to the error in the energy is worked out. A similar treatment of the problem of the transmission of matter waves through rounded potential barriers is formulated.

1. Introduction

Zwaan's[1] scheme for deriving the Kramers connection formulas[2] used in the Brillouin-Wentzel-Kramers (B. W. K.) method is simple and illuminating, but not entirely rigorous. In the attempt to put Zwaan's argument on a rigorous basis the author has developed a new way of studying the relation of the Stokes phenomenon to the B. W. K. method whose value extends beyond the scope of the connection formulas themselves.

Consider a one-dimensional problem in wave mechanics with the potential energy $V(z)$ and the classical local momentum

[1] A. Zwaan, *Intensitaten im Ca-Funkenspektrum*, Dissertation, Utrecht, 1929.

[2] H. A. Kramers, Zeits. f. Physik **39**, 828 (1926); H. A. Kramers and G. P. Ittman, Zeits. f. Physik **58**, 217 (1929).

$$p(z, E)=\{2\mu[E-V(z)]\}^{\frac{1}{2}}.$$

The Schrödinger equation takes the form

$$\psi''+\lambda^2p^2\psi=0 \quad (1)$$

with λ set equal to $2\pi/h$. The basic approximation functions of the B. W. K. method are

$$f_i(z, E)=p(z, E)^{-\frac{1}{2}}e^{\iota w(z, E)}; \quad f_v(z, E)=p(z, E)^{-\frac{1}{2}}e^{-\iota w(z, E)}; \quad (2)$$

$$w(z, E)=\lambda\int_{z_0}^{z} p(\zeta, E)d\zeta. \quad (3)$$

Here ι denotes $\sqrt{-1}$. The point z_0 can be chosen at pleasure but is ordinarily placed on the axis of reals. f_i and f_v are readily shown to be solutions of the differential equation

$$f''+[\lambda^2p^2-Q]f=0, \quad (4)$$

where Q denotes the function

$$Q=\frac{3}{4}\left(\frac{p'}{p}\right)^2-\frac{p''}{2p}\equiv\frac{1}{4}\left[\frac{5}{4}\left(\frac{V'}{E-V}\right)^2+\frac{V''}{E-V}\right]. \quad (5)$$

Wherever $|Q|\ll\lambda^2p^2$, Eq. (4) is very nearly the same as (1) and f_i, f_v may be said to be good approximate solutions of (1).

Let z' denote a simple real zero point of the function $E-V(z)$ (primes and double primes denote differentiation *except* when applied to the symbol z itself). For definiteness, we assume that $E-V$ is negative to the left of z' and positive to the right (cf. Fig. 1). At z', f_i, f_v and Q all become infinite. Hence f_i, f_v have no value as approximate solutions of (1) in the neighborhood of z'. We shall suppose, however, that there exist regions I and II to the left and right of z', respectively, where $|Q|\ll\lambda^2p^2$ and where the f's are good approximate solutions. Then if $\psi(z)$ is an exact solution of (1), we may reasonably assume that in any not too large portion of I, $\psi(z)$ is approximately equal to some suitable linear combination of f_i and f_v. Thus, if α_i and α_v are suitably chosen constants,

$$\alpha_if_i+\alpha_vf_v=\psi(z) \qquad \textit{in region I.}$$

Similarly

$$\beta_if_i+\beta_vf_v=\psi(z) \qquad \textit{in region II.}$$

It is well known that the coefficients β_i, β_v are not equal to α_i, α_v, respectively. The shift in the values of the coefficients is called the Stokes phenomenon and the problem is to determine the magnitude of this shift in terms of the values on one side of z', say in I.

Kramers method of attack is based on the assumption that $V(z)$ is sensibly linear in the interval between I and II. Under these circumstances $\psi(z)$ can be approximated by a suitable Bessel's function over the interval in question and this function constitutes a bridge between the B. W. K. approximations in I and II. Langer[3] has introduced a different type of approximation function involving a Bessel's function of the argument $w(x)$ defined in Eq. (3). The Langer approximation function degenerates into the B. W. K. type in I and II, but does not go to pieces near z'. This method is very powerful, but has some disadvantages with respect to that developed in the present article.

Zwaan's derivation of the connection formulas follows closely a line of thought previously developed by Stokes.[4] It is based on the introduction of complex values of the independent variable and the existence of a path Γ joining I and II, but passing around z' in the complex plane so that the inequality $|Q|\ll\lambda^2p^2$ is good along its entire length. If $V(z)$ is analytic on the axis of reals between I and II, and if its analytic extension into the upper half plane yields no roots of $V(z)-E$ near z', we may assume that such a path can be found.[5]

The essential feature of the situation to which Zwaan calls attention is that along a portion of any such path Γ the approximation f_i becomes

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[3] R. E. Langer, Trans. Am. Math. Soc. **33**, 23 (1931); **34**, 447 (1932); Bull. Am. Math. Soc. **40**, 545 (1934).

[4] Sir George Gabriel Stokes, *Math. and Phys. Papers*, Vol. 4, pp. 77–109 and 283–298.

[5] Let us assume that $V(z)-E$ is a polynomial of degree n, or can be accurately represented by such a polynomial in the interval between I and II. Then, if z_k denotes the kth root, $E-V=C\prod_{k=1}^{n}(z-z_k)$. It follows that

$$(E-V)'/(E-V)=\sum_{k=1}^{n}\frac{1}{z-z_k};$$

$$(E-V)''/(E-V)=\left(\frac{E-V'}{E-V}\right)^2-\sum_{k=1}^{n}\frac{1}{(z-z_k)^2}.$$

Hence

$$\frac{Q}{p^2}=\frac{1}{8\mu C\Pi_k(z-z_k)}\left[\frac{1}{4}\left(\sum_{k=1}^{n}\frac{1}{z-z_k}\right)^2+\sum_k\frac{1}{(z-z_k)^2}\right].$$

Inspection of this expression for Q/p^2 proves the above statement for the case that n is small.

very small, while f_v becomes very large. Hence f_v and f_i are said to be *dominant* and *subdominant*, respectively, in the region in question. Where this is true changes in the subdominant coefficient can take place even when $|Q|\ll\lambda^2p^2$. On the other hand the coefficient of the dominant approximation function cannot change much, since a small change in this coefficient makes a very large difference in the function. Zwaan assumes that the latter coefficient is constant.

2. The Differential Equations for the Coefficients

Granted that the changes do take place, we propose here to investigate them by setting up explicit differential equations for the variation in the coefficients. The formulation of these equations, which forms the essential contribution of the present paper, is reminiscent of a method of attack on one-dimensional problems in quantum mechanics due to Hill.[6]

Let $\psi(z)$ be an exact solution of (1). Let $u=a_if_i+a_vf_v$ be a linear combination of f_i and f_v which is fitted to ψ at the arbitrary point $z=z_1$. The best fit will evidently be obtained if u and du/dz are made equal to ψ and $d\psi/dz$, respectively, at the point z_1. In that case

$$a_if_i(z_1)+a_vf_v(z_1)=\psi(z_1); \tag{6}$$

$$a_if_i'(z_1)+a_vf_v'(z_1)=\psi'(z_1). \tag{7}$$

As the Wronskian determinant $f_if_v'-f_vf_i'$ does not vanish (it has the constant value $-4\pi\iota/h=-2\lambda\iota$) it is possible to choose a_i and a_v so as to satisfy these equations at an arbitrary point z_1 where f_i and f_v are uniquely defined. The coefficients so obtained will then vary *continuously* as z_1 moves around the path Γ, reducing to α_i and α_v where Γ meets the axis of reals in I and to β_i, β_v where Γ meets the axis of reals in II.

