Double Beta-Disintegration

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From the Fermi theory of β-disintegration the probability of simultaneous emission of two electrons (and two neutrinos) has been calculated. The result is that this process occurs sufficiently rarely to allow a half-life of over 10^{17} years for a nucleus, even if its isobar of atomic number different by 2 were more stable by 20 times the electron mass.

1. INTRODUCTION

IN a table showing the existing atomic nuclei it is observed that many groups of isobars occur, the term isobar referring to nuclei of the same atomic weight but different atomic number. It is unreasonable to assume that all isobars have exactly the same energy; one of them therefore will have the lowest energy, the others are unstable. The question arises why the unstable nuclei are in reality metastable, that is, why, in geologic time, they have not all been transformed into the most stable isobar by consecutive β-disintegrations.

The explanation has been given by Heisenberg[1] and lies in the fact that the energies of nuclei of fixed atomic weight, plotted against atomic number, do not lie on one smooth curve, but, because of the peculiar stability of the α-particle are distributed alternately on two smooth curves, displaced by an approximately constant amount against each other (the minimum of each curve is therefore at, roughly, the same atomic number). For even atomic weight the nuclei of even atomic number lie on the lower curve, those with odd atomic number on the higher one. One β-disintegration then brings a nucleus from a point on the lower curve into one of the upper curve, or *vice versa*. The nuclei on the upper curve are all of them unstable. But it may happen that a nucleus on the lower curve, in the neighborhood of the minimum, even though it is not the most stable one, cannot emit a single β-particle, since the resultant isobar, whose energy lies on the upper curve, has higher energy. This nucleus would then be metastable, since it cannot go over into a more stable one by consecutive emission of two electrons. This explanation is borne out by the fact that almost only isobars of even difference in atomic number occur.

A metastable isobar can, however, change into a more stable one by simultaneous emission of two electrons. It is generally assumed that the frequency of such a process is very small. In this paper the propability of a disintegration of that kind has been calculated.

The only method to attack processes involving the emission of electrons from nuclei is that of Fermi[2] which associates with the emission of an electron that of a neutrino, a chargeless particle of negligible mass. Thereby it is possible to explain the continuous β-spectrum and yet to have the energy conserved in each individual process by adjusting the momentum of the neutrino. In this theory the treatment of a β-disintegration is very similar to that of the emission of light by an excited atom.

A disintegration with the simultaneous emission of two electrons and two neutrinos will then be in strong analogy to the Raman effect, or, even more closely, to the simultaneous emission of two light quanta,[3] and can be calculated in essentially the same manner, namely, from the second-order terms in the perturbation theory. The process will appear as the simultaneous occurrence of two transitions, each of which does not fulfill the law of conservation of energy separately.

The following investigation is a calculation of the second-order perturbation, due to the interaction potential introduced by Fermi between neutrons, protons, electrons and neutrinos. As far as possible the notation used is that of Fermi. For a more detailed discussion and justification of this mathematical form and the assumptions involved reference must be made to Fermi's paper.

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[3] M. Goeppert-Mayer, Ann. d. Physik (V) **9**, 273 (1931).

2. The Mathematical Apparatus

The nucleus is assumed not to contain any electrons and neutrinos but to be built up out of neutrons and protons only. Neutron and proton are regarded as not essentially different from one another, but to represent two different quantum states of the heavy particle. The two kinds of light particles outside of the nucleus, the electrons and neutrinos are treated according to the method of superquantization. The stationary states of the electrons are taken to be those of positive energy H_s in the Coulomb field of the nucleus, described by four Dirac functions $\psi_s=(\psi_s^1, \psi_s^2, \psi_s^3, \psi_s^4)$. Since the neutrinos are not affected by the field of the nucleus their eigenfunctions are represented by plane Dirac waves $\varphi_\sigma=(\varphi_\sigma^1, \varphi_\sigma^2, \varphi_\sigma^3, \varphi_\sigma^4)$ with energies K_σ. The Pauli principle is assumed to hold for both electrons and neutrinos, so that the number N_s of electrons in a stationary state s and the number M_σ of neutrinos in a stationary state σ can be 0 or 1 only.

