to that which we have given to (25) except for the ambiguity in (35). *π_t is no longer an integral of the equation of motion* since it does not commute with $\bar{H}$. This constitutes a special case illustrating a general feature of the covariant form of Dirac's equation. It is due to the fact that the α's and γ's are now functions of the coordinates. Two equations which are equivalent to each other are not interpretable with equal ease because, as a result of the special choice of γ^i, one does not admit the same integrals as the other one. Since (35) and (25) are equivalent, there must exist, of course, a spin transformation which connects the two equations with each other. The transformation is given by the matrices (accurate to first order terms in ω)

$$
\begin{aligned}
S&=\cos\frac{\omega t}{2}\cdot 1-i\sin\frac{\omega t}{2}\cdot\sigma_z-\frac{\omega'}{2}\left(y\cos\frac{\omega t}{2}+x\sin\frac{\omega t}{2}\right)\alpha_x+\frac{\omega'}{2}\left(x\cos\frac{\omega t}{2}-y\sin\frac{\omega t}{2}\right)\alpha_y,\\
S^{-1}&=\cos\frac{\omega t}{2}\cdot 1+i\sin\frac{\omega t}{2}\cdot\sigma_z+\frac{\omega'}{2}\left(y\cos\frac{\omega t}{2}+x\sin\frac{\omega t}{2}\right)\alpha_x-\frac{\omega'}{2}\left(x\cos\frac{\omega t}{2}-y\sin\frac{\omega t}{2}\right)\alpha_y.
\end{aligned}
\tag{37}
$$

The Hamiltonians are connected by the relation $H=S^{-1}\bar{H}S$ $(\bar{7}a)$

and the eigenfunctions by $\psi=S^{-1}\Psi.$ $(\bar{7}b)$

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The Efficiency of the Tube Counter

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The efficiency of counting ionizing particles by the tube counter is investigated, taking account of the electrical behavior of the counter and the random nature of the arrival of the particles. Expressions are obtained for the efficiency of counting, average recovery time, average voltage impulse delivered to the amplifier, and the average number of particles counted per unit time, N, in terms of the number of particles arriving per unit time, N_0, the time constant of the counter, and one other parameter, d, which is constant for a given counter. Methods for measuring d (which for most counters will lie between 1.3 and five) are indicated. The efficiency of counting decreases with an increase of N_0, rapidly at first, then more slowly, so as to have the asymptotic value $1/d$. The determination of N_0 from N, and the general question of counter efficiency are discussed.

THE behavior of Geiger-Müller and similar types of counters has been studied experimentally by various investigators.[1] As the negative potential on the tube, V_T, is increased from zero, the average number of counts per unit time, the source being constant, behaves as in Fig. 1. Counting begins at the "threshold voltage," V_S, rises rapidly to a value N, and remains nearly constant for a range of voltage which depends on the individual counter. In a well-built counter, the flat region of the curve extends over a considerable range of voltage. The counter is operated in this region, since any slight variation of supply voltage will then have negligible effect on the rate of counting. At higher voltages, the

[1] (The first two references give a good bibliography of the previous work.) Burger Scheidlin, Ann. d. Physik **12**, 283 (1932); Schulze, Zeits. f. Physik **78**, 92 (1932); Medicus, Zeits. f. Physik **74**, 350 (1932); Curtis, Bur. Standards J. Research **10**, 229 (1933); Brunetti and Ollano, Nuovo Cimento **10**, 92 (1933); Greiner, Zeits. f. Physik **81**, 543 (1933); Hummel, Zeits. f. Physik **76**, 483 (1932); Physik. Zeits. **34**, 331 (1933); Kuhn, Zeits. f. Instrumentenk. **54**, 415 (1934); Danforth, Phys. Rev. **46**, 1026 (1934); J. Frank. Inst. **219**, 108 (1935); Wernow, Trav. de l'Inst. d'Etat de Radium **2**, 30 (1933) (abstracted in Physik. Berichte **15**, 1448 (1934)); H. Teichmann, Physik. Zeits. **35**, 637 (1934); Bosch, Ann. d. Physik **19**, 65 (1934); Janossy, Zeits. f. Physik **88**, 372 (1934); Henning and Schade, Zeits. f. Physik **90**, 597 (1934); A relevant article on photon counters is Werner, Zeits. f. Physik **90**, 384 (1934); **92**, 705 (1934).

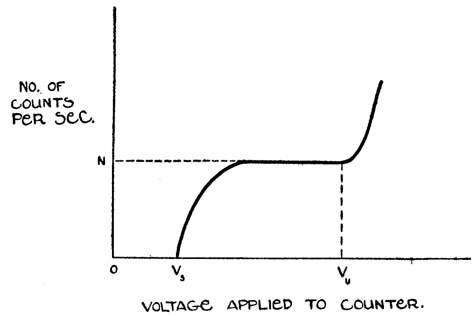

FIG. 1. Sensitivity curve for a tube counter.

curve begins to rise steeply, the conditions for breakdown are rapidly reached, and conditions are no longer suitable for counting.

