# Note on the Transmutation Function for Deuterons

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We consider the effect of the finite size and ready polarizability of the deuteron on the probability of transmutations involving the capture of the neutron. These have as a consequence that the Coulomb repulsion of the nucleus is less effective than for alpha-particles or protons, and that the corresponding transmutation functions increase less rapidly with deuteron energy. We treat the collision by the adiabatic approximation and obtain quantitative results for this energy dependence which are in good agreement with experiment.

Many elements can be rendered radioactive by deuteron bombardment, the reaction involving the capture of a neutron and the liberation of a proton:

$$ {}_nA^m+H^2\rightarrow {}_nA^{m+1}+H^1. \quad (1)$$

Four reactions of this type have been studied in detail by Lawrence, McMillan and Thornton.[1] The transmutation functions which are found increase smoothly with deuteron energy, but the increase is far less rapid than we should expect on the basis of the familiar considerations of Gamow[2] on the penetration of charged particles through the potential barrier of the nuclear Coulomb field. To account even roughly for the observations on this basis, we are forced to assume that the Coulomb potential breaks down at very large distances ($\sim 1.5\times 10^{-12}$ cm for copper). The transmutation function is thus anomalous when compared to that for protons and alpha-particles, and it is natural to associate this anomaly with the structure of the deuteron, particularly its low stability. We wish to show in this paper that when this is taken into account, it does in fact provide a satisfactory explanation of the experiments.

For neutron capture to be possible the neutron must have an appreciable probability of coming within the range of the nuclear forces. But this condition can be satisfied even when the center of mass of the deuteron lies beyond the range of these forces. It is this possibility which leads to an explanation of the fact that even with such a highly charged element as copper nuclear transmutations can occur for deuteron energies of the order of 2 MV.

The quantitative treatment of the corresponding collision problem is complicated, not only by our ignorance of the detailed forces involved, but by the complete inapplicability of the Born approximation. For the velocity of the deuteron is not large compared to the internal velocity of proton and neutron; the effective time of collision of the deuteron is long compared to its period. We have thus to use the adiabatic approximation: the relative motion of proton and neutron is approximately given by the solution of the wave equation when the distance of the center of mass of the deuteron from the nucleus is held fixed; the center of mass moves in an effective potential which is the energy $\mathcal{E}(X)$ of the relative motion; and the perturbation which induces the inelastic impacts of the transmutation is the term in the kinetic energy neglected in this zeroth approximation. In fact the cross section for the transmutation is then given, with a proper normalization of the wave functions, by

$$\sigma=1/\hbar^2\left|\int\int dp\,dn\,\bar{\psi}_f(p,n)\right.$$
$$\left.\times[\hbar^2/4M\Delta_X+W-I-\mathcal{E}(X)]\psi_i(x,X)\right|^2. \quad (2)$$

Here $p$ and $n$ are the coordinates of proton and neutron, $x=p-n$ their relative coordinates, $X$ the coordinates of the center of mass of deuteron measured from the nucleus as origin. Further $\psi_i$ is the approximate adiabatic wave function for the initial state (normalized to unit flux); $\psi_f$ the wave function for the final state, in which the neutron is captured and the proton is flying off with a considerable kinetic energy; and $W$, $I$, $2M$ are kinetic energy, binding energy and mass of

[1] Lawrence, McMillan and Thornton, preceding article. We are greatly indebted to the authors for the opportunity of seeing their experimental results, and for many helpful discussions.

[2] Gamow, *Atomic Nuclei and Radioactivity* (Oxford University Press, 1931).

the deuteron. We wish to evaluate the dependence of this cross section on the energy of the deuteron, insofar as this can be done without a detailed knowledge of the structure of the nucleus and the nature of the nuclear forces.