Dropping the subscript 1 in Eqs. (6) and (7), solving for a_i, and differentiating with respect to z, we obtain

$$\frac{da_i}{dz}=(\iota/2\lambda)(f_v''\psi-f_v\psi''). \tag{8}$$

On reduction with the aid of Eqs. (1) and (4)

$$\frac{da_i}{dz}=\frac{\iota}{2\lambda}Qf_v\psi=\frac{\iota}{2\lambda}\frac{Q}{p}[a_i+e^{-2\iota w}a_v]. \tag{9}$$

Similarly

$$\frac{da_v}{dz}=-\frac{\iota}{2\lambda}Qf_i\psi=-\frac{\iota}{2\lambda}\frac{Q}{p}[a_v+e^{2\iota w}a_i]. \tag{10}$$

These differential equations are *exact*. Their integration is equivalent to the integration of Eq. (1), although they blow up at the zeros of $p(z, E)$. If an exact integration were possible, it would be a simple matter to work out the exact eigenfunctions and eigenvalues of (1). The primary usefulness of Eqs. (9) and (10), however, lies in the possibility of deriving useful approximations from them.

3. The Approximate Integration of Eqs. (9) and (10)

Introducing a fixed point ξ, let us make the transformation

$$\begin{aligned} b_i(z,\xi)&=a_i(z)e^{-\iota F(z,\xi)};\\ b_v(z,\xi)&=a_v(z)e^{+\iota F(z,\xi)}, \end{aligned} \tag{11}$$

where

$$F(z,\xi)=\frac{1}{2\lambda}\int_\xi^z\frac{Q}{p}dz. \tag{11a}$$

This change of variables leads to the following simpler system of differential equations:

$$\begin{aligned} \frac{db_i}{dz}&=\frac{\iota}{2\lambda}\frac{Q}{p}e^{-2\iota(F+w)}b_v;\\ \frac{db_v}{dz}&=-\frac{\iota}{2\lambda}\frac{Q}{p}e^{2\iota(F+w)}b_i. \end{aligned} \tag{12}$$

Clearly b_i and b_v will be sensibly equal to a_i and a_v, respectively, at any point z which can be connected with ξ by a path along which $|Q/p|$ is small enough. It is convenient to define an index of quality μ_Λ for a path Λ in the complex plane by the equation

$$\mu_\Lambda=\frac{1}{\lambda}\int_\Lambda\left|\frac{Q}{p}\right|ds=\frac{1}{\lambda^2}\int_\Lambda\left|\frac{Q}{p^2}\right||dw|. \tag{13}$$

We shall hereafter refer to a path as good, provided that its index of quality is much less than unity. Any short path which gives a wide berth to the

[6] E. L. Hill, Phys. Rev. **38**, 1258 (1931).

zeros of $p(z, E)$ can ordinarily be assumed to be "good" in this sense. Since

$$\left|\frac{1}{\lambda}\int_\Lambda \frac{Q}{p} dz\right| \leqslant \mu_\Lambda,$$

it follows that $|F(z, \xi)| \leqslant \frac{1}{2}\mu_\Lambda$ if z and ξ are any two points on Λ. b_i and b_v are thus sensibly equal to a_i and a_v at any point z which can be connected with ξ by a good path.

It will simplify the discussion of the integration of Eqs. (12) if we assume at first a definite path of integration Λ, of total length l, leading from an initial point Z_0 to a final point Z_1, and having the property that the imaginary part of w is a monotonically decreasing function of the path length s measured from Z_0. In other words, we assume that $|e^{\iota w}|$ increases monotonically as we move from Z_0 to Z_1 along Λ. With these restrictions we introduce the real variable of integration s and the notation

$$w(z) = \omega(s), \quad \iota Q(z)/2\lambda p(z) = \Sigma(s), \qquad dz/ds = \varphi(s), \quad F(z, \xi) = \theta(s, \xi). \tag{14}$$

Eqs. (12) take the form

$$db_i/ds = \Sigma(s)\varphi(s)e^{-2\iota(\theta+\omega)}b_v, \tag{15}$$

$$db_v/ds = -\Sigma(s)\varphi(s)e^{2\iota(\theta+\omega)}b_i. \tag{16}$$

Under favorable circumstances b_i can be computed approximately for points on Λ by replacing b_v in (15) by its initial value $b_v^0 \equiv b_v(Z_0)$. Let $\tilde{b}_i(z)$ denote the corresponding approximation for $b_i(z)$. Then

$$\tilde{b}_i(s) = b_i^0 + b_v^0 \int_0^s \Sigma(s_1)\varphi(s_1)e^{-2\iota[\theta(s_1)+\omega(s_1)]}ds_1. \tag{17}$$

This approximation can be tested by computing an upper bound on $|b_i(Z_1) - \tilde{b}_i(Z_1)|$. We have

$$b_i(Z_1) - \tilde{b}_i(Z_1) = \int_0^l [b_v(s) - b_v^0]\Sigma(s)\varphi(s)e^{-2\iota(\theta+\omega)}ds. \tag{18}$$

But from (16)

$$b_v(s) - b_v^0 = -\int_0^s b_i(s_1)\Sigma(s_1)\varphi(s_1)e^{2\iota[\theta(s_1)+\omega(s_1)]}ds_1. \tag{19}$$

Hence

$$\begin{aligned} b_i(Z_1) - \tilde{b}_i(Z_1) &= -\int_0^l ds\,\{\Sigma(s)\varphi(s)e^{-2\iota[\theta(s)+\omega(s)]}\int_0^s b_i(s_1)\Sigma(s_1)\varphi(s_1)e^{-2\iota[\theta(s_1)+\omega(s_1)]}ds_1\} \\ &= -\iint_T b_i(s_1)\Sigma(s_1)\Sigma(s)\varphi(s_1)\varphi(s)e^{-2\iota[\theta(s)-\theta(s_1)+\omega(s)-\omega(s_1)]}ds_1 ds, \end{aligned} \tag{20}$$

where T is a triangle in the s, s_1 plane defined by the inequalities $0 \leqslant s_1 \leqslant s$; $0 \leqslant s \leqslant 1$. Since $|e^{\iota w}|$ increases monotonically along Λ, $|e^{-2\iota[\omega(s)-\omega(s_1)]}| \leqslant 1$ at every point in T. $\varphi(s)$ is of unit absolute value and

$$|e^{-2\iota[\theta(s)-\theta(s_1)]}| = |e^{-(\iota/\lambda)\int_{s_1}^s [Q(s')/p(s')]\varphi(s')ds'}| \leqslant e^{\mu_\Lambda}.$$

Introducing the symbol M_i for the maximum value of $|b_i|$ on Λ, we readily derive the inequality

$$|b_i(Z_1) - \tilde{b}_i(Z_1)| < M_i e^{\mu_\Lambda} \iint_T |\Sigma(s_1)||\Sigma(s)| ds_1 ds. \tag{21}$$

The function $|\Sigma(s)||\Sigma(s_1)|$ being symmetrical with respect to its arguments, the above integral over T is half the corresponding integral over the complete square $0 \leqslant s \leqslant 1$; $0 \leqslant s_1 \leqslant 1$. Thus

$$|b_i(Z_1) - \tilde{b}_i(Z_1)| < \tfrac{1}{2}M_i e^{\mu_\Lambda}\{\int_0^l |\Sigma(s)|ds\}^2 = \mu_\Lambda^2 e^{\mu_\Lambda}(M_i/8). \tag{22}$$

We conclude that in the case of a good path leading uphill with respect to $|e^{\iota w}|$ the approximation (17) will lead to an error which is small in comparison with the maximum value of $|b_i|$ on the path.