When an ionizing particle passes through the counter, held at a potential in the region $V_S \rightarrow V_U$, the ions formed cause a discharge during which a certain amount of charge is transferred to the wire. In general, three types of behavior have been found. In the first, there is a small change in the wire potential, considerably less than that required to reduce the potential across the tube to V_S. The voltage impulse given the amplifier in this case is small compared to that in the other two, which we shall call types I and II. Fig. 2 shows the general behavior of these types of discharge, whose characteristics are beautifully shown in the photographs of Danforth.[1] The negative potential of the wire, v, is plotted against time. When $v = V_T - V_S$, the potential across the counter is the threshold potential. Thus, if $v > V_T - V_S$, no discharge is possible. In type I, there will then be no further discharge until v has fallen to $V_T - V_S$. The recovery time, τ, is the time necessary for this to take place. τ depends on the discharge which preceded it. The greater v is at the moment the discharge occurs, the less will be the initial potential of the wire, v_0, and the smaller will be the recovery time following the discharge. In type II, the counter will register no further particle until after the time T corresponding to the straight portion of the curve, Fig. 2b. The value of T can also depend on the value of v at the moment the last discharge occurred. In either case, there is a statistical distribution of τ's, but not a completely random distribution. The problem is a particular case of chain statistics. The efficiency of counting parti-

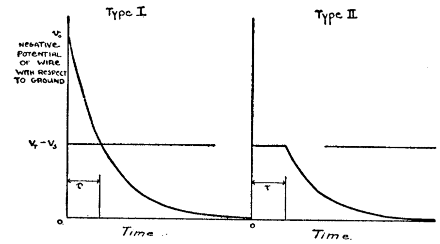

FIG. 2. Recovery time of counters.

cles under such circumstances will be investigated below. The question has been considered under the assumption of a constant τ.[2] For a counter in actual operation, τ is not constant; the statistical distribution of τ's will be different in the two cases, type I and type II, hence they will be treated separately.

Consider a situation in which, on the average, N_0 ionizing particles arrive per unit time. The chance of arrival of the particles is the same at any moment, i.e., the arrivals are distributed at random. The probability that the interval between the arrival of two particles lies between t_1 and $t_1 + dt_1$ is $N_0 \exp(-N_0 t_1) \cdot dt_1$. The probability that t_1 is more than any definite t_0 is $\int_{t_0}^{\infty} N_0 \exp(-N_0 t_1) \cdot dt_1 = \exp(-N_0 t_0)$. Let $\omega(\tau) d\tau$ be the probability that after a count the recovery time of the counter lies between τ and $\tau + d\tau$ when counting such a distribution of particles. Any particular τ_0 depends both on the recovery time for the previous discharge, and on the interval which elapsed after the last discharge, before the next particle arrived. If $t_1 < \tau_0$, the particle was not counted. Thus, in a great many discharges for which $\tau = \tau_0$, the fraction of the next particles to arrive which will be counted will be $\exp(-N_0 \tau_0)$. Those which arrive after an interval $t_1 > \tau_0$, thus being counted, cause a new discharge, whose recovery time $\tau' = \tau'(\tau_0; t_1)$. The relation $\tau' = \tau'(\tau_0; t_1)$ may be determined from the mechanism of the discharge. The probability that

[2] Johnson and Street, J. Frank. Inst. **215**, 239 (1933); Locher, J. Frank. Inst. **216**, 553 (1933); Volz, Zeits. f. Physik **93**, 539 (1935). If the counter be regarded as a mechanism which is not affected at all by any particle traversing it during its recovery time, its efficiency for constant τ is $(1 + N_0 \tau)^{-1}$, which is equivalent to the expression given in the first reference. The expression in the second reference gives too high an efficiency for particles arriving at random. The expression used by Volz allows for the ionization produced by the particle traversing the counter during its recovery time, and has the advantage of being mathematically tractable.

the new recovery time will lie between τ' and $\tau'+d\tau'$, is then: the product of the probability of the occurrence of a τ, by the probability the succeeding interval before the arrival of the next particle *counted* is such that $\tau'=\tau'(\tau;\ t_1)$, or $t_1=t_1(\tau;\ \tau')$ summed over all τ. Since for a particular τ only the fraction $\exp(-N_0\tau)$ are counted, the latter probability is

$$\frac{N_0 \exp(-N_0t_1)\cdot dt_1}{\exp(-N_0\tau)},\quad t_1\geqslant\tau;\quad 0,\quad t_1<\tau.$$

Therefore,

$$\omega(\tau')d\tau'=\int_{\text{all }\tau}\omega(\tau)d\tau\frac{N_0\exp(-N_0t_1)\cdot dt_1}{\exp(-N_0\tau)},\qquad t_1\geqslant\tau. \quad (1)$$

The range dt_1 must be that range, which, for any definite τ, corresponds to the range $d\tau'$.

$$\omega(\tau')d\tau'=\int_{\text{all }\tau}\omega(\tau)d\tau\frac{N_0\exp(-N_0t_1)}{\exp(-N_0\tau)}\left(\frac{\partial t_1}{\partial\tau'}\right)d\tau',\qquad t_1\geqslant\tau. \quad (2)$$