A quantum state N of the total system is then determined by these numbers N_s, M_σ and by the quantum state n, with energy W_n of the nucleus; n means in this case not only the quantum states in the ordinary sense but includes the quantum states of all possible isobars. The energy of a quantum state N is given by

$$E_N=\sum_s N_s H_s+\sum_\sigma M_\sigma K_\sigma+W_n. \tag{1}$$

The interaction energy between the heavy particles within the nucleus and the light particles without is constructed in such a way that the transition from neutron to proton is necessarily accompanied by the emission of an electron and a neutrino and *vice versa*. A matrix element of H corresponding to the transition of a neutron with eigenfunction u_n to a proton with eigenfunction v_m is different from 0 only if at the same time two numbers N_s, M_σ change from 0 to 1, and is then given by

$$H_M^N=H_{m\cdots 1_s\cdots 1_\sigma\cdots}^{n\cdots 0_s\cdots 0_\sigma\cdots}$$

$$=(-1)^{N_1+\cdots+N_{s-1}+M_1+\cdots+M_{\sigma-1}}\tilde{\psi}_s\delta\varphi_\sigma^* H_{nm}, \tag{2}$$

with the abreviation

$$\tilde{\psi}\delta\varphi^*=-\psi^{*1}\varphi^{*2}+\psi^{*2}\varphi^{*1}+\psi^{*3}\varphi^{*4}-\psi^{*4}\varphi^{*3}, \tag{3}*$$

$$H_{nm}=g\int v_m^* u_n d\tau. \tag{4}$$

The functions ψ and φ are taken at the place of the nucleus, the assumption being made that they do not vary considerably over its range. For an "allowed" transition $\int v_m^* u_n d\tau$, taken over the volume of the nucleus, has the order of magnitude 1. The value of the proportionality constant g can be roughly obtained by equating calculated and observed intensity of β-disintegration. Fermi determines it to be

$$g=4\times 10^{-50}\ \text{cm}^3\ \text{ergs}. \tag{5}$$

3. Calculations

The probability of simultaneous emission of two electrons and two neutrinos is obtained from the second order terms of the perturbation theory. In this approximation the amplitude of the transition probability is given by

$$a_M^N=\sum_K \frac{H_K^N H_M^K}{E_K-E_N}\left\{\frac{e^{(2\pi i/h)(E_M-E_N)t}-1}{E_M-E_N}-\frac{e^{(2\pi i/h)(E_M-E_K)t}-1}{E_M-E_K}\right\}. \tag{6}$$

We want to consider transitions of such a type that in the beginning state, N, no electrons or neutrinos are present; in the end state, M, the charge of the nucleus has been increased by 2, that is two neutrons have been transformed into protons, and two electrons have been emitted into states s and t, two neutrinos into states σ and τ. The intermediate state K then must be such that one neutron has changed into a proton, one electron and one neutrino have been emitted. This means that the intermediate state k of the nucleus is a quantum state of the isobar with atomic number larger by 1 than the original nucleus. The assumption that ordinary β-disintegration is energetically impossible means that $W_{kn}=W_k-W_n>-mc^2$; the first denominator E_K-E_N in (6) will therefore always be positive. The process of double β-disintegration is, however, energetically permissible if $W_{mn}=W_m-W_n<-2mc^2$. In this case the second

* A star denotes the conjugate complex of a quantity.

denominator, E_M-E_N, the difference of energy between beginning and end state, will, for those processes for which the energy is conserved, approach zero, and only then has $a_M{}^N$ appreciable values. As in the theory of radiation the second term in the bracket in (6) can then be neglected with respect to the first, since E_M-E_K has always a finite value.

