Linear Modifications in the Maxwell Field Equations

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Expressions, accurate to the first order in e^2, are obtained for the charge and current densities which, according to positron theory, are induced in vacuum by an electromagnetic field. Because the corresponding correction terms in the Maxwell field equations involve integral operators, it does not seem possible to treat the modified field equations by Hamiltonian methods.

THE theory of the positron introduces certain modifications in the classical electromagnetic field equations of Maxwell, which result in the equations of the field being no longer equivalent to a system of linear second order differential equations. These modifications arise from the interaction of the electromagnetic field with the continuous distribution of electrons in negative energy states envisaged by the theory. The interaction is treated by expanding the interaction terms in the Hamiltonian function in powers of the electronic charge; in the following we shall consider only effects in e^2. In this order the linearity of the equations is still preserved and the effect of the interaction can conveniently be expressed as the induction of a charge and current density in vacuum by the electromagnetic field.

STATIC FIELDS

We shall first investigate, using the methods recently formulated by Dirac[1] and Heisenberg,[2] the charge and current densities induced by a static field. The equations take their simplest form when written in terms of the Fourier amplitudes of the quantities involved; the Fourier amplitude of a function $f(\mathbf{r})$ corresponding to a propagation vector $\mathbf{k}$ will be written simply f.

The Fourier amplitude of the charge induced by an electrostatic field which is everywhere small compared to the critical field $F_c = m^2c^3/e\hbar$ can be written[3]

$$\delta j_0 = -(\alpha/4\pi^3)\chi(k)A_0 = -(\alpha/\pi^2k^2)\chi(k)j_0, \quad (1)$$

where $j_0(\mathbf{r})$ is the charge density, $A_0(\mathbf{r})$ is the scalar potential, k is the magnitude of the propagation vector $\mathbf{k}$, and

$$\chi(k) = \int \frac{\epsilon\epsilon' - (q^2+1-\frac{1}{4}k^2)}{\epsilon\epsilon'(\epsilon+\epsilon')} e^{i(q\cdot R)} d\mathbf{q}, \quad (2)$$

$$\epsilon = [1+(\mathbf{q}+\tfrac{1}{2}\mathbf{k})^2]^{\frac{1}{2}}, \quad \epsilon' = [1+(\mathbf{q}-\tfrac{1}{2}\mathbf{k})^2]^{\frac{1}{2}}.$$

The second form of (1) is obtained from the first by means of the equation $\Delta A_0(\mathbf{r}) = -4\pi j_0(\mathbf{r})$. From the integral (2) certain singular terms and normalization terms are still to be subtracted, after which the "off diagonal distance," $\boldsymbol{R}$, is to be put equal to zero. Explicit expressions for these subtractitive terms are given in Heisenberg's paper. Introducing the abbreviations $a^2 = 1+\frac{1}{4}k^2$, $v = \mathbf{q}\cdot\mathbf{k}/(q^2+a^2)$, we can write

$$\chi(k) = \tfrac{1}{2}\int (q^2+a^2)^{-\frac{1}{2}} v^{-1}\{(1+v)^{\frac{1}{2}} - (1-v)^{\frac{1}{2}} + (1+v)^{-\frac{1}{2}} - (1-v)^{-\frac{1}{2}}\} e^{i(q\cdot R)} d\mathbf{q}$$
$$+ \tfrac{1}{4}k^2 \int (q^2+a^2)^{-\frac{3}{2}} v^{-1}\{(1-v)^{-\frac{1}{2}} - (1+v)^{-\frac{1}{2}}\} e^{i(q\cdot R)} d\mathbf{q}. \quad (3)$$

On expanding the integrand of the first integral in (3) in powers of v we obtain, for this integral, the expression[4]

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[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. **30**, 150 (1934).

[2] W. Heisenberg, Zeits. f. Physik **90**, 209 (1934).

[3] See reference 2. It should be noted that we employ rational units, measuring length in terms of the Compton wavelength $\hbar/mc$, time in terms of $\hbar/mc^2$, and mass in terms of the electronic mass m.