The existence of a probability distribution of the τ's requires $\omega(\tau')$ to be the same function of τ', which $\omega(\tau)$ is of τ. (2) is an homogeneous integral equation for $\omega(\tau)$, with the kernel

$$K(\tau',\tau)=\left[\frac{N_0\exp(-N_0t_1)}{\exp(-N_0\tau)}\cdot\frac{\partial t_1}{\partial\tau'}\right], \quad (3)$$

in which t_1 is a function depending on the type of discharge. In both of the cases to be considered $K(\tau',\tau)$ is of the form $f(\tau')$. Therefore,

$$\omega(\tau')=\int_0^{\tau_{\text{max.}}}\omega(\tau)d\tau f(\tau') \quad (4)$$

$$=Hf(\tau')\qquad (\partial H/\partial\tau'=0). \quad (4')$$

Substituting (4′) in (4), we see that a necessary condition for the existence of a solution, is

$$1=\int_0^{\tau_{\text{max.}}}f(\tau')d\tau'. \quad (5)$$

When (5) is true, (4′) is the solution of (4). Evidently

$$\int_0^{\tau_{\text{max.}}}\omega(\tau)d\tau=1.$$

$$\therefore\quad H=1,\quad\text{and}\quad\omega(\tau)=f(\tau). \quad (6)$$

The total number of counts that occur when N_0 particles per unit time arrive at the counter, is the number whose intervals equal or exceed the (variable) recovery time.

$$N=\int_0^{\tau_{\text{max.}}}\omega(\tau)N_0\exp(-N_0\tau)\cdot d\tau \quad (7)$$

and the fraction of incident particles counted is then

$$N/N_0=\int_0^{\tau_{\text{max.}}}f(\tau)\exp(-N_0\tau)\cdot d\tau. \quad (8)$$

The average and mean square recovery times are

$$\bar{\tau}=\int_0^{\tau_{\text{max.}}}\tau f(\tau)d\tau,\quad \overline{\tau^2}=\int_0^{\tau_{\text{max.}}}\tau^2 f(\tau)d\tau. \quad (9)$$

The average voltage impulse delivered to the amplifier, namely, the average rise of voltage of the wire during a discharge, is

$$v_A=\int_0^{\tau_{\text{max.}}}(v_0'-v_1)f(\tau)d\tau. \quad (10)$$

The quantities "v" are defined below.

Discharges of Type I

The maximum potential to which the wire rises during a discharge is v_0. The decay through the system, R and C,[3] takes place according to the usual relation

$$v=v_0\exp(-t/RC). \quad (11)$$

When $v\leq V_T-V_S$, a new discharge may occur. That is,

$$V_T-V_S=v_0\exp(-\tau/RC). \quad (12)$$

The statistical variation of the v_0's conditions that of the τ's. If every discharge were to result in the raising of the wire to the same potential v_0, the τ's would have a value (constant so long as N_0 were constant) $\tau=RC\ln(v_0/[V_T-V_S])$ (where, in general, v_0, and therefore τ, depend on the value of N_0), and

[3] R and C will mean the total resistance and capacity from wire to ground, respectively.

$$N/N_0 = \exp(-N_0\tau) = [v_0/(V_T - V_S)]^{-N_0RC}. \quad (13)$$

Danforth[1] has found in discharges of type I, the following:[4]

i. $v_0' - (V_T - V_S)$ increases with V_T;
ii. for the same V_T, v_0' decreases with increase of C; (from the examples given, $v_0' - (V_T - V_S) \propto 1/C$);
iii. v_0' is smaller if the discharge occurs before v from the previous discharge has become zero.

To these considerations, we add the fact that below the threshold voltage, the yield of new ions to continue the discharge decreases greatly. Thus, when the previous v has become zero, the relation will be

$$\begin{aligned} v_0' - (V_T - V_S) &= (A/C)(V_T - V_S), \\ A &= A(V_T) \\ v_0' &= ((A+C)/C)(V_T - V_S) \\ &= (C_1/C)(V_T - V_S), \quad C_1 > C. \end{aligned} \quad (14)$$

If C_1 be considered as the effective capacity of the system during discharge, this would correspond to a discharge which ceases when $V = V_S$, the quantity $q = C_1(V_T - V_S)$ being the amount carried over to the wire. This charge would be stored on C (which includes the capacity of the wire) and in the space between the wire and the tube. When the discharge ceases, the ions in the latter region are driven across by the remaining field, leaving only the small sheath about the wire, raising v from $V_T - V_S$ to v_0, and decreasing C_1 to C. The actual relations during discharge, of course, may be quite different, and the relation above still hold.

If the discharge occurs at a time t_1 after the last pulse, when the previous $v = v_1 + 0$, the voltage across the tube would be $V_T - v_1$, and the expression for v_0' would be

$$v_0' = v_1 + q/C = v_1 + (C_1/C)[(V_T - v_1) - V_S], \quad (14')$$

where $\qquad v_1 = v_0 \exp(-t_1/RC). \quad (11')$

v_0 is the maximum voltage on the wire in the last impulse. Let the recovery times of the two discharges be τ' and τ, respectively; then

$$\begin{aligned} V_T - V_S &= v_0 \exp(-\tau/RC) \\ &= v_0' \exp(-\tau'/RC). \end{aligned} \quad (15)$$