Let w_0 denote the value of w at the point Z_0. The maximum absolute value of the factor $e^{-2\iota\omega(s_1)}$ of the integrand of Eq. (17) is $|e^{-2\iota w_0}|$. Furthermore on the path Λ $|e^{-2\iota\theta}| \leqslant e^{2|\theta|} < e^{\mu_0+\mu_\Lambda}$ where μ_0 denotes the index of quality of the best path leading from ξ to Z_0. We assume that μ_0 is a small quantity of the order of μ_Λ. Eq. (17) yields

$$|\tilde{b}_i(Z_1) - b_i^0| < \tfrac{1}{2}\mu_\Lambda e^{\mu_0+\mu_\Lambda}|b_v^0||e^{-2\iota w_0}|.$$

Combining this inequality with (22), we obtain

$$|b_i(Z_1)| < |b_i^0| + \tfrac{1}{2}\mu_\Lambda e^{\mu_0+\mu_\Lambda}|b_v^0|\,|e^{-2\iota w_0}| + \mu_\Lambda^2 e^{\mu_\Lambda}(M_i/8). \tag{23}$$

Let Z_m denote the point on Λ where $b_i(z)$ becomes equal to M_i, and let μ_Λ' denote the quantity $1/\lambda \int_{Z_0}^{Z_m}|Q/p|ds$, where the integration is carried out along Λ. Then $\mu_\Lambda' \leqslant \mu_\Lambda$. Clearly an equation similar to (23) applies to the point Z_m. Thus

$$M_i < |b_i^0| + (\mu_\Lambda'/2)e^{\mu_0+\mu_\Lambda'}|b_v^0|\,|e^{-2\iota w_0}| + \mu_\Lambda'^2 e^{\mu_\Lambda'}(M_i/8).$$

Transposing and neglecting terms of higher power in the small quantities μ_0, μ_Λ, we obtain $M_i < |b_i^0| + \frac{1}{2}\mu_\Lambda|b_v^0|\,|e^{-2\iota w_0}|$. It follows that in this approximation

$$|b_i(Z_1) - b_i^0| < |b_i(Z_1) - \tilde{b}_i(Z_1)| + |\tilde{b}_i(Z_1) - b_i^0| < \tfrac{1}{2}\mu_\Lambda|b_v^0|\,|e^{-2\iota w_0}| + \tfrac{1}{8}\mu_\Lambda^2|b_i^0|. \tag{24}$$

Consider next the change in a_i. We have

$$|a_i(Z_1) - a_i^0| = |e^{\iota F(Z_0,\,\xi)}|\,|b_i(Z_1)e^{\iota/2\lambda \int_\Lambda (Q/p)dz} - b_i^0|,$$

where

$$|F(Z_0,\xi)| < \tfrac{1}{2}\mu_0 \quad \text{and} \quad \left|\frac{1}{2\lambda}\int_\Lambda \frac{Q}{p}dz\right| < \tfrac{1}{2}\mu_\Lambda.$$

Hence, neglecting small quantities of higher order,

$$|a_i(Z_1) - a_i^0| < e^{\mu_0/2}\{e^{\mu_\Lambda/2}|b_i(Z_1) - b_i^0| + |b_i^0|(\mu_\Lambda/2)\} < \tfrac{1}{2}\mu_\Lambda[|b_i^0| + |b_v^0|\,|e^{-2\iota w_0}|].$$

Finally, it follows from (11) that to this same approximation $|b_i^0| - |a_i^0| < |a_i^0|(\mu_0/2)$. Consequently

$$|a_i(Z_1) - a_i^0| < \tfrac{1}{2}\mu_\Lambda[|a_i^0| + |a_v^0|\,|e^{-2\iota w_0}|]. \tag{25}$$

We conclude that the variation in a_i is small in comparison with the larger of the two quantities $|a_i^0|$ and $|a_v^0|\,|e^{-2\iota w_0}|$.

In applying the above result two special cases are to be considered. The first is that of a good uphill path restricted by the additional requirement that $|e^{\iota w}| \geqslant 1$ along its entire length. (It will be convenient to refer to regions in which $|e^{\iota w}| > 1$ as *mountains*, and to regions in which $|e^{\iota w}| < 1$ as *valleys*. Clearly f_i is dominant and f_v subdominant on the upper levels of the mountains. These roles are reversed for the lower levels of the valleys.) In this case $|e^{-2\iota w_0}| < 1$ and $|a_i(Z_1) - a_i^0|$ is small compared with the average of the quantities $|a_i^0|$ and $|a_v^0|$. Under these circumstances we shall say that a_i is sensibly constant along the path, although if $|a_i^0| \ll |a_v^0|$, it may vary by a large percentage of its initial absolute value.

In order to deal with the variation in a_i along a good path leading from Z_0 clear over a mountain to Z_1, but not entering the adjacent valleys, it is convenient to think of the path as made up of two parts leading to the crest from opposite sides. Applying the above argument to each part of the path we come to the conclusion that a_i is *sensibly constant* over the entire path in comparison with the largest of the four quantities $|a_i^0|$, $|a_v^0|$, $|a_i(Z_1)|$, $|a_v(Z_1)|$.

A second important case of the application of Eq. (25) is that of a good path leading uphill, but beginning in a valley, where $|e^{\iota w}| < 1$ and f_v is dominant. If the valley is deep $|e^{-2\iota w_0}|$ is a very large number, but if $|a_v^0|\,|e^{-2\iota w_0}| \leqslant |a_i^0|$, we have the inequality

$$|a_i(Z_1) - a_i^0| < \mu_\Lambda|a_i^0| \tag{26}$$

and can say that $a_i(z)$ is sensibly constant along the good path Λ. Evidently $a_i(z)$ will also be sensibly constant along a good path Λ extending across a valley from one mountain to another, provided that $|a_v(z)||e^{-2\iota w(z)}| \leqslant |a_i(z)|$ at the lowest point of the path.

Turning to $a_v(z)$, we readily derive formulas parallel to those for $a_i(z)$ provided that we perform the integrations in the opposite direction on Λ, i.e., downhill with respect to $|e^{\iota w}|$. Thus, if we take a_i^0 and a_v^0 as the values of a_i and a_v at the upper end of the path, we obtain instead of (25) the inequality

$$|a_v(Z_1) - a_v^0| < \tfrac{1}{2}\mu_\Lambda\{|a_v^0| + |a_i^0||e^{2\iota w_0}|\}. \quad (27)$$

In general the coefficient of the dominant approximation f_i or f_v is sensibly constant over a good path Λ on which $|e^{\iota w}| - 1$ does not change sign. The coefficient of the subdominant term is also sensibly constant over such a path provided that the absolute value of the dominant term at the point of maximum dominance is zero, or smaller than the product (evaluated at the same point) of the absolute value of the coefficient of the subdominant term multiplied by the smaller of the two quantities $|e^{\pm 2\iota w}|$. However, since we have defined the phrase "sensibly constant" somewhat loosely, it is perhaps best to base all rigorous arguments directly on the inequalities (25) and (27).