[4] The Bessel function $K_n(z)$ which appears in (4) is that defined by Whittaker and Watson, *Modern Analysis*, §17.71. The integration formula used in obtaining (4) is given by Watson, *Theory of Bessel Functions*, §13.6. Watson's definition of $K_n(z)$ differs from that of *Modern Analysis* by a factor $(-1)^n$.

$$-\sum_{n=1}^{\infty}\frac{(-1)^n(4n)!}{2^{4n}[(2n)!]^2}\frac{2n}{2n+1}(\mathbf{k}\cdot\mathrm{grad}_R)^{2n}\int\frac{e^{i(q\cdot R)}}{(q^2+a^2)^{2n+\frac{1}{2}}}d\mathbf{q}$$

$$=\pi\sum_{n=1}^{\infty}\frac{(-1)^n 2n}{2^{2n-2}(2n+1)!}(\mathbf{k}\cdot\mathrm{grad}_R)^{2n}\frac{R^{2n-1}}{a^{2n-1}}K_{2n-1}(aR). \quad (4)$$

If one now examines the ascending series for $R^{2n-1}K_{2n-1}(aR)$ one sees that there are terms of two types: (a) an infinite series, whose leading term is of the form $R^{4n-2}(\log\frac{1}{2}aR+\text{const.})$; (b) a polynomial in R of degree $4(n-1)$. Hence, when the operator $(\mathbf{k}\cdot\mathrm{grad}_R)^{2n}$ is applied to terms of type (a), and $\mathbf{R}$ is permitted to approach zero, one obtains the result zero when $n>1$. From the terms of type (b) one obtains a contribution only from the term of degree $2n$. This contribution is

$$\frac{(-1)^{n-1}}{4}\frac{(n-2)!}{n!}a[(\mathbf{k}\cdot\mathrm{grad}_R)^{2n}R^{2n}]_{R=0}$$
$$=\frac{(-1)^{n-1}}{4}\frac{(n-2)!(2n)!}{n!}ak^{2n}.$$

The terms after the first in (3) thus give

$$\frac{\pi k^2}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+5)}\left(\frac{k}{2a}\right)^{2n+2}. \quad (5)$$

The second integral in (3) can be handled in the same way; the terms after the first are found to be

$$\frac{\pi k^2}{2}\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+3)}\left(\frac{k}{2a}\right)^{2n+2}. \quad (6)$$

Adding (5) and (6) and performing the summation, one obtains the result

$$\tfrac{2}{3}\pi k^2\log a$$
$$-\tfrac{1}{2}\pi k^2\int_0^1(1-x^2)\log[1+\tfrac{1}{4}k^2(1-x^2)]dx. \quad (7)$$

The first term in (4) and in the analogous series for the second integral in (3) are readily shown to give just the singular and normalization terms which, according to the theory, must be deleted, and in addition a term, $-\frac{2}{3}\pi k^2\log a$, which cancels the first term in (7). Hence

$$\chi(k)=$$
$$-\tfrac{1}{2}\pi k^2\int_0^{\pi/2}\cos^3\psi\log[1+\tfrac{1}{4}k^2\cos^2\psi]d\psi. \quad (8)$$

The calculation of the current density induced by a magnetostatic field is very similar. This current is given by

$$\delta\mathbf{j}=(\alpha/\pi^2k^2)\int\{[\epsilon\epsilon'+(q^2+1-\tfrac{1}{4}k^2)]\mathbf{j}-2\mathbf{q}(\mathbf{q}\cdot\mathbf{j})+\tfrac{1}{2}\mathbf{k}(\mathbf{k}\cdot\mathbf{j})\}[\epsilon\epsilon'(\epsilon+\epsilon')]^{-1}e^{i(q\cdot R)}d\mathbf{q} \quad (9)$$