The summation over the intermediate states of the light particles can be made explicitly; there are only four intermediate states possible, namely, electron s may have been emitted before or after electron t, and neutrino σ before or after neutrino τ. The power of -1 in the operators $a_s{}^*$, $b_\sigma{}^*$ introduces a difference in sign in the different cases. The result is:

$$a_M{}^N=\pm\sum_k H_{nk}H_{km}\left\{\left[\frac{1}{W_{kn}+H_s+K_\sigma}+\frac{1}{W_{kn}+H_t+K_\tau}\right]\tilde{\psi}_s\delta\varphi_\sigma{}^*\tilde{\psi}_t\delta\varphi_\tau{}^*\right.$$
$$\left.-\left[\frac{1}{W_{kn}+H_s+K_\tau}+\frac{1}{W_{kn}+H_t+K_\sigma}\right]\tilde{\psi}_s\delta\varphi_\tau{}^*\tilde{\psi}_t\delta\varphi_\sigma{}^*\right\}\frac{\exp\left[(2\pi i/h)(W_{mn}+H_s+H_t+K_\sigma+K_\tau)t\right]-1}{W_{mn}+H_s+H_t+K_\sigma+K_\tau}. \quad (7)$$

To obtain the transition probability $|a_M{}^N|^2$ has to be summed over all possible states s, σ, t and τ. The first step to this is to average the absolute value squared of (7) over all possible directions of the plane waves of the neutrino. This is done most easily by assuming that the neutrinos are confined to a large space Ω. In that case, neglecting the neutrino mass, $\overline{\varphi_\sigma{}^\nu\varphi_\sigma{}^\mu}=(1/4\Omega)\delta_{\nu\mu}$. We obtain terms of the type

$$\overline{|\tilde{\psi}_s\delta\varphi_\sigma{}^*|^2}=(1/4\Omega)(\tilde{\psi}_s\psi_s),\quad \overline{\tilde{\psi}_s\delta\varphi_\sigma{}^*\cdot\tilde{\psi}_t{}^*\delta\varphi_\sigma}=(1/4\Omega)(\tilde{\psi}_s\psi_t),$$

where the abbreviation $(\tilde{\psi}_s\psi_t)=\sum_\nu\psi_s{}^\nu{}^*\psi_t{}^\nu$ has been used.

Furthermore, from the properties of the ψ's, the relativistic eigenfunctions of the Coulomb field, it follows that, if the spin of the state s is parallel to that of t,

$$|(\tilde{\psi}_s\psi_t)|^2=(\tilde{\psi}_s\psi_s)(\tilde{\psi}_t\psi_t);$$

if the spins of states s and t are antiparallel

$$|\tilde{\psi}_s\psi_t|^2=0.\dagger$$

We therefore obtain

$$|a_M{}^N|^2=\frac{1}{16\Omega^2}A(\tilde{\psi}_s\psi_s)(\tilde{\psi}_t\psi_t)\cdot\frac{4\sin^2\left[(\pi/h)(W_{mn}+H_s+H_t+K_\sigma+K_\tau)t\right]}{(W_{mn}+H_s+H_t+K_\sigma+K_\tau)^2},$$

where A is the following abbreviation:

For parallel spin of states s and t,

$$A=A_1=\left|\sum_k H_{nk}H_{km}\left[\frac{1}{W_{kn}+H_s+K_\sigma}+\frac{1}{W_{kn}+H_t+K_\tau}-\frac{1}{W_{kn}+H_s+K_\tau}-\frac{1}{W_{kn}+H_t+K_\sigma}\right]\right|^2. \quad (8a)$$

For antiparallel spin

$$A=A_2=\left|\sum_k H_{nk}H_{km}\left[\frac{1}{W_{kn}+H_s+K_\sigma}+\frac{1}{W_{kn}+H_t+K_\tau}\right]\right|^2$$
$$+\left|\sum_k H_{nk}H_{km}\left[\frac{1}{W_{kn}+H_s+K_\tau}+\frac{1}{W_{kn}+H_t+K_\sigma}\right]\right|^2. \quad (8b)$$

Here the sum over k runs over all possible quantum states of the intermediate isobar.