4. Derivation of the Connection Formulas

We assume that the potential function in the neighborhood of the classical turning point z' has the form indicated in Fig. 1. Introducing polar coordinates r, θ with the origin at z', we write p^2 in the form

$$p^2 = 2\mu(E - V) = re^{\iota\theta}\varphi(z),$$

where $\varphi(z)$ is assumed to be bounded from zero in a wide region of the complex plane about z'. We further assume the existence of a path passing around z' in the upper half plane (cf. Fig. 3) enclosing no complex zeros of p^2 and good in the technical sense that $\mu_\Lambda \ll 1$. Let P_1 and

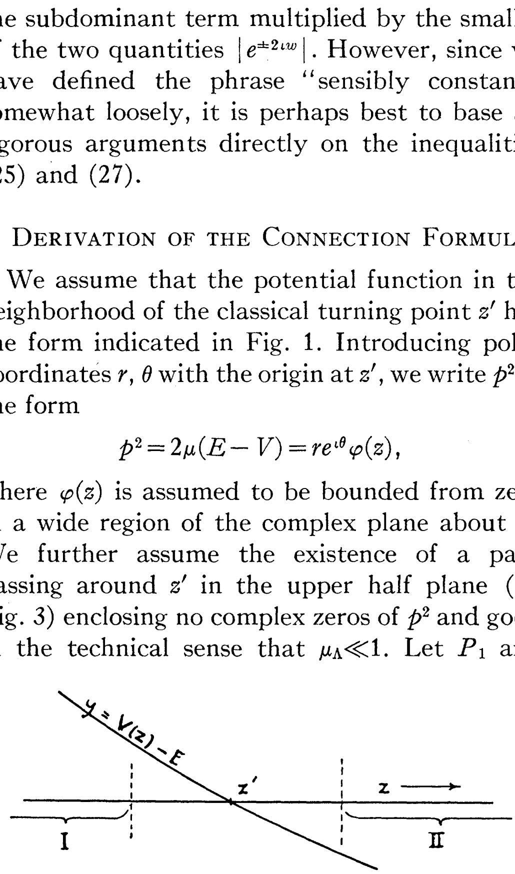

Fig. 1.

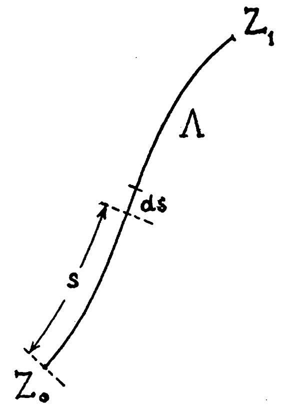

Fig. 2.

P_2 denote the end points of Λ located in regions I and II, respectively.

To avoid ambiguity due to the multiple-valued character of p, $p^{\frac{1}{2}}$, and w, we restrict the angle θ to the range $0 \leqslant \theta \leqslant 2\pi$ and define p and $p^{\frac{1}{2}}$ by the equations

$$p = r^{\frac{1}{2}}e^{\iota\theta/2}\varphi^{\frac{1}{2}} : \quad p^{\frac{1}{2}} = r^{\frac{1}{4}}e^{\iota\theta/4}\varphi^{\frac{1}{4}}.$$

$\varphi^{\frac{1}{4}}$ can be treated as single-valued. Let the point z_0 [Eq. (3)] be identified with z'. Then in the region I we have

$$p = \iota|p| : \quad p^{\frac{1}{2}} = e^{\iota\pi/4}|p|^{\frac{1}{2}} : \qquad (28)$$
$$w = -\iota\lambda\int_z^{z'}|p|d\zeta.$$

In the region II to the right of z'

$$p = |p|; \quad p^{\frac{1}{2}} = |p|^{\frac{1}{2}} : \quad w = \lambda\int_{z'}^{z}|p|d\zeta. \quad (29)$$

A cut extending from z' to the right along the axis of reals insures that we have a single branch of $w(z)$ to deal with.

In order to visualize the problem it is convenient to map the level lines of $|e^{\iota w}|$. Eqs. (28) and (29) show directly that the axis of reals passes through a mountain to the left of z', but that the portion of the axis of reals to the right of z' is a level line with $|e^{\iota w}| = 1$. Writing $z = x + \iota y$; $p = |p|e^{\iota\chi}$, the differential equation for the level lines is seen to be

$$dy/dx = \iota\frac{(p - p^*)}{(p + p^*)} = -\tan\chi, \quad (30)$$

where p^* is the complex conjugate of p. Treating φ in first approximation as constant over the

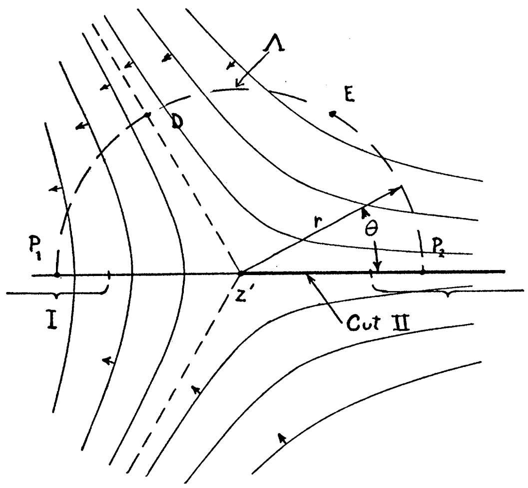

FIG. 3. Level lines of $|e^{\iota w}|$ for linear potential function. $|e^{\iota w}|$ is unity along cut and dotted lines radiating from Z'. Arrows indicate the uphill direction.

neighborhood of z' we obtain the equation $dy/dx = -\tan \frac{1}{2}\theta$ for the neighborhood in question. Fig. 3 shows the approximate form of the level lines derived from the above differential equation. Three lines radiate from the branch point z' and from every other simple zero point of $E-V(z)$. The exact forms of the contour lines will, of course, depend on $\varphi(z)$, but the distortion produced by φ is sure to be small if φ is smooth in the neighborhood of z'.

We identify the point ξ of Eqs. (11) with P_1, thus insuring that the a's and b's are sensibly equal on Λ. Let D denote the point where Λ crosses the line $|e^{\iota w}|=1$ which extends into the upper half plane. Then a_v will be sensibly constant between D and P_2 because f_v is dominant over this portion of Λ. It will also be constant over P_1D if a_i vanishes, or is very small, at P_1. It follows that if $\alpha_i \equiv a_i(P_1)=0$, $\alpha_v \cong \beta_v$. This fact, together with the reality condition gives the first connection formula.