By (15), (11′), and (14′), the relation between τ', τ and t_1, is

$$\begin{aligned} (V_T - V_S)\exp(\tau'/RC) &= (V_T - V_S)\exp[(\tau - t_1)/RC] \\ &\quad + (C_1/C)(V_T - V_S)\{1 - \exp[(\tau - t_1)/RC]\} \end{aligned}$$

$$e^{-t_1/RC} = e^{-\tau/RC}\frac{d - e^{\tau'/RC}}{d-1}; \quad d = C_1/C > 1. \quad (16)$$

Since $d > 1$, the maximum τ' occurs when $v_1 = 0$, i.e., $t_1 = \infty$. In that case (14) holds, and comparison with (15) shows

$$d = \exp(\tau_{\max.}/RC) \quad (17)$$

and therefore, (16) incorporates in itself the fact that $t_1 \geqslant \tau$.

$$\begin{aligned} K(\tau', \tau) &= N_0 \exp[-N_0 t_1(\tau; \tau') + N_0\tau] \cdot \partial t_1/\partial\tau' \\ &= N_0 \exp(N_0\tau) \\ &\quad \cdot \left[e^{-\tau/RC}\left(\frac{d - e^{\tau'/RC}}{d-1}\right)\right]^{N_0RC}\left[\frac{e^{\tau'/RC}}{d - e^{\tau'/RC}}\right] \\ &= f(\tau'), \end{aligned}$$

$$f(\tau') = N_0\left[\frac{d - e^{\tau'/RC}}{d-1}\right]^b\left[\frac{e^{\tau'/RC}}{d - e^{\tau'/RC}}\right] \qquad [b = N_0RC]. \quad (18)$$

The condition (5) is satisfied, as may be seen by the substitution

$$w = \left[\frac{d - \exp(\tau'/RC)}{d-1}\right] \quad (19)$$

in the integrand. By this same substitution, and the substitution

$$x = [\exp(-\tau'/RC)] \cdot w,$$

we have,

$$N/N_0 = \int_0^1 b x^{b-1} dx/(1 + (d-1)x), \quad (20)$$

[4] Primed quantities refer to the following discharge, unprimed to the preceding one.

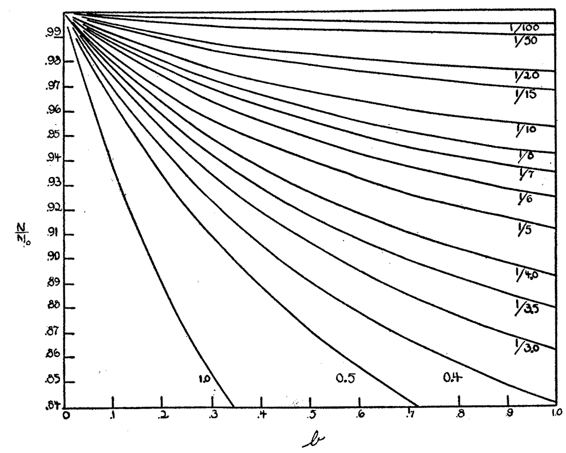

FIG. 3a.

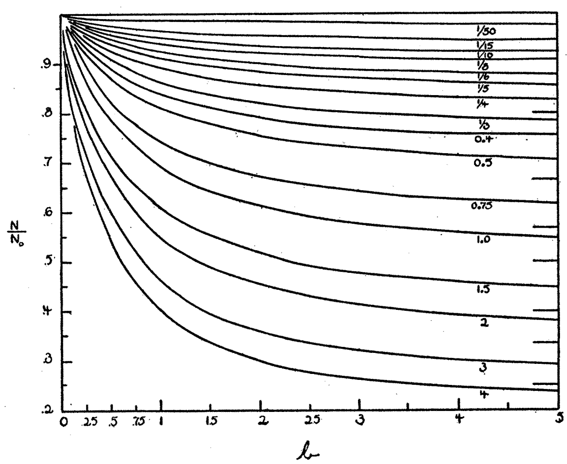

FIG. 3b.

FIG. 3. Efficiency of counting for discharges of type I. The abscissa is $b(=NRC)$. The value of "$d-1$" appears directly below each curve.

$$\bar{\tau}=bRC\ln d\int_0^1 \ln\left[1-(d-1)/d)w\right]w^{b-1}dw, \quad (21)$$

$$\overline{\tau^2}=b(RC)^2(\ln d)^2\int_0^1\{\ln\left[1-((d-1)/d)w\right]\}^2 \times w^{b-1}dw. \quad (21')$$

In order to evaluate (10) we have

$$(v_0'-v_1)=q/C=d[(V_T-V_S)-v_1]$$
$$=d(V_T-V_S)[1-\exp(\tau-t_1)/RC],$$

which by (19), gives as the expression for the average voltage impulse delivered to the amplifier,

$$v_A=d(V_T-V_S)/(b+1)$$
$$=(v_0)_{\text{max.}}/(N_0RC+1). \quad (21'')$$

For the experimentalist, the important quantity is N/N_0. The integral in (20) is not a standard form, and, except for special values of the constants, recourse must be had to series expansion. The series below are convenient.