The third term in the bracket vanishes in virtue of the condition $\mathrm{div}\mathbf{j}(\mathbf{r})=0$, i.e., $\mathbf{k}\cdot\mathbf{j}=0$. The remaining terms can be handled by the method used to evaluate (2). The only new point of interest arises in connection with the second term in the bracket, which apparently gives a dependence of the ith component of the induced current on the jth component of the inducing current. From this term, in the series analogous to (4) one obtains terms of the form

$$[\mathrm{grad}_R(\mathbf{j}\cdot\mathrm{grad}_R)(\mathbf{k}\cdot\mathrm{grad}_R)^{2n}R^{2n+2}]_{R=0}$$
$$=(2n+2)(2n)![k^{2n}\mathbf{j}+2n\mathbf{k}(\mathbf{k}\cdot\mathbf{j})k^{2n-2}].$$

Again using $\mathbf{k}\cdot\mathbf{j}=0$, we see that the contribution of j_j to δj_i in fact vanishes. One readily finds

$$\delta\mathbf{j}=-(\alpha/\pi^2k^2)\chi(k)\mathbf{j}, \quad (10)$$

with $\chi(k)$ as given by (8). The equality of the quite different looking integrals which appear in (1) and (9) might have been easily foretold by considerations, similar to those given in the following section, of the transformation properties of charge and current under a Lorentz transformation. The origin of the different representation of the function $\chi(k)$ in the two cases is to be found in the different meanings attached to canonical momentum in the presence and absence of a vector potential.

On recombining the Fourier components, we find for the charge and current densities induced by a static field[5]

$$\delta j_\lambda(\mathbf{r}') = \int U(r)\Delta \mathbf{j}_\lambda(\mathbf{r}'')d\mathbf{r}'', \ (\lambda = 0, 1, 2, 3), \quad (11)$$

where $\mathbf{r} = \mathbf{r}' - \mathbf{r}''$,

$$U(r) = -(\alpha/16\pi^4)\int_0^{\pi/2} \cos^3 \psi d\psi \int e^{i(k\cdot r)} \log\,(1+\tfrac{1}{4}k^2 \cos^2 \psi)k^{-2}d\mathbf{k}$$

$$= -(\alpha/4\pi^3 r)\int_0^{\pi/2} \cos^3 \psi d\psi \int_0^\infty \sin kr \log\,(1+\tfrac{1}{4}k^2\cos^2\psi)k^{-1}dk$$

$$= (\alpha/4\pi^2 r)\int_0^{\pi/2} \cos^3 \psi Ei(-2r \sec \psi)d\psi.$$

The exponential-integral function is obtained by deforming the path of the k integration to the imaginary axis, $\int_0^\infty = \frac{1}{2}[\int_0^{i\infty+0} + \int_0^{-i\infty+0}]$.

Varying Fields

The expressions (1) and (10) for the charge and current densities induced by a static field can readily be generalized to the case of an arbitrarily varying field. Let us consider a single Fourier component of the charge and current, $j_\lambda e^{i[(k\cdot r)-k_0 t]}$, $\lambda = 0, 1, 2, 3$. If $k^2 > k_0^2$ we can make a Lorentz transformation to a system moving with velocity $\mathbf{v} = k_0\mathbf{k}/k^2$; in this system the Fourier component becomes $j_\lambda' e^{i(K\cdot r')}$, where the length of the vector $\mathbf{K}$ is $K = |k^2 - k_0^2|^{\frac{1}{2}}$. This however can be regarded as a Fourier component of a static field, which induces a charge and current given by (1) and (10). Upon transforming back to the original coordinate system we thus have