Let us cast the argument in rigorous form. Denoting the point on Λ where $|e^{\iota w}|$ has its minimum value by E and the corresponding values of a_i and a_v by γ_i and γ_v, respectively, we apply Eq. (27) first to the arc P_1E and then to P_2E. Let μ_1 and μ_2 be the values of μ for these two sections of the path Λ. Then

$$\begin{aligned} |\gamma_v-\alpha_v| &< \tfrac{1}{2}\mu_1\{|\alpha_v|+|\alpha_i|\,|e^{2\iota w(P_1)}|\}, \\ |\gamma_v-\beta_v| &< \tfrac{1}{2}\mu_2\{|\beta_v|+|\beta_i|\}. \end{aligned} \tag{31}$$

Consider the application of these equations to a solution of Eq. (1) which is real on the axis of reals and which is of the f_v type at P_1. From the reality condition it follows that $\beta_i=\beta_v^*$ and $|\beta_i|=|\beta_v|$. If the function is to be of f_v type at P_1 we may assume that $|\alpha_i|\,|e^{2\iota w(P_1)}| \leqslant |\alpha_v|$. Hence

$$\begin{aligned} |\alpha_v-\beta_v| &< |\gamma_v-\alpha_v|+|\gamma_v-\beta_v| \\ &< \mu_1|\alpha_v|+\mu_2|\beta_v| < \mu_\Lambda\{|\alpha_v|+|\beta_v|\}. \end{aligned}$$

We can accordingly identify α_v and β_v in first approximation.

It follows from (28) that if α_v is given the value $e^{\iota\pi/4}$ the product $\alpha_v f_v$ is real in I and describes a real ψ function. Then $\beta_v=\beta_i^*=e^{\iota\pi/4}$. Thus the connection formula

$$\underbrace{e^{\iota\pi/4}f_v \equiv |p|^{-\frac{1}{2}}e^{-\lambda\int_z^{z'}|p|\,d\zeta}}_{Region\ I} \longrightarrow \underbrace{2p^{-\frac{1}{2}}\cos\left[\lambda\int_{z'}^{z}p\,d\zeta-\frac{\pi}{4}\right]}_{Region\ II} \tag{32}$$

is established.

In this formula, following Langer, we draw the arrow from left to right to indicate that the approximate validity of the left-hand member implies that of the right, but that the converse statement is not true. In order to make clear the justification of this one-way street sign and to prepare the way for the derivation of the second connection formula, we observe that in virtue of the homogeneous linear character of Eqs. (9) and (10), the functions $a_i(z)$, $a_v(z)$ must be homogeneous linear in the constants of integration α_i, α_v. Hence β_i and β_v must be homogeneous linear functions of α_i and α_v. Thus a complete exact solution of the whole connection problem would involve a complete and exact determination of the matrix $\|g\|$ of the equations

$$\beta_i=g_{ii}\alpha_i+g_{iv}\alpha_v, \qquad \beta_v=g_{vi}\alpha_i+g_{vv}\alpha_v. \tag{33}$$

Our approximate solution of Eqs. (9) and (10) for the special case that $\alpha_i=0$ shows that

$$g_{vv}=1+\delta; \qquad g_{iv}=-\iota(1+\delta^*), \tag{34}$$

where δ is a small quantity of the order of magnitude of μ_Λ which we have neglected in (32).

Additional information regarding the matrix $\|g\|$ is obtainable from the fact that the Wronskian of any two exact solutions of the Schrödinger equation, say ψ and $\bar{\psi}$, is constant along the axis of reals.[7] Thus if

$$\psi = \alpha_i f_i + \alpha_v f_v; \qquad \bar{\psi} = \bar{\alpha}_i f_i + \bar{\alpha}_v f_v$$

in I, it follows that in I

$$W[\psi, \bar{\psi}] \equiv \psi \frac{\partial \bar{\psi}}{\partial z} - \bar{\psi} \frac{\partial \psi}{\partial z} = 2\iota\lambda(\alpha_v \bar{\alpha}_i - \alpha_i \bar{\alpha}_v) = \text{constant}.$$

The Wronskian of the same pair of solutions in II takes the form

$$W[\psi, \bar{\psi}] = 2\iota\lambda(\alpha_v \bar{\alpha}_i - \alpha_i \bar{\alpha}_v)(g_{vv} g_{ii} - g_{iv} g_{vi}).$$

Equating these two expressions for the Wronskian, we see that *the determinant of* $\|g\|$ *is unity.* Hence the inverse equations to (33) are

$$\alpha_i = g_{vv}\beta_i - g_{iv}\beta_v, \qquad \alpha_v = -g_{vi}\beta_i + g_{ii}\beta_v. \quad (35)$$

It follows from the reality condition that if $\beta_i = \beta_v^*$, the quantities $\alpha_i e^{\iota\pi/4}$ and $\alpha_v e^{\iota\pi/4}$ are real. Hence

$$g_{vi} = -\iota g_{ii}^*; \qquad g_{iv} = -\iota g_{vv}^*. \quad (36)$$

Combining the last relation with the requirement that $|g|$ be equal to unity, we obtain the following relation between g_{ii} and δ:

$$g_{ii}(1+\delta) + g_{ii}^*(1+\delta^*) = 1. \quad (37)$$

If we neglect small quantities of the order of μ_Λ in comparison with unity, we may conclude that the real part of g_{ii} is $\frac{1}{2}$. The imaginary part of g_{ii} cannot be determined even approximately without an explicit evaluation of an integral of the form given in Eq. (17) over a path on which the exponential factor is very large. However, if we apply the inequalities (31) to the evaluation of $|\beta_v|$ when $\alpha_v = 0$ we can derive the upper bound

$$|g_{ii}| = |g_{vi}| < \mu_\Lambda |e^{2\iota w(P_1)}| = \mu_\Lambda \exp\left[2\lambda \int_{P_1}^{z'} |p| d\zeta\right].$$

[7] The writer is indebted to Dr. Eugene Feenberg for the observation that the second connection formula can be derived from the first by means of the properties of the Wronskian.

It does not seem necessary to include the details of this simple proof here.

Let us now consider the validity of the inverse of Eq. (32). If the relation in question were exact, we could, of course, invert it; but it is actually only approximate. We know from the derivation that if the left-hand member fits the function $\psi(z)$ exactly at P_1, the right-hand member fits the same function approximately at P_2 and along the neighboring portion of the axis of reals. The inverse relation would be the statement that if the right-hand member fits $\psi(z)$ exactly at P_2, the left-hand member is approximately correct at P_1. With the aid of our partial determination of the matrix $\|g\|$ we can test the validity of this statement. We accordingly assume as in the right-hand member of (32) that $\beta_i = \iota\beta_v = e^{\iota\pi/4}$. It follows that the corresponding values of α_v and α_i are not $e^{\iota\pi/4}$ and 0, but $e^{\iota\pi/4}[1-\delta^* + g_{ii}(\delta^* - \delta)]$ and $(\delta - \delta^*)e^{-\iota\pi/4}$. Although the correction to the value of α_i is small, its product by the dominant approximation function f_i is not necessarily small. Hence the left-hand member of (32) may be entirely incorrect if the right-hand member is exact.

The second connection formula is a direct consequence of (35) and our information regarding $\|g\|$. Consider an exact solution of (1) which has the form

$$\psi(x) = 2p^{-\frac{1}{2}} \cos(w+\gamma) = e^{\iota\gamma} f_i + e^{-\iota\gamma} f_v$$

in the neighborhood of P_2. By (35) the corresponding values of α_i and α_v are, if we neglect small quantities of the order of μ_Λ,

$$\alpha_i = 2e^{\iota\pi/4} \cos\left(\gamma - \frac{\pi}{4}\right):$$

$$\alpha_v = e^{\iota\pi/4}\left[2x \cos\left(\gamma - \frac{\pi}{4}\right) - \sin\left(\gamma - \frac{\pi}{4}\right)\right].$$

Here we have introduced the symbol x, for the unknown, but finite, imaginary part of g_{ii}. If γ is not too close to the critical value $-\pi/4$ (cf. the *first* connection formula) so that $|\cos(\gamma - \pi/4)|$ is not *much* smaller than $|\sin(\gamma - \pi/4)|$, the product $\alpha_v f_v$ is small compared with $\alpha_i f_i$. Hence we have the connection formula

$$\underbrace{\cos\left(\gamma-\frac{\pi}{4}\right)|p|^{-\frac{1}{2}}e^{+\lambda\int_z^{z'}|p|\,d\zeta}}_{\text{Region I}}\leftarrow\underbrace{p^{-\frac{1}{2}}\cos\left[\lambda\int_{z'}^{z}p\,d\zeta+\gamma\right]}_{\text{Region II}}. \tag{38}$$

This is usually specialized by writing $\gamma=\pi/4$, but the possibility of a generalization has been indicated by Langer (last reference, Note 3).