$$\frac{N}{N_0}=b\sum_0^\infty\frac{(-1)^n(d-1)^n}{n+b}$$

$$=1-b\sum_1^\infty\frac{(-1)^n(d-1)^n,}{(n+b)}, \qquad 2>d\geqslant 1 \quad (22)$$

$$=\frac{b}{2}\left[Z'\left(\frac{b+1}{2}\right)-Z'\left(\frac{b}{2}\right)\right], \quad d=2;$$

$$\text{where } Z'(a)=\frac{d}{da}\ln\Gamma(a)$$

$$=\frac{b}{d}\sum_0^\infty\frac{(d-1)^n}{d^n}\,\frac{n!}{(b+n)(b+n-1)\cdots(b+1)b}. \quad (22')$$

For small values of d, the computation of (22) is more convenient, and it has the advantage of being an oscillating series. However, for $d\geqslant 2$, the expansion of the denominator in the integrand as $\Sigma(-1)^n(d-1)^nx^n$ is no longer uniformly convergent, and integration term by term is not possible. The formal series (22), it will be noted, is not convergent in this case. To obtain (22′) we set

$$b\int_0^1\frac{x^{b-1}dx}{1+(d-1)x}=\frac{b}{d}\int_0^1\frac{x^{b-1}dx}{1+((d-1)/d)(x-1)}$$

$$=\frac{b}{d}\sum_0^\infty\left(\frac{d-1}{d}\right)^n\int_0^1(1-u)^{b-1}u^n du.$$

The series follows immediately.

The rapidity of convergence of the series (22) may be increased by a transformation of the Kummer type. Applied once, for instance, the series becomes

$$\frac{N}{N_0}=\frac{1}{d}+\frac{b(d-1)}{d}\sum_0^\infty\frac{(-1)^n(d-1)^n}{(b+n)(b+n+1)}.$$

Fig. 3 shows the behavior of the function N/N_0 as b is increased, for different values of the parameter d. Fig. 5 is a plot of NRC $(=bN/N_0)$ against b, showing the variation in the number of particles counted, with the number arriving, for different values of d. For the larger values of b, the curves are practically linear due to the approximate constancy of N/N_0 in this region. As the number of ionizing particles to be counted increases from zero, the efficiency of counting decreases steadily. In the limit, for a very large number of particles arriving per unit time, the ratio approaches the value $1/d$.

$$\frac{N}{N_0}=b\int_0^1\frac{x^{b-1}dx}{1+(d-1)x}=\frac{1}{d}\int_0^b\frac{(1-v/b)^{b-1}dv}{1-((d-1)/d)v/b},$$

$$\operatorname*{Lim}_{b\to\infty}\left(\frac{N}{N_0}\right)=\frac{1}{d}\int_0^\infty e^{-v}dv=\frac{1}{d}.$$

In Fig. 3, the asymptotes are indicated by short lines directly under the curves. If, by some curious chance, the particles should arrive at equal intervals, the recovery time would be the same for each discharge.[5] However, it may easily be seen by Eq. (16), that τ would be constant only so long as the number of particles arriving per unit time remained the same. An increase of N_0 (the particles still coming at equal intervals) would decrease τ. Thus N/N_0 would not decrease exponentially to zero, as predicted by the assumption of a constant τ.

For a given system, $RC \ln d$ is equal to $\tau_{\text{max.}}$, i.e., the value τ takes when the potential of the wire drops to zero before the arrival of the next particle. Therefore d does not vary with the rate of arrival of the particles. Both b and d are of dimensions zero. Any change in the unit of time used is immaterial, as it should be.

The trend of N/N_0 has experimental significance, but the limit is not closely approached in practice, since b will not usually take large values. (A possible exception occurs in the case of showers, where, for a short time, b may become large.) As the number of incident particles per second (and hence b) increases, the amount of charge available in each discharge, and the maximum potential of the wire, decrease to the point where the amplifier does not supply the mechanical recorder with sufficient impulse to register the counts. For example, the voltage impulse delivered to the amplifier decreases to one-fifth of the value for slow rates of counting, when N_0RC has become equal to four. The values of b for which the mechanical recorder fails to respond, in general, will correspond to values of N_0 of such order that the tube counter is not the logical detector for the incident particles. With a counter of the usual construction, RC will be of the order of 0.1 to 0.001 sec. A value of $b=5$ would mean the arrival of at least fifty ionizing particles per second. For the detection of such a stream, an electrometer or electroscope would profitably be used. For this reason, values of b greater than $b=5$ are not included in the figures.

Discharges of Type II

The discharge in this case is of the form in Fig. 2. This takes place at higher tube voltages, and with larger C. The charge involved raises the wire to V_T-V_S, and keeps it there for a time T, after which v falls exponentially to zero. τ is given by T. During the time T, a current $I=(V_T-V_S)/R$ is passing through the tube. The mechanism is less certain. The fact that the process occurs at higher voltages suggests a temporarily self-sustaining discharge. The treatment below will be quite general, however. The following assumptions will be made:

1. There exists a maximum potential to which the wire can rise, $\bar{v}$. For the present the only assumption that will be made in regard to $\bar{v}$ is that $\bar{v}\geqslant V_T-V_S$. This is logical, since if $\bar{v}$ were less than V_T-V_S, the next incident particle would be registered even if it were to arrive immediately afterwards. τ' would be zero, which, of course, is not true.
2. The total quantity of electricity which passes in the discharge is of the form $q=C_1(V_T-V_S-v_1)$. C_1 depends on V_T, R and C, but not on v_1. C_1 must be greater than C in order to have the straight line portion of the discharge occur. The dependence of C_1 on R may be expected since R affects the value of the current through

[5] Eq. (13) referred to the case where τ is assumed constant, the particles arriving at random. In the case of particles coming at equal intervals, it is not permissible to assume the expression $\exp(-N_0\tau)$.

the tube in the steady state. If the counter is operated on the flat portion of its sensitivity curve, the variation of C_1 with V will be negligible. Thus, for a given counter, C_1 will have a definite value. We define $A = C_1 - C$.