† This is true only for ψ functions with the angular quantum number $j=1/2$ in the neighborhood of $r=0$, with which we are concerned. But all other ψ's are negligible at these distances from the origin.

The summation over all possible states of the neutrinos is now replaced by an integration over the energies K_σ, K_τ. The number of neutrino states whose energy lies between K_σ and $K_\sigma+dK_\sigma$ is given by

$$(8\pi\Omega/h^3c^3)K_\sigma^2 dK_\sigma.$$

In the first integration, i.e., over K_τ, only those energies contribute which lie in the neighborhood of the place $E_M-E_N=0$, which corresponds to conservation of energy. The integral becomes proportional to the time t in the usual manner. By neglecting the dependence of A on the energies, the probability of emission of one electron into state s, one into the state t becomes

$$P^{st}=(1/t)\sum_{\sigma,\tau}|a_M{}^N|^2=(8\pi^4/h^7c^6)A(\tilde{\psi}_s\psi_s)(\tilde{\psi}_t\psi_t)\int K_\sigma^2(W_{mn}+H_s+\dot{H}_t+K_\sigma)^2dK_\sigma$$
$$=(4\pi^4/15)(A/h^7c^6)(\tilde{\psi}_s\psi_s)(\tilde{\psi}_t\psi_t)(W_{nm}-H_s-H_t)^5. \quad (9)$$

Finally the summation over s and t is replaced by a double integration over the energy, that is over the continuous spectrum of the electron in the Coulomb field of the nucleus. It is somewhat more convenient to express the energies in units of mc^2, namely, to introduce the dimensionless quantities $mc^2\epsilon=W_{nm}>0$, $mc^2h_s=H_s$, $mc^2h_t=H_t$.

According to Fermi the value of ψ at the outer edge of the nucleus, that is at a distance $\rho=9\times10^{-13}$ cm from the center has to be used; the sum of $(\tilde{\psi}_s\psi_s)$ over all states with a fixed direction of spin and energy in the range dh_s is given by:

$$\sum_{dh_s}\psi_s(\rho)\psi_s(\rho)=\frac{16\pi}{|\Gamma(3+2S)|^2}\frac{m^3c^3}{h^3}\left(\frac{4\pi mc\rho}{h}\right)^{2S}\cdot h_s(h_s^2-1)^{\frac{1}{2}+S}$$
$$\exp\left[\pi\gamma\frac{h_s}{(h_s^2-1)^{\frac{1}{2}}}\right]\left|\Gamma\left(1+S+i\gamma\frac{h_s}{(h_s^2-1)^{\frac{1}{2}}}\right)\right|^2dh_s, \quad (10)$$

where Z is atomic number, $\gamma=Z/13F$, $S=(1-\gamma^2)^{\frac{1}{2}}-1$. S is for any occurring value of nuclear charge negative and small compared to 1; $\gamma=0.3$, $S=-0.05$ correspond to $Z\backsimeq41$. (10) can therefore be approximated by a much simpler form by neglecting S against $1/2$ in the exponent of (h_s^2-1) and S against 1 in the Γ-function, since

$$\exp\left(\pi\gamma\frac{h_s}{(h_s^2-1)^{\frac{1}{2}}}\right)\left|\Gamma\left(1+i\gamma\frac{h_s}{(h_s^2-1)^{\frac{1}{2}}}\right)\right|^2\backsimeq2\pi\gamma\frac{h_s}{(h_s^2-1)^{\frac{1}{2}}}.$$