In addition to the restriction on γ, the only condition for the validity of (32) and (38) is that there shall exist a path Λ in the upper half plane connecting P_1 with P_2, enclosing no complex zeros of p^2 and having the property that $\mu_\Lambda \ll 1$.

Both connection formulas in the form given above presuppose that the slope of the real curve $y=V(z)$ is negative at z'. In order to obtain the connection formulas for the case in which $V(z)$ has a positive slope at the classical turning point z' it is only necessary to make the transformation $z\rightarrow -z$ in (32) and (38) and to interchange the roles of the regions I and II.

5. Limiting Case Where One of the Connected Points Is at Infinity

So far we have assumed that the points P_1 and P_2 are located at finite distances from z' and can be connected by a good path Λ of finite length. In applying the B. W. K. method to energy level problems, however, we frequently wish to assume that $\psi(z)$ vanishes at $z=\pm\infty$. In this case it becomes necessary to discuss the changes in the coefficients a_i, a_v which occur in a path which extends to infinity along the axis of reals. Consider, for example, a case in which $p(z)$ is imaginary and bounded from zero for all real values of z to the left of the turning point z' of Fig. 1. Let it be required to find the values of β_i, β_v appropriate to P_2 when a_i is known to vanish at $z=-\infty$. The required transformation of the a's can be thought of as made in two steps, of which the first gives the values at P_1 in terms of those at $-\infty$ and the second is the one already studied giving the values at P_2 in terms of those at P_1.

Consider first the value of a_i at P_1, which as before we shall call α_i. It follows from Eq. (9) that

$$\alpha_i=\frac{\iota}{2\lambda}\int_{-\infty}^{P_1}Qf_v\psi dz. \tag{39}$$

As both f_v and ψ approach zero exponentially for large negative values of z, $|\alpha_i|$ is clearly very small. We compute an upper limit. Let z_1 denote the value of z at P_1. Eq. (39) is equivalent to

$$\alpha_i=\frac{\iota}{2\lambda^2}\int_{w(-\infty)}^{w(z_1)}\frac{Qe^{-\iota w}}{p^{\frac{3}{2}}}\psi dw.$$

Hence

$$|\alpha_i|<\tfrac{1}{2}\int_{w(-\infty)}^{w(z_1)}\left|\frac{Q}{\lambda^2p^{\frac{3}{2}}}\right|e^{-|w|}|\psi|\,|dw|.$$

Let m denote the maximum value of $|Q/\lambda^2p^{\frac{3}{2}}|$ in the range of integration. This we may assume to be small. Since $|\psi(z)|$ decreases monotonically as we pass from z_1 out to $-\infty$ along the axis of reals,

$$\begin{aligned}|\alpha_i|&<\frac{m}{2}|\psi(z_1)|\int_{w(z_1)}^{w(-\infty)}e^{-|w|}d|w|\\&<\frac{m}{2}|\psi(z_1)|e^{-|w(z_1)|}\\&<\frac{m}{2|p(z_1)^{\frac{1}{2}}|}[|\alpha_i|+|\alpha_v|e^{-2|w(z_1)|}].\end{aligned} \tag{39a}$$

If $m/|p(z_1)^{\frac{1}{2}}|$ is much less than unity, $|\alpha_i| \ll|\alpha_v|e^{-2|w(z_1)|}$.

By analogy with the corresponding problem for a finite path one expects that α_v is sensibly equal to the limiting value of $a_v(z)$ as z moves out to $-\infty$. It is not necessary, however, to investigate this point in order to deal with the eigenvalue-eigenfunction problem.

Consider the application of Eqs. (33) to the determination of β_i and β_v when $a_i(z)$ vanishes at the negative end of the axis of reals. In view of the upper bounds on $|\alpha_i|$ and $|g_{vi}|$

$$|g_{vi}\alpha_i|<\frac{m}{2|p(z_1)^{\frac{1}{2}}|}\mu_\Lambda|\alpha_v g_{vv}|.$$

Thus, by Eqs. (33), if P_1 is connected with P_2

by a good path, and if $|Q(z)/\lambda^2 p(z)^{\frac{3}{2}} p(z_1)^{\frac{1}{2}}|$ is small between P_1 and $-\infty$, β_v has sensibly the same value as if α_i were zero, i.e., β_v is sensibly equal to α_v. We conclude that the connection formula (32) remains valid when the point P_1 is moved out to infinity along the negative axis of reals, provided that $m/|p(z_1)^{\frac{1}{2}}|\ll 1$.

6. Evaluation of the Low Energy Levels of a One-Dimensional Oscillator

The usual wave mechanics derivation of the Sommerfeld "phase integral" quantum condition (with half-integral quantum numbers) is based on the use of the connection formulas and loses its validity when applied to the lower energy levels. Thus in the case of the Planck ideal linear oscillator, where $V(z)=\frac{1}{2}kx^2$, we have

$$Q=k(E+6V)/(E-V)^2. \tag{40}$$

The minimum value of $|Q/\lambda^2 p^2|$ consistent with the reality of p occurs at the origin and is $k/(8\lambda^2\mu E^2)$. In the case of the lowest energy level the value is $\frac{1}{2}$ and the corresponding minimum value of $\lambda^{-1}\int|Q/p|ds$ for any path joining the region of classical vibration with any point outside this region is $\frac{1}{3}$. As this is of the order of unity, appreciable departures from the connection formulas are to be expected for this lowest energy level, and it is surprising to note that the value of E given by the phase integral formula is in *exact* agreement with the rigorous value obtained by other methods. Thus one is led to inquire whether there is not some way of validating the Sommerfeld formula without assuming the existence of good paths joining the region of classical vibration with the regions of imaginary momentum on the axis of reals.

Birkhoff[8] has recently sketched a proof of the phase integral formula applicable to the lower energy levels. Partly, however, due to the fact that Birkhoff's technique is unfamiliar to physicists, and partly because his proof rests on a plausible, but as yet unverified assumption, it seems worth while to give an independent derivation here.

Let the problem under discussion be that of finding the discrete energy levels of a one-dimensional oscillator with the fundamental interval $-\infty<z<+\infty$ and a single potential valley in which the classical local momentum $p(z, E)$ takes on real values. We shall assume at first that $V(z)$ has the special parabolic form $V=\frac{1}{2}kz^2$ and consider later on the modification of the argument for an anharmonic oscillator.