The relation between t_1, τ and τ', is obtained as follows: The particle arrives when $v = v_1$. Since it is not registered if $t_1 < T$, the value of v_1 will be less than $\bar{v}$. In raising the wire to $\bar{v}$, an amount of electricity $C(\bar{v} - v_1)$ is used. During the rest of the discharge, an amount $q_1 = C_1(V_T - V_S - v_1) - C(\bar{v} - v_1)$ will pass. This will flow to ground through R as a current $I = \bar{v}/R$, and in time T an amount $(\bar{v}/R)T$ flows to ground. This is equal to q_1, since the exponential decrease of wire potential is due to the loss of the charge on the wire and condenser, which has held them at $\bar{v}$.

$$T = \frac{q_1}{I} = C_1 R \frac{V_T - V_S - v_1}{\bar{v}} - CR\left(1 - \frac{v_1}{\bar{v}}\right). \quad (23)$$

The recovery time is $\tau = T + \tau_1$, where $\tau_1 = RC \ln (\bar{v}/(V_T - V_S))$ is the time required for $\bar{v}$ to decay to $V_T - V_S$. The potential of the wire when $t > T$, is

$$v = \bar{v} \exp [-(t - T)/RC]$$

and we have

$$V_T - V_S = \bar{v} \exp [-\tau_1/RC]$$

so that

$$v = (V_T - V_S) \exp [(\tau_1 + T - t)/RC].$$

The new recovery time is then, by (23)

$$\tau' = \tau_1' + T' = RC \ln [\bar{v}/(V_T - V_S)]$$
$$- ((V_T - V_S)/\bar{v}) R(C_1 - C) \exp [(\tau - t_1)/RC]$$
$$+ C_1 R((V_T - V_S)/\bar{v}) - CR \quad (24)$$

so that the relation between τ', τ, and t_1 is

$$\exp(-t_1/RC) = [\exp(-\tau/RC)] \times \left[\frac{RC(\ln x) + (RC_1/x) - RC - \tau'}{R(C_1 - C)/x}\right], \quad (25)$$

where

$$x = \bar{v}/(V_T - V_S).$$

The kernel of the integral equation for $\omega(\tau)$ is then,

$$N_0 \left[\frac{RC\{(\ln x) - 1\} + (RC_1/x) - \tau'}{R(C_1 - C)/x}\right]^{N_0 RC - 1} \cdot \left[\frac{RC}{R(C_1 - C)/x}\right] = \frac{b}{g}\left[\frac{e - \tau'}{g}\right]^{b-1}, \quad (26)$$

the two equations being a definition of e and g.

For the existence of a solution, by (5),

$$\int_0^{\tau_{\max.}} \frac{b}{g}\left[\frac{e - \tau}{g}\right]^{b-1} d\tau = 1,$$

where $\tau_{\max.}$ is the τ corresponding to an infinite t_1; by (25), $\tau_{\max.} = e$. The condition becomes

$$b \int_0^{e} \left[\frac{e - \tau}{g}\right]^{b-1} \frac{d\tau}{g} = [u^b]_0^{e/g} = 1$$

so that for a solution to exist, e/g must be equal to one, which reduces to the equation

$$\phi(x) = \ln x + 1/x = 1; \quad (x = \bar{v}/(V_T - V_S)). \quad (27)$$

The function $\phi(x)$ has a minimum for $x = 1$, at which its value is one. For all other values of x more than zero, the value of the function is more than one. The condition (27) therefore requires that

$$\bar{v} = V_T - V_S. \quad (28)$$

This is exactly what is found by Schulze, and by Danforth.

When $\bar{v} = V_T - V_S$;

$$\tau_1 = 0; \quad \tau = T, \quad e = g = R(C_1 - C).$$

With these values of the constants,

$$\frac{N}{N_0} = \int_0^e \frac{b}{e}\left[1 - \frac{\tau}{e}\right]^{b-1} \exp(-N_0 \tau) \cdot d\tau$$
$$= \int_0^1 b[1 - u]^{b-1} \exp(-fu) \cdot du \quad (29)$$
$$f = b(d - 1),$$

where b and d are the same as before.

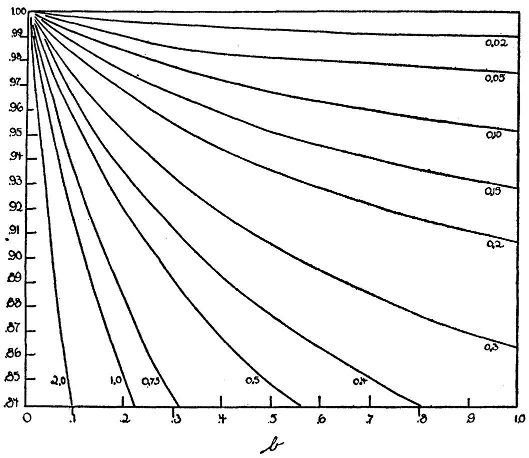

FIG. 4a.