$$\delta j_\lambda = -(\alpha/\pi^2 K^2)\chi(K)j_\lambda. \quad (12)$$

This equation has been established only when $k^2 > k_0^2$, i.e., only for spacelike $\mathbf{K}$. In order to establish this result also for timelike $\mathbf{K}$, we have only to note that, for time varying potentials, the modification in the expression for the induced charge and current consists in replacing the factor $1/(\epsilon+\epsilon')$ in (2) by $(\epsilon+\epsilon')/[(\epsilon+\epsilon')^2 - k_0^2]$. Since this is true irrespective of whether $\mathbf{K}$ is spacelike or timelike, it can immediately be concluded that (12) holds in either case. It should be noted however, that in consequence of our definition of K as $K = |k^2 - k_0^2|^{\frac{1}{2}}$, K^2 must be replaced by $-K^2$ throughout (12) if $\mathbf{K}$ is timelike. The only difficulty which can arise comes from the possible vanishing of the denominator $(\epsilon+\epsilon')^2 - k_0^2$. Since the minimum value of $(\epsilon+\epsilon')^2$ is $4+k^2$, this cannot occur if $\mathbf{K}$ is spacelike, or if $\mathbf{K}$ is timelike and $K<2$. That is to say, only the Fourier components of the field for which $\mathbf{K}$ is timelike and $K>2$ are capable of producing pairs, since only under these conditions can both energy and momentum be conserved. In performing integrations over such components the pair-production singularity must be avoided by deforming the path of the K integration into the complex plane.

Heisenberg's results (40) and (44) may readily be obtained from (12) by writing $j_0 = j_0(\mathbf{k})\delta(k_0 - k_0')$, and retaining only the first term in the expansion of $\chi(K)$ in powers of K^2.

We can now write the induced charge and current as an integral over four-space,

$$\delta j_\lambda(\mathbf{s}') = (\alpha/32\pi^5)\int_0^{\pi/2} \cos^3 \psi d\psi \int \Lambda(s, \psi)\Box\Box j_\lambda(\mathbf{s}'')d\mathbf{s}'', \quad (13)$$

where the four-vector $\mathbf{s} = \mathbf{s}' - \mathbf{s}''$, $s = |r^2 - t^2|^{\frac{1}{2}}$, and

$$\Lambda(s, \psi) = \int e^{i(K\cdot s)} \log\,(1 \pm a^2K^2)K^{-4}d\mathbf{K}. \quad (14)$$

Here $a = \frac{1}{2}\cos\psi$ and the plus or minus sign is to be taken in (14) according as $\mathbf{K}$ is spacelike or timelike.

We now introduce hyperbolic coordinates:

[5] A discussion of this formula and some of its applications is given in the following paper by E. A. Uehling. The expression for the induced charge was originally obtained by Dr. Uehling by a somewhat different method of calculation.

The three space components of $\mathbf{K}$ are expressed in terms of the usual polar coordinates k, θ, Φ; while

$$k_0 = \pm K \cosh\varphi, \quad k = K\sinh\varphi,$$

$$d\mathbf{K} = K^3 \sinh^2\varphi \sin\theta dK d\varphi d\theta d\Phi,$$

$0 < \varphi < \infty$, when $\mathbf{K}$ is timelike; and

$$k_0 = K\sinh\varphi, \quad k = K\cosh\varphi,$$

$$d\mathbf{K} = K^3\cosh^2\varphi \sin\theta dK d\varphi d\theta d\Phi,$$

$-\infty < \varphi < \infty$, when $\mathbf{K}$ is spacelike. We then have,[6] for timelike $\mathbf{s}$,

$$\begin{aligned}\Lambda(s, \psi) = 4\pi \int_0^\infty \log(1+a^2K^2)K^{-1}dK \int_{-\infty}^{\infty} e^{isK\sinh\varphi}\cosh^2\varphi d\varphi \\ +8\pi\int_0^\infty \log(1-a^2K^2)K^{-1}dK\int_0^\infty \cos(sK\cosh\varphi)\sinh^2\varphi d\varphi \\ = 8\pi\left[s^{-1}\int_0^\infty K_1(sK)\log(1+a^2K^2)K^{-2}dK + \tfrac{1}{2}\pi s^{-1}\int_0^\infty Y_1(sK)\log(1-a^2K^2)K^{-2}dK\right], \quad (15)\end{aligned}$$