Let the complex z plane have a cut extending from the left-hand classical turning point z' through the right-hand turning point z'' to $+\infty$ along the axis of reals. The functions p, $p^{\frac{1}{2}}$ and w are uniquely defined over the cut plane by the requirement that they conform to Eqs. (28) for all negative real values of $z-z'$. They will then conform to Eqs. (29) on the upper lip of the cut between z' and z''. For positive real values of $z-z''$ we have

$$\begin{aligned}&\text{Upper lip:}\quad p=-\iota|p|;\quad p^{\frac{1}{2}}=e^{-\iota\pi/4}|p^{\frac{1}{2}}|;\quad w=\lambda\int_{z'}^{z''}|p|d\zeta-\iota\lambda\int_{z''}^{z}|p|d\zeta;\\&\text{Lower lip:}\quad p=-\iota|p|;\quad p^{\frac{1}{2}}=-e^{-\iota\pi/4}|p^{\frac{1}{2}}|;\quad w=-\lambda\int_{z'}^{z''}|p|d\zeta-\iota\lambda\int_{z''}^{z}|p|d\zeta.\end{aligned} \tag{41}$$

We designate by III that portion of the region of imaginary momentum on the axis of reals to the right of z'' where $|Q/\lambda^2p^2|\ll 1$ (cf. Fig. 4). It follows from Eqs. (41) that $|e^{\iota w}|$ is very large in both of the regions I and III. The level lines of $|e^{\iota w}|$ are readily sketched in and show that any path connecting I with III and avoiding the classical portion of the axis of reals must cross a single valley between the ridges on which it terminates. This is true whether the path lies in the upper or lower half plane.

Let z_1 and z_3 be any two points on the axis of reals in I and III, respectively. It follows from (40) that in the case of the parabolic potential function under consideration it is always possible to join z_1 and z_3 by a path Λ for which $\mu_\Lambda\ll 1$. In more general cases such a path will exist provided

[8] G. D. Birkhoff, Bull. Am. Math. Soc. 39, 696 (1933).

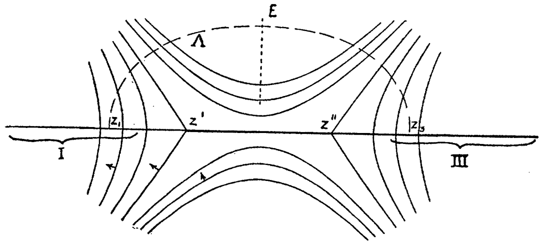

FIG. 4. Level lines of $|e^{\iota w}|$ for harmonic oscillator.

there are no complex zeros of $V-E$ near to z' and z''.

Let us now assume that we have to do with an eigenvalue of the energy $E=E_n$ and a corresponding eigenfunction $\psi_n(z)$. Then ψ_n vanishes at both ends of the axis of reals. Since $f_i(z)$ becomes infinite at both ends of this axis, a_i must vanish there. It follows from our previous analysis that a_i is very small at both of the points z_1 and z_3. Let E denote the point where the path Λ crosses the bottom of the valley, i.e., the point on Λ where $|e^{\iota w}|$ takes on its minimum value. Integration from z_1 to E shows that a_v is sensibly constant along the corresponding portion of Λ. Similarly integration from z_3 to E shows that it is constant along the rest of Λ. If we reflect the path Λ in the axis of reals we get a second path connecting z_1 with z_3 along which a_v is constant neglecting small quantities of the order of $\mu_\Lambda a_v(z_1)$. The existence of these two paths shows that a_v takes on the same values on the upper and lower edges of the cut at z_1. ψ_n is, of course, single-valued along the entire axis of reals. Hence $f_v(z)$ must have the same value on the upper and lower edges of the cut at z_3. Equating the two values of f_v in question and using the corresponding values of $p^{\frac{1}{2}}$ and w given in Eqs. (41), we obtain

$$\exp\{\lambda \textstyle\int_{z'}^{z''}|p|d\zeta\} = -\exp\{-\lambda \textstyle\int_{z'}^{z''}|p|d\zeta\},$$

or the familiar equation

$$\oint p dz = (n+\tfrac{1}{2})h. \tag{42}$$

The accuracy of the result is limited only by the quality of the path Λ. But in the case of the ideal linear oscillator it is possible to reduce the value of μ_Λ below any assignable quantity by choosing the path Λ as a semicircle of sufficiently large radius with the origin as its center. Hence the above equation is *exact* for the case under discussion.

Turning now to the problem of the *anharmonic oscillator*, we observe that in order to secure single-valued functions f_i, f_v it is necessary to introduce cuts extending from the complex roots of $E-V$ to infinity. The argument then goes through exactly as in the case of the harmonic oscillator except that the path Λ must lie inside all the roots above mentioned in order to avoid crossing the cuts. Consequently it is not possible to reduce the value of μ_Λ below a certain minimum determined by the location of the roots of $E-V$. The formula (42) is essentially inexact for anharmonic oscillators.

The uncertainty δE in the energy due to the fact that μ_Λ and m do not actually vanish is readily calculated if we retain terms of the first order in these two infinitesimals. Let $\tilde{a}_i(z_3)$ and $\tilde{a}_v(z_3)$ denote the values of $a_i(z)$ and $a_v(z)$ on the lower edge of the cut at z_3. Equating the approximate expressions for ψ on the two edges of the cut at z_3 in terms of f_i, f_v we obtain

$$e^{\iota(\pi+\lambda J)} = \frac{a_v(z_3)+e^{2\lambda \int_{z''}^{z_3}|p|d\zeta}\tilde{a}_i(z_3)}{\tilde{a}_v(z_3)+e^{2\lambda \int_{z''}^{z_3}|p|d\zeta}a_i(z_3)}.$$

Let ϵ_v and $\tilde{\epsilon}_v$ denote the small quantities $a_v(z_3)-a_v(z_1)$, $\tilde{a}_v(z_3)-a_v(z_1)$, respectively. To a first approximation

$$\iota(\pi+\lambda J) = \log_e\left\{1+\frac{\epsilon_v-\tilde{\epsilon}_v+[\tilde{a}(z_3)-a_i(z_3)]e^{2\lambda\int_{z''}^{z_3}|p|d\zeta}}{a_v(z_1)}\right\}.$$

Introducing appropriate upper bounds for ϵ_v, $\tilde{\epsilon}_v$, $|a_i(z_3)|$, $|\tilde{a}_i(z_3)|$ from the inequalities (27) and (39a), we obtain

$$|\pi+\lambda J-2\pi k| < \mu_\Lambda + m/|p(z_3)^{\frac{1}{2}}|,\ k=0, 1, 2, \cdots,$$

where m is the maximum value of $|Q/\lambda^2 p^{\frac{3}{2}}|$ between z_3 and $+\infty$. Let ΔE denote the quantity $h\partial E/\partial J$ which measures the approximate spacing of the energy levels given by the Sommerfeld formula. Then

$$\frac{\delta E}{\Delta E} = \frac{1}{2\pi}\left(\mu_\Lambda + \frac{m}{|p(z_3)^{\frac{1}{2}}|}\right).$$

The writer has carried through a calculation of μ_Λ and $m/|p(z_3)^{\frac{1}{2}}|$ for the Morse curve potential function corresponding to the normal state of the

H_2 molecule. For simplicity of computation the energy E was so chosen as to bring the classical turning points z' and z'' together. Λ was taken to be a semicircle with radius equal to half the distance from z' to the nearest complex root of $p^2(E, z)$. Under these circumstances μ_Λ is approximately 0.015 and $m/|p(z_3)^{\frac{1}{2}}|$ is 7.5×10^{-5}. The corresponding value of $\delta E/\Delta E$ is 0.0025, showing that the B. W. K. approximation should give the energy correct to at least two or three tenths of a percent of the spacing of adjacent levels. Of course the actual error involved should be appreciably less than this upper limit.