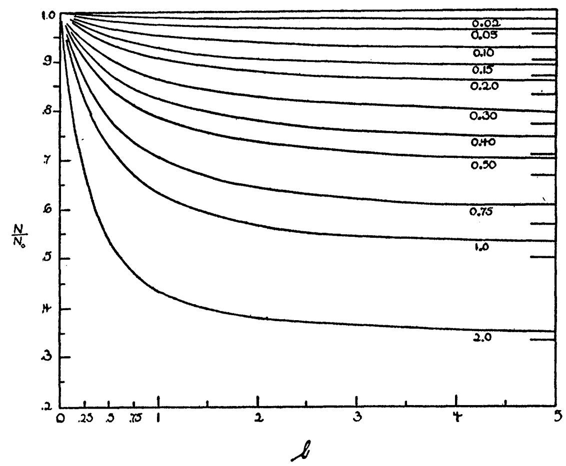

FIG. 4b.

FIG. 4. Efficiency of counting for discharges of type II. The abscissa is $b(=NRC)$. The value of "$d-1$" appears directly below each curve.

$$\bar{\tau}=\frac{b}{e}\int_0^e\left[1-\frac{\tau}{e}\right]^{b-1}\tau d\tau=\frac{e}{b+1}$$

$$=\frac{R(C_1-C)}{N_0RC+1}=\frac{RA}{N_0RC+1}, \qquad (30)$$

$$\overline{\tau^2}=\frac{e^2}{(b+1)(b+2)}. \qquad (30')$$

The average voltage impulse applied to the grid of the amplifier is obtained as follows:

$$v_0'-v_1=\bar{v}[1-\exp((T-t_1)/RC)]$$

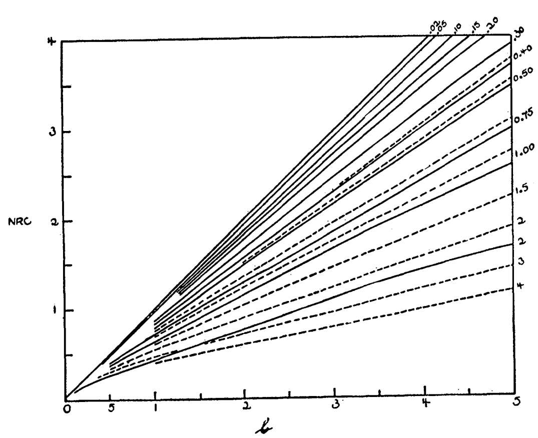

FIG. 5. The number of particles counted in relation to the number arriving. (The unit of time is RC.) The dashed lines refer to discharges of type I, the solid lines to those of type II. For small values of "$d-1$," the two cannot be drawn separately to this scale. The value of "$d-1$" is given at the right of each curve.

$$=(V_T-V_S)[1-(1-\tau'/e)],$$

$$v_A=(V_T-V_S)/(b+1) \qquad (30'')$$

$$=(v_0)_{\text{max.}}/(N_0RC+1) \text{ as before.}$$

The behavior of the function N/N_0 and the function NRC are given in Figs. 4 and 5. Again the integral for N/N_0 is not a standard form, and must be evaluated by series expansion. Expanding the exponential, the expression below is obtained:

$$\frac{N}{N_0}=b\sum_0^\infty\frac{(-f)^n}{n!}\frac{\Gamma(b)\Gamma(n+1)}{\Gamma(n+b+1)}$$

$$=1-\sum_1^\infty\frac{(-1)^nb^n(d-1)^n}{(b+n)(b+n-1)\cdots(b+1)}. \qquad (29')$$

This series converges rapidly, and is in convenient form. The values of N/N_0 lie lower than in the previous case, which means physically that the excess charge, q_1 takes longer to leak off through R, due to the lower potential of the wire during the recovery period; thus on the average, the τ's are longer, resulting in more lost counts. By a limiting process exactly analogous to that used in the previous case, it may be shown that N/N_0 again approaches the value $1/d$ as a limit when b becomes large.

The average recovery time, of course, decreases with increase of N_0. Its behavior with regard to

C and R depends upon how A varies with C and R. If A varies according to some such relation as $A=KC/R$ where K varies very slowly with change of C or R, the expression becomes

$$\bar{\tau}=KC/(N_0RC+1).$$

Such a relation would be in accord with the experimental results in this type of discharge, namely, that $\bar{\tau}$ increases with decrease of R, or increase of C. A general behavior of this nature would be expected if the straight line portion of Fig. 2b is regarded as a temporarily self-sustaining discharge, which ceases when the potential applied to the tube of the counter happens to fall below the value V_T due to natural fluctuations in the source of supply, or some similar accident. A knowledge of the exact mechanism is not necessary, though, as in actual practice, for a given counter, C and R are constant, and the variation of A with V_T is negligible.