The reactions (1) are highly exothermic, and we may therefore neglect the dependence of the final wave function for the proton on the deuteron energy; except in the immediate neighborhood of the nucleus we can take for the final wave function of the proton a plane wave. Further, the final wave function for the neutron will vanish except in the immediate neighborhood of the nucleus. Since the neutron is far more stably bound in the nucleus than in the deuteron, it is reasonable to neglect the finite extension of the corresponding wave function. The effect of this neglect will be to make the value of the binding energy of the deuteron which best fits the experimental curves somewhat lower than the true value, but the error should be small, especially for an atomic number as high as that of copper. We shall take then

$$\psi_f \sim e^{i/\hbar (2ME)^{\frac{1}{2}} p}\delta(n), \tag{3}$$

except for very small values of $p$. Here $E$ is the energy imparted to the proton by the disintegration.

The essential energy dependence of $\sigma$ is determined by the form of $\psi_i$. The adiabatic approximation gives

$$\psi_i = u(x, X)\varphi(X), \tag{4}$$

where $u(x, X)$ is the wave function for the relative coordinates $x$, when the center of mass $X$ is fixed, and $\varphi(X)$ is the wave function for the motion of the center of mass. If $V_0$ be the potential between neutron and proton, and $V_N$ the nuclear potential, then these functions are determined by the wave equations

$$\{\hbar^2/M\Delta_x + \mathcal{E}(X) - V_0(x) - V_N(x, X)\}u = 0, \tag{5}$$

$$\{\hbar^2/4M\Delta_X + W - I - \mathcal{E}(X)\}\varphi = 0. \tag{6}$$

According to the familiar arguments of Wigner, $V_0$ is given by an extremely narrow and deep trough, and is equivalent, as Bethe and Peierls[3] have observed, to the boundary condition for

$$\partial \ln u/\partial x)_{x=0} = -(MI)^{\frac{1}{2}}/\hbar. \tag{7}$$

For $V_N$ we shall take the Coulomb repulsion of the proton, and neglect the specifically nuclear forces in the immediate neighborhood of the nucleus; for these the adiabatic approximation can hardly be valid, and they could have a sensible effect on the transmutation function only if resonance occurred; the experimental curves afford no evidence for this.

According to (3) we need consider $u$ only for $n=0$, $x=2X$. If we now determine $u$, $\mathcal{E}$ and $\varphi$ by the Wentzel-Kramers-Brillouin method, we find, with sufficient approximation,

$$\begin{aligned} u(2X, X) &= a(X) \exp\left\{-\frac{2X(WI)^{\frac{1}{2}}}{\hbar v}\theta\left(\frac{Ze^2}{XI}-1\right)\right\} \\ \mathcal{E}(X) &= -I + Ze^2/X \\ \varphi(X) &= \frac{b(W, X)}{v}\exp\left\{-\frac{2Ze^2}{\hbar v}f\left(\frac{XW}{Ze^2}\right)\right\}. \end{aligned} \tag{8}$$

Here $Z$ is the nuclear charge, $v$ the deuteron velocity, $W=Mv^2$ the deuteron energy, and the functions $f$ and $\theta$ are defined by

$$\begin{aligned} f(\alpha) &= \cos^{-1}(\alpha)^{\frac{1}{2}} - (\alpha(1-\alpha))^{\frac{1}{2}}, \\ \theta(z) &= \frac{1+z}{(z)^{\frac{1}{2}}}\cot^{-1}(z)^{\frac{1}{2}} - 1, && z>1 \\ &= (2(1-z))^{\frac{1}{2}} - 1 + \frac{1+z}{(z)^{\frac{1}{2}}}\left(\tan^{-1}\left(\frac{2z}{1-z}\right)^{\frac{1}{2}} - \tan^{-1}(z)^{\frac{1}{2}}\right), && 0<z<1 \\ &= (2(1-z))^{\frac{1}{2}} - 1 + \frac{1+z}{(-z)^{\frac{1}{2}}}\left(\tanh^{-1}\left(\frac{-2z}{1-z}\right)^{\frac{1}{2}} - \tanh^{-1}(-z)^{\frac{1}{2}}\right). && z<0 \end{aligned} \tag{9}$$

[3] Bethe and Peierls, Proc. Roy. Soc. **A148**, 146 (1935).