and for spacelike $\mathbf{s}$,

$$\begin{aligned}\Lambda(s, \psi) = 2\pi\int_0^\infty \log(1+a^2K^2)K^{-1}dK\int_{-\infty}^{\infty}\cosh^2\varphi d\varphi\int_0^\pi e^{isK\cosh\varphi\cos\theta}\sin\theta d\theta \\ +4\pi\int_0^\infty \log(1-a^2K^2)K^{-1}dK\int_0^\infty \sinh^2\varphi d\varphi\int_0^\pi \cos(sK\sinh\varphi\cos\theta)\sin\theta d\theta \\ = -8\pi\left[\tfrac{1}{2}\pi s^{-1}\int_0^\infty Y_1(sK)\log(1+a^2K^2)K^{-2}dK + s^{-1}\int_0^\infty K_1(sK)\log(1-a^2K^2)K^{-2}dK\right]. \quad (16)\end{aligned}$$

The singularities which arise from the possibility of pair production make their appearance in the second integrals in (15) and in (16), which represent the contribution of timelike $\mathbf{K}$, when $K = 1/a$. In these integrals one is to suppose a cut made in the K plane from $1/a$ to infinity, and the symbol $\int_0^\infty$ is to be understood as meaning $\frac{1}{2}[\int_0^{\infty+i0} + \int_0^{\infty-i0}]$. The term in $1/sK$ in the ascending series for $K_1(sK)$ and for $Y_1(sK)$ does not converge at the lower limit of integration; however this singularity is only apparent, disappearing if the lower limit in both the K_1 and Y_1 integrals is put equal to ϵ, and ϵ is allowed to approach zero. The integrals are also singular when $s=0$; this singularity we must investigate in detail.

When $s \neq 0$, the contour of the second integral in (16) may be deformed to the imaginary axis,

$$\int_0^\infty \equiv \tfrac{1}{2}\left[\int_0^{\infty+i0} + \int_0^{\infty-i0}\right] = \tfrac{1}{2}\left[\int_0^{i\infty} + \int_0^{-i\infty}\right];$$

we then obtain just the negative of the first integral in (16). Hence $\Lambda(s, \psi)$ vanishes, as it obviously must, when $\mathbf{s}$ is outside the light cone; no effects are propagated with a velocity greater than that of light.

The first integral in (15) can be treated in a similar manner provided $s \neq 0$; making the deformation

$$\int_0^\infty = \tfrac{1}{2}\left[\int_0^{i\infty+0} + \int_0^{-i\infty+0}\right]$$

we obtain the negative of the second integral in (15), and in addition another term,

$$(\pi^2/2s)\int_{1/a}^\infty J_1(sK)K^{-2}dK,$$

which results from the difference in the values

[6] The integral representations of the Bessel functions which are used in the derivation of (15) and (16), and, later, in the Appendix, can be obtained, by use of the recurrence formulae, from the representation of $Y_0(x)$ given in *Modern Analysis*, Chap. XVII, Example 26, and from the representations of $K_0(x)$ given in Example 40.

of log $(1-a^2K^2)$ on the two sides of the cut.

We can thus write

$$\Lambda(s,\psi)=f(r,\psi)\delta(t+r)+\begin{cases}0, & \text{s spacelike.}\\ 4\pi^3 as^{-1}\int_1^\infty J_1(sK/a)K^{-2}dK, & \text{s timelike.}\end{cases} \quad (17)$$

The coefficient of the delta function can be most readily evaluated by noting that the integral over time of the second term in (17) vanishes. This integral is[7]

$$8\pi^3 a\int_1^\infty K^{-2}dK\int_0^\infty J_1(sK/a)(s^2+r^2)^{-\frac{1}{2}}ds$$
$$=8\pi^3 a\int_1^\infty I_{\frac{1}{2}}(rK/2a)K_{\frac{1}{2}}(rK/2a)K^{-2}dK=0,$$

since $K_{\frac{1}{2}}(z)\equiv 0$. Hence $f(r,\psi)=\int_{-\infty}^{\infty}\Lambda(s,\psi)dt$.