The Determination of the Number of Incident Ionizing Particles

In order to determine N_0 from the registered value N, the values of d and of the product RC must be obtained for the particular counter. Using the product of RC and the registered number of particles per unit time as ordinate, the value of b may be read from the curve with the correct value of d, in Fig. 6. Division by RC then gives the true value, N_0. If the unit of time is taken to be RC, N gives N_0 directly.

If it is possible to obtain a trace of some actual discharges, as, for example, by a Braun tube, both constants may be obtained directly from the trace. RC is of course obtained from the rate of decay, and the parameter d may be obtained from either of the relations

$$d=\exp\ (\tau_{\text{max.}}/RC)=[v_0/(V_T-V_S)]_{v_1=0}.$$

If a trace is not obtainable, R and C may be measured directly. In that case, d can be obtained as follows:[6] A source of ionizing particles is brought to different distances from the counter, and values of N obtained. If the radiation at any one position is continued long enough, the inverse square law may be assumed to hold for the number of incident particles per unit time. This will give the ratio of the true number of incident particles, and the measured values will give the ratio of the N's. Knowing RC and the approximate number of incident particles, the value of d which gives a curve with the correct ratio of the N's for the chosen value of the ratio of the N_0's may be determined. (If desired, the more accurate value of one N_0 may be obtained from this curve, and the determination of d repeated, to increase accuracy.)

[6] The use of an electrometer to determine the average charge passing per count suggests itself. Then $d=C_1/C$, where C is measured, and $C_1=q/(V_T-V_S)$. If this is done, care must be taken that the number of counts per second is small, as q depends on the number of discharges per second, and the formula for C_1 is only valid when a long interval occurs between discharges.

Discussion

The efficiency of counting depends only on b and d. A change in the potential which the experimenter applies to the counter affects only the quantity A and therefore d. From the sensitivity curve of the counter, Fig. 1, it is evident that a counter operated on the flat portion of its sensitivity curve will show no variation of efficiency with change of applied voltage. Therefore d is to be regarded as constant for a given counter.

The value of N/N_0 decreases rapidly at first, and then more slowly, approaching asymptotically the value $1/d$ in both cases above. Thus the number counted always increases with the number arriving. This behavior would be expected physically, since the potential of the wire varies between narrower and narrower limits as N_0 increases, and therefore the recovery time becomes smaller and smaller. In the limit the ratio of $\bar{\tau}$ to the average period between the arrival of particles, namely $(1/N_0)$, approaches the value $(d-1)$ for both cases. Now, for a large number of particles, the ratio of $\bar{\tau}$ to $1/N$ (that is, the ratio of the average recovery period to the average period between counts) should be equal to the number of lost counts divided by the total number to be counted. Since, in the limit, $N/N_0=1/d$,

$$\frac{d-1}{d}=\frac{N_0-N}{N_0}=\frac{\bar{\tau}}{1/N}=\frac{\bar{\tau}}{d(1/N_0)};$$

so that $N_0\bar{\tau}$ should actually have the limiting value which it does.

In the actual counter, the registered number of counts will sometimes decrease as N_0 is increased. This is due to the smallness of the charge involved in each discharge, when N_0 becomes large. Thus the effect is to be traced to amplifier and mechanical registering device performance. For example, with a telephone message counter, five or six counts per second is very near the maximum obtainable, and a relatively large amount of charge must pass through it on each impulse to cause it to register. Commercial counters are on the market which will give somewhat over a hundred counts per second. Various laboratories have designed and are using counters using cheap watch mechanisms, capable of counting up to fifty or sixty evenly spaced impulses per second, accurately and conveniently. One such, designed in this laboratory, resolves impulses less than 0.01 sec. apart, and is accurate over the whole range up to its maximum count. This more than covers any experimental region which would be desired, and reduces the question of efficiency to amplifier and tube counter performance.[7]

In any counter there are a certain number of background counts (natural, cosmic, and radioactive contamination in the laboratory). With proper design and care, these may be reduced to as low as from one to five per minute. The correction in the case of a small number of background counts would consist, to sufficient accuracy, in the subtraction of this number from the number of registered counts. If much radioactive contamination is present in the laboratory, or, for any reason, the background radiation N' is of the same order as N, a better procedure would be to determine the true N's for the background, and the count separately, and then subtract. If the background becomes of the same order as the count, the usefulness of the counter decreases greatly.[8]

Throughout the above, the term, "ionizing particle" has been used. The number of particles which actually produce ionization in the counter is determined both by the actual number of particles, and by the ionization efficiency. The latter quantity will vary with the nature and velocity of the particles.[9] For the theoretical explanation of natural phenomena, however, the value of N, and the nature of the source are important. Corrections are then possible. If only the registered number is known, the true number of incident particles cannot, in general, be determined, since the counter of each investigator will have a different efficiency, and a different variation of efficiency with the rate of counting. Only for high rates of counting will the efficiency be sensibly constant, as may be seen from the curves given.

[7] For a good discussion of this, see the article by Locher, reference 2.

[8] The useful sensitivity is given by the expression $(N-N')/(N+N')^{\frac{1}{2}}$. Evans and Mugele, Phys. Rev. **47**, 427 (1935).

[9] In this connection, see Chase, Phys. Rev. **36**, 984 (1930).