7. Higher Order Approximations

It is well known that our functions f_i, f_v can be obtained from (1) by making the transformation $\psi = e^{i\lambda\int y dz}$, expanding y formally in powers of h, or $1/\lambda$, and taking only the first two terms of the series. By taking n terms of the series, one obtains functions $f_i^{(n)}$, $f_v^{(n)}$ having properties similar to those of f_i and f_v. The series are semiconvergent and hence the $f^{(n)}$'s, regarded as approximate solutions of (1), at first improve in quality as n increases, and then grow worse. They obey equations of the form

$$\frac{d^2 f^{(n)}}{dz^2} + [\lambda^2 p^2 - Q^{(n)}] f^{(n)} = 0,$$

where $Q^{(n)}(z, E)$ is a polynomial in $1/\lambda$ beginning with a term of degree $n-2$. If $Q \equiv Q^{(2)}$ is small at any point of the complex plane, $|Q^{(n)}|$ will be smaller than $|Q|$ provided that $n>2$, but not too large. Thus if $|Q/p^2|$ is small along a path Λ joining z_1 and z_3, $|Q^{(n)}/p^2|$ will be still smaller. We infer that the argument used to prove that, in the case of an anharmonic oscillator and an eigenvalue of E, the function f_v has sensibly the same values on the upper and lower edges of the cut in the region III, is applicable with a higher degree of precision to $f_v^{(n)}$. It follows at once that the energy level condition[9]

$$\oint y_v^{(n)} dz = mh, \quad m = 0, 1, 2, \cdots \qquad (43)$$

is even more accurate than (42) for small values of n greater than 2.

[9] Cf. J. L. Dunham, Phys. Rev. **41**, 713 (1932). Dunham's derivation of (43) is fundamentally sound, though no more rigorous than Zwaan's proof of the connection formulas.

8. Modification of the B. W. K. Method for the Radial Equation in the Two-Body Problem

In applying the B. W. K. method to the radial equation of the two-particle problem one meets with an apparent difficulty in that the B. W. K. approximations do not have the right character to fit the exact solutions of the differential equation at the left-hand boundary point $r=0$. As Kramers has pointed out, however, it is possible to fit the boundary conditions at both ends of the fundamental region if we modify the approximation formulas by the addition of the term $h^2/32\pi^2\mu r^2$ to the potential energy. The added term is negligible except in the immediate neighborhood of the origin, and it is ordinarily possible to find a good path Λ leading from the origin around the classical turning points (there is only one when the angular momentum is zero) in the complex plane to a point on the axis of reals outside the region of classical motion where the approximation function is good. In this manner it is possible to apply to the two-particle problem essentially the same reasoning as we have used above for the anharmonic oscillator in the interval $-\infty < z < +\infty$. One obtains the energy level formula (42) with $p(r, E)$ modified by the substitution of $(l+\frac{1}{2})^2$ for $l(l+1)$ in the term which gives the contribution of centrifugal force to the effective potential energy of the radial vibration. Further details regarding this application of our technique will be published elsewhere.[10]

9. The Transmission of Progressive Matter Waves Through a Potential Barrier

The transmission of matter waves through potential barriers has been the subject of much discussion,[11] but so far as the writer has been able to discover there has been no application of the B. W. K. method to this problem that is properly applicable to cases in which the energy of the incident particles is not very different from the maximum potential energy of the barrier.

Let us assume a rounded potential hill of approximately parabolic form. Consider first the

[10] In a book on the general principles of quantum mechanics now in preparation.

[11] Cf., e.g., N. H. Frank and L. A. Young, Phys. Rev. **38**, 80 (1931).

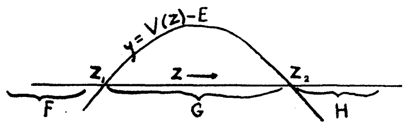

FIG. 5.

case in which the top of the hill projects above the energy level of the incident particles. We divide the axis of reals of the complex z plane into three parts F, G, H, separated by the classical turning points z_1, z_2 as indicated in Fig. 5. Single-valued functions f_i, f_v are obtained by introducing a cut along the axis of reals from z_1 to $+\infty$ and by choosing those branches of p and $p^{\frac{1}{2}}$ which are negative real and negative imaginary, respectively, in F. p, $p^{\frac{1}{2}}$, and w are to be analytic over the cut plane. p is then real and positive in H and positive imaginary in G. On both sides of the hill the function

$$f_i e^{-2\pi\iota Et/h} = p^{-\frac{1}{2}} \exp\left\{\frac{2\pi\iota}{h}\left[\int_{z'}^{z} p d\zeta - Et\right]\right\}$$

describes an outgoing wave, while $f_v e^{-2\pi\iota Et/h}$ describes an incoming wave. The level lines of $|e^{\iota w}|$ for a parabolic potential function are shown in the figure. f_v is dominant in most of the first and third quadrants, while f_i is dominant in most of the second and fourth quadrants.

Let us assume that progressive waves are incident on the potential hill from the left side only. There will then be reflected as well as incident waves in F, but only transmitted waves in H. We accordingly assume that a_v vanishes in the first quadrant where f_v is dominant. Let the regions F and H be connected by a good path Λ. Then a_i will be sensibly constant along this path. Thus we have a connection formula of the type

$$f_i + cf_v \to f_i, \tag{44}$$

where c is a constant whose value has yet to be determined. As a matter of fact we can evaluate only the modulus of c. This is done by using the constancy of the current density

$$D[\psi] = \frac{h}{4\pi i}\left(\psi^* \frac{d\psi}{dz} - \psi \frac{d\psi^*}{dz}\right)$$

on the axis of reals. Choosing z_1 as the origin of the integral w, we give $e^{\iota w}$ the absolute value 1 in the region F and the absolute value e^{-K} in H where K denotes the integral $(2\pi/h)\int_{z_1}^{z_2}|p|d\zeta$. The right-hand member of (40) now represents a current of density e^{-2K} in the direction of the positive z axis, while the left-hand member represents a current with the density $cc^* - 1$ in the same direction. Equating these two currents, we find that $cc^* = 1 + e^{-2K}$. The corresponding transmission coefficient is

$$T = e^{-2K}/cc^* = 1/(1+e^{2K}). \tag{45}$$

This formula is free from all restrictions depending on the height of the hill provided that $E - V_{\max} < 0$. Like Eq. (42) it is easily seen to hold for any analytic potential hill which yields no zeros of $E - V$ near z_1 and z_2 and hence permits a good path Λ joining the regions F and H.

If the maximum potential energy is less than the energy of the incident particles, a parabolic potential function yields two imaginary roots of $E - V(z)$, $z = \pm\iota\{2(E - V_{\max})/k\}^{\frac{1}{2}}$, where k denotes the negative of V''. Denoting these roots by z_1 and z_2 and the integral $(2\pi/h)\int_{z_1}^{z_2}|p|\,|d\zeta|$ by K', it is not difficult to prove that the transmission coefficient is

$$T = 1/(1+e^{-2K'}). \tag{46}$$

The reader will observe that the formulas (45) and (46) join continuously in the limiting case where $z_1 = z_2 = 0$, giving the common value $\frac{1}{2}$ for T.

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