The evaluation of this integral, using the expressions (15) and (16) for $\Lambda(s,\psi)$, is given in the Appendix; we find

$$f(r,\psi)=8\pi^2 r^{-1}\int_0^\infty \sin Kr\log(1+a^2K^2)K^{-3}dK$$
$$=-8\pi^3 a^2 r^{-1}\int_1^\infty e^{-Kr/a}K^{-3}dK, \quad (18)$$

the second expression being obtained from the first by deforming the path of integration to the imaginary axis.

Eq. (13) can now be written

$$\delta j_\lambda(\mathbf{s}')=\int \Lambda_1(r)\{\Box\Box j_\lambda(\mathbf{s}'')\}d\mathbf{r}''+\int\Lambda_2(s)\Box\Box j_\lambda(\mathbf{s}'')d\mathbf{s}'', \quad (19)$$

where $\{\Box\Box j_\lambda(\mathbf{s}'')\}$ is the retarded value of $\Box\Box j_\lambda(\mathbf{s}'')$,

$$\Lambda_1(r)=-\frac{\alpha}{16\pi^2 r}\int_0^{\pi/2}\cos^5\psi d\psi\int_1^\infty e^{-2Kr\sec\psi}K^{-3}dK,$$

$$\Lambda_2(s)=\frac{\alpha}{8\pi^2 s}\int_0^{\pi/2}\cos^4\psi d\psi\int_1^\infty J_1(2Ks\sec\psi)K^{-2}dK.$$

The second integral in (19) is to be extended only over the interior of the light cone, and only over past values of the time. Our formalism of course, has made no distinction between past and future; this distinction first arises when we introduce boundary conditions to validate our neglect of surface integrals. A similar choice is involved in the first integral of (19), namely, in the use of the retarded, rather than the advanced value of $\Box\Box j_\lambda(\mathbf{s}'')$.

The second term in (19) represents an effect which is propagated with a velocity less than that of light. This can be pictured as due to the propagation of the electromagnetic pulse from an element of charge and current at $(\mathbf{r}'', t'')$ to within a few Compton wavelengths of $\mathbf{r}'$, and here the production (real or virtual) of a pair; one particle of this pair then reaches the point $\mathbf{r}'$, at a later time of course than the electromagnetic pulse.

It has already been shown that when $j_\lambda(\mathbf{s}'')$ is independent of time the second term in (19) vanishes. On transferring, by partial integration, one Laplacian from $j_\lambda(\mathbf{s}'')$ to $\Lambda_1(r)$, one again obtains the formula (11). Another interesting case is that of a charge and current $j_\lambda(\mathbf{s}'')=j_\lambda e^{i[(k\cdot r)-k_0 t]}$. If $\mathbf{K}$ is spacelike one can transform to a Lorentz frame in which $k_0'=0$; in this frame the entire contribution is again from the first term in (19). However if $\mathbf{K}$ is timelike one finds that in the Lorentz frame in which $\mathbf{k}'=0$ the entire contribution comes from the second term in (19). In either case, of course, one obtains just (12).

The linear corrections introduced by positron theory in the equations of the electromagnetic field can be represented by replacing the field equation div $\mathbf{E}(\mathbf{s}')=4\pi j_0(\mathbf{s}')$ by

$$\operatorname{div}\mathbf{E}(\mathbf{s}')-\int\Lambda(s)\Box\Box\operatorname{div}\mathbf{E}(\mathbf{s}'')d\mathbf{s}''=4\pi j_0(\mathbf{s}'), \quad (20)$$

where $\Lambda(s)=(\alpha/32\pi^5)\int_0^{\pi/2}\Lambda(s,\psi)\cos^3\psi d\psi$, and by a similar modification in the field equations involving $\mathbf{j}$. If $\mathbf{E}(\mathbf{s}')$ is expanded in powers of e^2, the first correction term is equivalent to (19), while the higher terms represent the effect of the charge induced by the induced charge, and so forth. These latter effects, of course, fiıst appear when the interaction of the electron distribution with itself is taken into account. To the order in

[7] Watson, *Theory of Bessel Functions*, §13.6.

which we have been working, the correction terms vanish in the absence of a charge or current. This, however, will no longer be true in the next approximation, wherein the terms of order e^4, cubic in the fields and their derivatives, are taken into account. In consequence of the occurrence of integral operators in (20), it does not appear to be possible to treat the modified field equations by Hamiltonian methods. The situation here is analogous to that encountered in classical electrodynamics, where, when one attempts to eliminate the electromagnetic field from the equations of motion for the charge, one is also led to integral operators.