Further $a(X)$, $b(W, X)$ are slowly varying functions. Far from the turning points $X_1 = Ze^2/2I$, $X_2 = Ze^2/W$, they are given by

$$a(X) = (I - Ze^2/2X)^{-\frac{1}{4}};\ b(W, X) = (Ze^2/X - W)^{-\frac{1}{4}}.$$

Since, as we shall see, the values of $X$ which contribute essentially to the matrix integrals for $\sigma$ vary little with $W$, the contribution of $a$ and $b$ to the transmutation function will be neglected.

We can thus write

$$\psi_i(2X, X) \sim 1/v$$

$$\exp\left\{-\frac{2Ze^2}{\hbar}\left(\frac{M}{I}\right)^{\frac{1}{2}} F\left(\frac{W}{I}, \frac{IX}{Ze^2}\right)\right\}, \quad (10)$$

where

$$F(q, y) = y\theta(1/y - 1) + q^{-\frac{1}{2}} f(qy). \quad (11)$$

Regarded as a function of $X$, $F$ has a rather sharp minimum. The position of this minimum varies little with deuteron energy in the range 1.5 MV to 3 MV, and lies roughly at $Ze^2/2I$. The breadth of the corresponding maximum of $\psi_i(2X, X)$ decreases slowly with $W$, but since this breadth is of the same order as the wavelength $\hbar/(2ME)^{\frac{1}{2}}$ of the proton wave function, this variation will affect the magnitude of the matrix integral very little. The effect of the variation of $a$ and $b$ and of the logarithmic derivatives of $u$ and $\phi$ which appear in the integrals for $\sigma$ could in principle be taken into account by obtaining accurate solutions of the wave equations (5), (6); but without a detailed knowledge of the energy of the proton $E$, and the form of the final neutron wave function, such refinements would be illusory. Throughout the range $I/2 < W < 2I$,

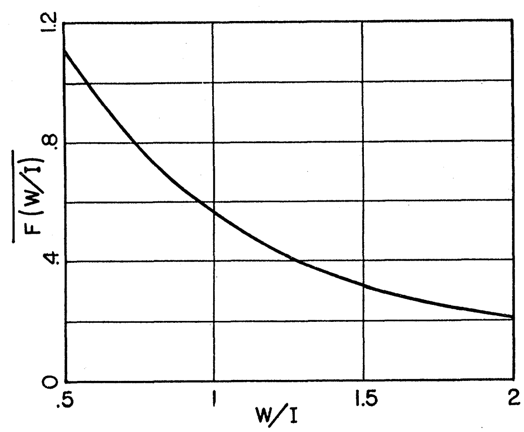


FIG. 1.

the variation of $\sigma$ with $W$ is given essentially by

$$\sigma \sim \frac{1}{v^2} \exp\left\{-\frac{4Ze^2}{\hbar}\left(\frac{M}{I}\right)^{\frac{1}{2}} \overline{F\left(\frac{W}{I}\right)}\right\}, \quad (12)$$

where $\overline{F(W/I)}$ is the minimum value of $F(W/I, IX/Ze^2)$. A plot of $\overline{F(W/I)}$ is given in Fig. 1.

The form of the transmutation functions given by (12) still depends upon the value of $I$. This is known to be roughly 2 MV, and this value gives satisfactory agreement with the experimental curves of Lawrence, McMillan and Thornton. In Fig. 2 of their paper Eq. (12) is plotted for three values of $I$ (1.5 MV, 2 MV, 2.4 MV) for the aluminum reaction. It is seen that the agreement with the experimental values is best for $I \sim 2$ MV. The approximations in the theoretical treatment would tend to favor rather too low a value of the binding energy of the deuteron.