The transition probability of the nucleus obtained by summation of (9) over all states and the two cases of parallel and antiparallel spin of the electrons, then turns out to be

$$P=\frac{4^6\pi^8\gamma^2}{15|\Gamma(3+2S)|^4}\frac{m^{11}c^{10}}{h^{13}}\left(\frac{4\pi mc\rho}{h}\right)^{4S}(A_1+A_2)\int_1^{\epsilon-1}h_s^2\int_1^{\epsilon-h_s}h_t^2(\epsilon-h_s-h_t)^5dh_tdh_s \quad (11)$$
$$=\frac{4^6\pi^8\gamma^2}{6\cdot7\cdot15|\Gamma(3+2S)|^4}\frac{m^{11}c^{10}}{h^{13}}\left(\frac{4\pi mc\rho}{h}\right)^{4S}(A_1+A_2)F(\epsilon-2),$$

where

$$F(x)=x^7\left(1+\frac{1}{2}x+\frac{1}{9}x^2+\frac{1}{9.10}x^3+\frac{1}{2\cdot9\cdot10\cdot11}x^4\right).$$

ϵ is the difference of energy between the beginning and end states of the nucleus in units mc^2. A_1 and A_2 are given by (8); they contain the dependence of the process on the energy levels of the intermediate nucleus. It turns out, however, that the values of those energies do not greatly influence the probability. Roughly one can put

$$A_1+A_2 \backsimeq \left|\frac{H_{nk}H_{km}}{W_k-W_n}\right|^2 \sim \frac{g^4}{m^2c^4},$$

with g given by (5). Numerical evaluation for $Z=31$ leads to

$$P=1.15\times10^{-35}F(\epsilon-2) \text{ sec.}^{-1}$$
$$=3.6\times10^{-28}F(\epsilon-2) \text{ year}^{-1}. \quad (12)$$

As seen from the general formula (11) P is almost independent of Z.

The value of F for some arguments is given in the following table:

$\epsilon=$	4	6	8	10
$F(\epsilon+2)=$	0.37×10^2	9.2×10^4	3.4×10^6	4.2×10^7

$\epsilon=$	12	20
$P(\epsilon-2)=$	3.3×10^8	1×10^{11}

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The Infrared Absorption Spectra of the Linear Molecules Carbonyl Sulphide and Deuterium Cyanide

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Carbonyl sulphide. The discovery of new vibration-rotation bands in the infrared absorption spectrum of carbonyl sulphide has made it possible to determine the vibrational energy level scheme of the molecule. The agreement between theory and experiment is quite satisfactory. *Deuterium cyanide.* The fundamental bands of deuterium cyanide ω_2 at 570 cm^{-1} and ω_3 at 2630 cm^{-1} have been measured with grating spectrometers. The former has a strong zero branch at 570.16 cm^{-1}, and the fine structure lines on each side are well separated. From the line spacing the moment of inertia of the molecule is found to be 22.92×10^{-40} gram cm^2. A comparison of this value with the corresponding one for HCN, i.e., 18.72×10^{-40} gram cm^2, permits the calculation of the internuclear distances. From the positions of these two bands, together with those of the observed fundamentals of HCN, the zeroth order quadratic potential energy expression is computed.

THE infrared absorption spectrum of carbonyl sulphide (COS) has been investigated by Cassie and Bailey[1] who found ten unresolved bands in the region 1–20μ using a prism spectrometer. Vegard[2] has deduced from x-ray measurements that the molecule is linear with interatomic separations C–O of 1.10 Angstrom units and C–S of 1.96 Angstrom units. The calculated moment of inertia is 178×10^{-40} gram cm^2.

The perturbed expression for the vibrational energy is

$$E \text{ (vibration)}=\nu_1V_1+\nu_2V_2+\nu_3V_3+X_{11}V_1^2$$
$$+X_{22}V_2^2+X_{33}V_3^2+X_{12}V_1V_2+X_{13}V_1V_3$$
$$+X_{23}V_2V_3+X_{ll}l^2+\text{constant}$$

where the V's are the vibrational quantum numbers, l is the azimuthal quantum number, and ν_1 to X_{ll} are constants. If ten bands which involve these constants in an independent way are located experimentally the ten constants may be calculated and the energy level system for the molecule determined. This is of fundamental importance because it provides the correlating network and serves to predict new absorption bands.

This problem is similar to that of HCN which has been developed through a number of experimental researches to a fairly complete solution.[3, 4, 5, 6, 7, 8] Recently Herzberg and Spinks[9]

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