I have discussed this subject many times with Professor J. R. Oppenheimer and I wish to thank him for his generous advice. I am indebted also to Dr. E. A. Uehling for his kindness in placing at my disposal his earlier calculation of the induced charge.

Appendix

Evaluation of the integral $f(r, \psi)=\int_{-\infty}^{\infty}\Lambda(s, \psi)dt$.

The time integral of the first terms in (15) and (16) is

$$16\pi\int_0^{\infty}\frac{dK}{K^2}\log(1+a^2K^2)\times\left[\int_0^{\infty}\frac{K_1(sK)}{(s^2+r^2)^{\frac{1}{2}}}ds-\frac{\pi}{2}\int_0^r\frac{Y_1(sK)}{(r^2-s^2)^{\frac{1}{2}}}ds\right].$$

The expression in brackets can be written

$$[\,]=\int_0^{\infty}K_1(Kr\sinh\theta)d\theta-\tfrac{1}{2}\pi\int_0^{\pi/2}Y_1(Kr\sin\theta)d\theta,$$

or, introducing integral representations of the Bessel functions,[6]

$$\begin{aligned}[\,]&=-\int_0^{\infty}\cosh t\,dt\Big[\int_0^{\infty}e^{-Kr\cosh t\sinh\theta}d\theta\\&\quad-\tfrac{1}{2}\int_0^{\pi}\sin(Kr\cosh t\sin\theta)d\theta\Big]\\&=\tfrac{1}{2}\pi\int_0^{\infty}Y_0(Kr\cosh t)\cosh t\,dt\\&=-\tfrac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos(Kr\cosh t\cosh s)\cosh t\,dtds.\end{aligned}$$

If we now set $t=u+v$, $s=u-v$, and use the addition formulae for the circular and hyberbolic functions, we find that the variables separate, and we are finally left with only integrals of the form

$$\int_{-\infty}^{\infty}\cos(Kry^2)dy\quad\text{and}\quad\int_{-\infty}^{\infty}\sin(Kry^2)dy.$$

One easily obtains the result

$$[\,]=\tfrac{1}{2}\pi\sin Kr/Kr$$

and hence (18).

It remains to show that the second terms in (15) and (16) contribute nothing. The time integral of these terms is

$$16\pi\int_0^{\infty}\frac{dK}{K^2}\log(1-a^2K^2)\times\left[\frac{\pi}{2}\int_0^{\infty}\frac{Y_1(Krs)}{(1+s^2)^{\frac{1}{2}}}ds-\int_0^1\frac{K_1(Krs)}{(1-s^2)^{\frac{1}{2}}}ds\right].\quad(21)$$

Again making use of an integral representation of Y_1, we write

$$\frac{\pi}{2}\int_0^{\infty}\frac{Y_1(Krs)}{(1+s^2)^{\frac{1}{2}}}ds=-\int_0^{\infty}\cosh t\,dt\int_0^{\infty}\frac{\sin(Krs\cosh t)}{(1+s^2)^{\frac{1}{2}}}ds.\quad(22)$$

If the sine is now expressed in terms of exponentials, and the paths of the resulting integrals over s are deformed to the imaginary axis, we find that (22) is equal to

$$-\int_0^1\frac{ds}{(1-s^2)^{\frac{1}{2}}}\int_0^{\infty}e^{-Krs\cosh t}\cosh t\,dt=\int_0^1\frac{K_1(Krs)}{(1-s^2)^{\frac{1}{2}}}ds;$$

this just cancels the second term in (21).